Proceedings of the Edinburgh Mathematical Society (2012) **55**, 65–78
DOI:10.1017/S0013091510000799

CONJUGACIES FOR IMPULSIVE EQUATIONS

LUIS BARREIRA AND CLAUDIA VALLS
Departamento de Matemática, Instituto Superior Técnico,
Av. Rovisco Pais, 1049001 Lisboa, Portugal
(barreira@math.ist.utl.pt; cvalls@math.ist.utl.pt)

(Received 27 May 2010)

Abstract For impulsive differential equations, we construct topological conjugacies between linear and nonlinear perturbations of non-uniform exponential dichotomies. In the case of linear perturbations, the topological conjugacies are constructed in a more or less explicit manner. In the nonlinear case, we obtain an appropriate version of the Grobman–Hartman Theorem for impulsive equations, with a simple and direct proof that involves no discretization of the dynamics.

Keywords: Grobman–Hartman Theorem; topological conjugacies; impulsive equations

2010 *Mathematics subject classification:* Primary 37D25; 37C15

1. Introduction

We consider the linear impulsive differential equation

$$x' = A(t)x, \quad t \neq \tau_i, \qquad \Delta x|_{t=\tau_i} = B_i x \tag{1.1}$$

in $X = \mathbb{R}^p$, and its perturbation

$$x' = A(t)x + f(t,x), \quad t \neq \tau_i, \qquad \Delta x|_{t=\tau_i} = B_i x + g_i(x). \tag{1.2}$$

Essentially, impulsive differential equations correspond to a smooth evolution that at certain times τ_i changes abruptly, as for example in a mechanical clock. There are many applications of these equations to mechanical and natural phenomena involving abrupt changes. We refer the reader to [**14, 22**] for an extensive list of references.

Assuming that the dynamics defined by (1.1) admits a non-uniform exponential dichotomy, for sufficiently small perturbations f and g_i we show that there exist topological conjugacies between the dynamics defined by (1.1) and (1.2). This means that if $T(t,s)$ and $R(t,s)$ are respectively the evolution operators of the two equations, then there exist homeomorphisms $h_t \colon X \to X$ for $t \in \mathbb{R}$ such that

$$h_t \circ T(t,s) = R(t,s) \circ h_s, \quad t,s \in \mathbb{R}.$$

We also show that the map $(t,x) \mapsto h_t(x)$ has at most discontinuities of the first kind in the first variable at the times τ_i (this means that the limits when $t \to \tau_i^{\pm}$ exist, although they are different). In other words, this is an appropriate version of the Grobman–Hartman Theorem for impulsive equations. Our proof is somewhat inspired by the work of Chicone and Swanson [**9**] for (non-impulsive) autonomous equations obtained from perturbing a uniform exponential dichotomy. We emphasize that the argument does not involve any discretization of the dynamics. In addition, we also consider a certain class of linear perturbations of (1.1) and we construct topological conjugacies between the evolution operators of this equation and of its linear perturbations. Our proofs of these results follow to some extent arguments in [**4**, **6**], which already consider the non-uniform case, although only for non-impulsive differential equations. We refer the reader to [**1**, **2**] for related results for impulsive differential equations in the case of uniform exponential dichotomies. We note that these works use quite different techniques from ours, most notably in the case of the Grobman–Hartman Theorem. The arguments concerning linear perturbations are related to the former work of Palmer in [**19**].

The problem of whether the linearization of the system along a given solution approximates well to the solution itself goes back to Poincaré, and can be described as looking for a change of variables that takes the system to a linear one. The original references for the Grobman–Hartman Theorem are [**10**–**13**]. Using the ideas in Moser's proof of the structural stability of Anosov diffeomorphisms [**16**], the theorem was extended to Banach spaces independently by Palis [**17**] and Pugh [**21**]. A version for non-autonomous differential equations was obtained by Palmer in [**18**]. Sternberg [**24**, **25**] showed that there are algebraic obstructions preventing the existence of conjugacies with a prescribed higher regularity (we refer the reader to [**7**, **8**, **15**, **23**] for related work).

The classical notion of (uniform) exponential dichotomy, essentially introduced by Perron [**20**], plays a central role in a substantial part of the theory of dynamical systems. On the other hand, this notion is too stringent for the dynamics and it is of interest to look for more general types of hyperbolic behaviour. This is precisely the motivation to introduce the notion of non-uniform exponential dichotomy in the case of impulsive differential equations. Indeed, essentially any linear equation $x' = A(t)x$ in a finite-dimensional space with non-zero Lyapunov exponents has a non-uniform exponential dichotomy (see [**5**] for details). Our work is thus also a contribution to the theory of non-uniform hyperbolicity (we refer the reader to [**3**] for a detailed exposition).

2. Exponential dichotomies

We consider the linear impulsive differential (1.1), for some $m \times m$ matrices $A(t)$ and B_i for each $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, and some jumping times

$$\cdots < \tau_{-2} < \tau_{-1} < 0 < \tau_1 < \tau_2 < \cdots$$

satisfying $\lim_{i\to\pm\infty} \tau_i = \pm\infty$,

$$p := \sup\left\{\frac{\operatorname{card}\{i \in \mathbb{Z}\colon s \leqslant \tau_i < t\}}{t-s} : t, s \in \mathbb{R}, t > s\right\} < \infty, \tag{2.1}$$

and

$$\inf_{i\in\mathbb{Z}} |\det(\mathrm{Id} + B_i)| > 0.$$

https://doi.org/10.1017/S0013091510000799 Published online by Cambridge University Press

We assume that $t \mapsto A(t)$ has at most discontinuities of the first kind at the times τ_i. In particular, these conditions ensure the existence and uniqueness of global left-continuous solutions of (1.1) [**22**].

We write the solutions in the form $x(t) = T(t,s)x(s)$ for $t, s \in \mathbb{R}$, where $T(t,s)$ is the associated linear evolution operator (defined by the former identity). Clearly,

$$T(t,s)T(s,r) = T(t,r) \quad \text{and} \quad T(t,t) = \mathrm{Id}$$

for every $t, s, r \in \mathbb{R}$. We say that (1.1) admits a *non-uniform exponential dichotomy* if there exist projections $P(t)$ for $t \in \mathbb{R}$ satisfying

$$T(t,s)P(s) = P(t)T(t,s), \quad t, s \in \mathbb{R}, \tag{2.2}$$

and there exist constants $a, b, D > 0$ and $\varepsilon \geqslant 0$ such that

$$\|T(t,s)P(s)\| \leqslant D\mathrm{e}^{-a(t-s)+\varepsilon|s|}, \quad t \geqslant s, \tag{2.3}$$

and

$$\|T(t,s)Q(s)\| \leqslant D\mathrm{e}^{-b(s-t)+\varepsilon|s|}, \quad s \geqslant t, \tag{2.4}$$

where $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection of $P(t)$. We then define the *stable* and *unstable subspaces* at time $t \in \mathbb{R}$ respectively by

$$E(t) = P(t)(X) \quad \text{and} \quad F(t) = Q(t)(X).$$

The following is an example of non-uniform exponential dichotomy in the particular case when $P(t) = \mathrm{Id}$.

Example 2.1. Given $\omega, a, b > 0$, we consider the impulsive equation

$$x' = (-\omega - at\sin t)x, \quad t \neq \tau_i, \qquad \Delta x|_{t=\tau_i} = bx. \tag{2.5}$$

We assume that

$$\omega > a + p\log(1+b), \tag{2.6}$$

where p is the lim sup in (2.1). The solutions of (2.5) are given by

$$x(t) = T(t,s)(1+b)^{\mathrm{card}\{i \in \mathbb{N}\colon \tau_i \leqslant t\}} x(s),$$

where

$$T(t,s) = \mathrm{e}^{-\omega t + \omega s + at\cos t - as\cos s - a\sin t + a\sin s}.$$

Moreover, for each $t \geqslant s \geqslant 0$ we have

$$\begin{aligned} T(t,s) &= \mathrm{e}^{(-\omega+a)(t-s)+at(\cos t-1)-as(\cos s-1)+a(\sin s-\sin t)} \\ &\leqslant D\mathrm{e}^{(-\omega+a)(t-s)+2as}, \end{aligned}$$

where $D = \mathrm{e}^{2a}$. Therefore, by (2.1), there exists $C > 0$ such that

$$\begin{aligned} |x(t)| &\leqslant CD\mathrm{e}^{(-\omega+a)(t-s)+2as}\mathrm{e}^{pt\log(1+b)}|x(s)| \\ &= CD\mathrm{e}^{-(\omega-a-p\log(1+b))(t-s)}\mathrm{e}^{(2a+p\log(1+b))s}|x(s)|. \end{aligned} \tag{2.7}$$

By (2.6), this shows that (2.5) admits a non-uniform exponential dichotomy with $P(t) = \mathrm{Id}$ for each $t \in \mathbb{R}$. Moreover, if $t = 2k\pi$ and $s = (2l-1)\pi$ with $k, l \in \mathbb{N}$, then

$$T(t,s) = \mathrm{e}^{(-\omega+a)(t-s)+2as}.$$

This implies that in general (that is, for an arbitrary sequence τ_i) the exponent $2a + p\log(1+b)$ in (2.7) cannot be made smaller by taking the constants D and $\omega - a$ sufficiently large.

3. Conjugacies under nonlinear perturbations

Now we consider the perturbed equation (1.2), where the functions $f\colon \mathbb{R}\times X \to X$ and $g_i\colon X \to X$ satisfy $f(t,0) = 0$ for each $t \in \mathbb{R}$, and $g_i(0) = 0$ for each $i \in \mathbb{Z}$. We assume that f is piecewise continuous in t at most with discontinuities of the first kind at the times τ_i, and that there exists a constant $\delta > 0$ such that

$$\|f(t,x) - f(t,y)\| \leqslant \delta\mathrm{e}^{-2\varepsilon|t|}\min\{1, \|x-y\|\} \tag{3.1}$$

and

$$\|g_i(x) - g_i(y)\| \leqslant \delta\mathrm{e}^{-2\varepsilon|\tau_i|}\min\{1, \|x-y\|\} \tag{3.2}$$

for each $t \in \mathbb{R}$, $i \in \mathbb{N}$ and $x, y \in X$. Under these assumptions, provided that δ is sufficiently small, (1.2) has global left-continuous solutions. We denote the corresponding evolution operator by $R(t,s)$ for $t, s \in \mathbb{R}$.

We denote by $\mathcal{X}$ the space of functions $\eta\colon \mathbb{R}\times X \to X$ at most with discontinuities of the first kind in the first variable at the times τ_i, such that

$$\|\eta\|_\varepsilon := \sup\{\mathrm{e}^{\varepsilon|t|}\|\eta_t\|_\infty : t \in \mathbb{R}\} < +\infty, \tag{3.3}$$

where $\eta_t = \eta(t,\cdot)$ and

$$\|\eta_t\|_\infty = \sup\{\|\eta_t(x)\| : x \in X\}.$$

We note that $\mathcal{X}$ is a Banach space with the norm $\|\cdot\|_\varepsilon$ in (3.3).

Now we establish the existence of topological conjugacies between the solutions of (1.1) and (1.2). Given $\nu > 0$, we set

$$r_\nu = \sup_{t\in\mathbb{R}} \sum_{i\in\mathbb{Z}} \mathrm{e}^{-\nu|\tau_i - t|}.$$

Theorem 3.1. *Assume that (1.1) admits a non-uniform exponential dichotomy with $\varepsilon < \min\{a,b\}$. If δ in (3.1) and (3.2) is sufficiently small, then there exists a unique $\eta \in \mathcal{X}$ such that*

$$h_t \circ T(t,s) = R(t,s) \circ h_s, \quad t, s \in \mathbb{R}. \tag{3.4}$$

Moreover, $h_t = \mathrm{Id} + \eta_t$ is a homeomorphism for each $t \in \mathbb{R}$.

https://doi.org/10.1017/S0013091510000799 Published online by Cambridge University Press

Proof. We separate the proof into several steps. Set

$$U(t,s) = P(t)T(t,s) \quad \text{and} \quad V(t,s) = Q(t)T(t,s),$$

with $P(t)$ and $Q(t)$ as in (2.3) and (2.4). We define an operator F in $\mathcal{X}$ by

$$\begin{aligned}
F(\eta)(t,x) = \int_{-\infty}^{t} & U(t,u)f(u, T(u,t)x + \eta(u, T(u,t)x))\,\mathrm{d}u \\
&+ \sum_{\tau_i < t} U(t,\tau_i^+)g_i(T(\tau_i,t)x + \eta(\tau_i, T(\tau_i,t)x)) \\
&- \int_{t}^{\infty} V(t,u)f(u, T(u,t)x + \eta(u, T(u,t)x))\,\mathrm{d}u \\
&- \sum_{t \leqslant \tau_i} V(t,\tau_i^+)g_i(T(\tau_i,t)x + \eta(\tau_i, T(\tau_i,t)x)).
\end{aligned} \tag{3.5}$$

Lemma 3.2. *The operator F is well defined and $F(\mathcal{X}) \subset \mathcal{X}$.*

Proof. By (2.3) and (2.4) we have

$$\begin{aligned}
\int_{-\infty}^{t} & \|U(t,u)\| \cdot \|f(u, T(u,t)x + \eta(u, T(u,t)x))\|\,\mathrm{d}u \\
&+ \sum_{\tau_i < t} \|U(t,\tau_i^+)\| \cdot \|g_i(T(\tau_i,t)x + \eta(\tau_i, T(\tau_i,t)x))\| \\
&+ \int_{t}^{\infty} \|V(t,u)\| \cdot \|f(u, T(u,t)x + \eta(u, T(u,t)x))\|\,\mathrm{d}u \\
&+ \sum_{t \leqslant \tau_i} \|V(t,\tau_i^+)\| \cdot \|g_i(T(\tau_i,t)x + \eta(\tau_i, T(\tau_i,t)x))\| \\
&\leqslant D \int_{-\infty}^{t} \mathrm{e}^{-a(t-u)+\varepsilon|u|} \|f(u, T(u,t)x + \eta(u, T(u,t)x))\|\,\mathrm{d}u \\
&\quad + D \sum_{\tau_i < t} \mathrm{e}^{-a(t-\tau_i)+\varepsilon|\tau_i|} \|g_i(T(\tau_i,t)x + \eta(\tau_i, T(\tau_i,t)x))\| \\
&\quad + D \int_{t}^{\infty} \mathrm{e}^{-b(u-t)+\varepsilon|u|} \|f(u, T(u,t)x + \eta(u, T(u,t)x))\|\,\mathrm{d}u \\
&\quad + D \sum_{t \leqslant \tau_i} \mathrm{e}^{-b(\tau_i-t)+\varepsilon|\tau_i|} \|g_i(T(\tau_i,t)x + \eta(\tau_i, T(\tau_i,t)x))\|,
\end{aligned}$$

and hence

$$\begin{aligned}
\mathrm{e}^{\varepsilon|t|} & \sup_{x \in X} \|F(\eta)(t,x)\| \\
&\leqslant D\delta \int_{-\infty}^{t} \mathrm{e}^{(-a+\varepsilon)(t-u)} \mathrm{e}^{2\varepsilon|u|} \mathrm{e}^{-2\varepsilon|u|}\,\mathrm{d}u \\
&\quad + D\delta \sum_{\tau_i < t} \mathrm{e}^{(-a+\varepsilon)(t-\tau_i)} \mathrm{e}^{2\varepsilon|\tau_i|} \mathrm{e}^{-2\varepsilon|\tau_i|}
\end{aligned}$$

$$
\begin{aligned}
&+ D\delta \int_t^{\infty} \mathrm{e}^{-(b-\varepsilon)(u-t)} \mathrm{e}^{2\varepsilon|u|} \mathrm{e}^{-2\varepsilon|u|} \, \mathrm{d}u \\
&+ D\delta \sum_{t \leqslant \tau_i} \mathrm{e}^{-(b-\varepsilon)(\tau_i-t)} \mathrm{e}^{2\varepsilon|\tau_i|} \mathrm{e}^{-2\varepsilon|\tau_i|} \\
= {}& D\delta \biggl(\frac{1}{a-\varepsilon} + r_{a-\varepsilon} + \frac{1}{b-\varepsilon} + r_{b-\varepsilon} \biggr) < +\infty.
\end{aligned}
$$

This implies the desired statement. □

Lemma 3.3. *Identity (3.4) holds if and only if* $F(\eta) = \eta$.

Proof. We first assume that the identity (3.4) holds. Since $h_t = \mathrm{Id} + \eta_t$, we can rewrite (3.4) in the form

$$\mathrm{Id} + \eta_t = R(t, s) \circ h_s \circ T(s, t). \tag{3.6}$$

On the other hand, by the variation-of-constants formula we have

$$R(t, s)(x) = T(t, s)x + \int_s^t T(t, u) f(u, R(u, s)(x)) \, \mathrm{d}u + \sum_{s \leqslant \tau_i < t} T(t, \tau_i^+) g_i(R(\tau_i, s)x). \tag{3.7}$$

Using again (3.4), we can thus rewrite (3.6) in the form

$$
\begin{aligned}
\eta_t(x) &= R(t, s)(h_s(T(s, t)x)) - x \\
&= T(t, s) h_s(T(s, t)x) + \int_s^t T(t, u) f(u, R(u, s)(h_s(T(s, t)x))) \, \mathrm{d}u \qquad (3.8) \\
&\quad + \sum_{s \leqslant \tau_i < t} T(t, \tau_i^+) g_i(R(\tau_i, s)(h_s(T(s, t)x))) - x \\
&= T(t, s) \eta_s(T(s, t)x) + \int_s^t T(t, u) f(u, h_u(T(u, t)x)) \, \mathrm{d}u \\
&\quad + \sum_{s \leqslant \tau_i < t} T(t, \tau_i^+) g_i(h_{\tau_i}(T(\tau_i, t)x)). \qquad (3.9)
\end{aligned}
$$

Applying $P(t)$ to (3.8) yields

$$
\begin{aligned}
P(t)\eta_t(x) = U(t, s)\eta_s(T(s, t)x) &+ \int_s^t U(t, u) f(u, h_u(T(u, t)x)) \, \mathrm{d}u \\
&+ \sum_{s \leqslant \tau_i < t} U(t, \tau_i^+) g_i(h_{\tau_i}(T(\tau_i, t)x)). \qquad (3.10)
\end{aligned}
$$

Since $\eta \in \mathcal{X}$, it follows from (2.3) that, for $t \geqslant s$,

$$\|U(t, s)\eta_s(T(s, t)x)\| \leqslant D\mathrm{e}^{-a(t-s)+\varepsilon|s|} \|\eta_s\|_\infty \leqslant D\mathrm{e}^{-a(t-s)} \|\eta\|_\varepsilon.$$

Therefore, taking the limit in (3.10) when $s \to -\infty$ we obtain

$$P(t)\eta_t(x) = \int_{-\infty}^t U(t, u) f(u, h_u(T(u, t)x)) \, \mathrm{d}u + \sum_{\tau_i < t} U(t, \tau_i^+) g_i(h_{\tau_i}(T(\tau_i, t)x)). \tag{3.11}$$

On the other hand, applying $T(s,t)$ to (3.8) and replacing x by $T(t,s)x$ we obtain

$$T(s,t)\eta_t(T(t,s)x) = \eta_s(x) + \int_s^t T(s,u)f(u,h_u(T(u,s)x))\,\mathrm{d}u + \sum_{s\leqslant\tau_i<t} T(s,\tau_i^+)g_i(h_{\tau_i}(T(\tau_i,s)x)). \tag{3.12}$$

Applying $Q(s)$ to (3.12) yields

$$Q(s)\eta_s(x) = V(s,t)\eta_t(T(t,s)x) - \int_s^t V(s,u)f(u,h_u(T(u,s)x))\,\mathrm{d}u - \sum_{s\leqslant\tau_i<t} V(s,\tau_i^+)g_i(h_{\tau_i}(T(\tau_i,s)x)). \tag{3.13}$$

Since $\eta\in\mathcal{X}$, it follows from (2.4) that for $t\geqslant s$,

$$\|V(s,t)\eta_t(T(t,s)x)\| \leqslant D\mathrm{e}^{-b(t-s)+\varepsilon|t|}\|\eta_t\|_\infty \leqslant D\mathrm{e}^{-b(t-s)}\|\eta\|_\varepsilon.$$

Therefore, taking the limit in (3.13) when $t\to+\infty$, we obtain

$$Q(s)\eta_s(x) = -\int_s^\infty V(s,u)f(u,h_u(T(u,s)x))\,\mathrm{d}u - \sum_{s\leqslant\tau_i} V(s,\tau_i^+)g_i(h_{\tau_i}(T(\tau_i,s)x)). \tag{3.14}$$

In view of (3.5), it follows from (3.11) and (3.14) that

$$F(\eta)(t,x) = P(t)\eta_t(x) + Q(t)\eta_t(x) = \eta_t(x) = \eta(t,x).$$

Now we assume that $F(\eta)=\eta$. Replacing (t,x) by $(t,T(t,s)x)$ in (3.5), we obtain

$$\begin{aligned}
&\eta(t,T(t,s)x)\\
&\quad= \int_{-\infty}^t U(t,u)f(u,h_u(T(u,s)x))\,\mathrm{d}u + \sum_{\tau_i<t} U(t,\tau_i^+)g_i(h_{\tau_i}(T(\tau_i,s)x))\\
&\qquad - \int_t^\infty V(t,u)f(u,h_u(T(u,s)x))\,\mathrm{d}u - \sum_{t\leqslant\tau_i} V(t,\tau_i^+)g_i(h_{\tau_i}(T(\tau_i,s)x))\\
&\quad= T(t,s)\eta_s(x) + \int_s^t T(t,u)f(u,h_u(T(u,s)x))\,\mathrm{d}u + \sum_{s\leqslant\tau_i<t} T(t,\tau_i^+)g_i(h_{\tau_i}(T(\tau_i,s)x)),
\end{aligned}$$

which thus yields

$$\begin{aligned}
h_t(T(t,s)x) &= T(t,s)x + \eta_t(T(t,s)x)\\
&= T(t,s)h_s(x) + \int_s^t T(t,u)f(u,h_u(T(u,s)x))\,\mathrm{d}u\\
&\quad + \sum_{s\leqslant\tau_i<t} T(t,\tau_i^+)g_i(h_{\tau_i}(T(\tau_i,s)x)).
\end{aligned}$$

Moreover, by the variation-of-constants formula (see (3.7)), we have

$$R(t,s)(h_s(x)) = T(t,s)h_s(x) + \int_s^t T(t,u)f(u,R(u,s)h_s(x))\,\mathrm{d}u + \sum_{s\leqslant\tau_i<t} T(t,\tau_i^+)g_i(R(\tau_i,s)h_s(x))$$

We note that $h_t(T(t,s)x)$ and $R(t,s)(h_s(x))$ satisfy the same variation-of-constants formula and coincide for $t=s$. Therefore, they are the same, and identity (3.4) holds. This completes the proof of Lemma 3.3. □

Lemma 3.4. *Provided that δ is sufficiently small, there exists a unique $\eta\in\mathcal{X}$ satisfying $F\eta=\eta$.*

Proof. It is sufficient to prove that the operator F is a contraction. For each $\eta,\xi\in\mathcal{X}$ we have

$$\begin{aligned}
&F(\eta)(t,x) - F(\xi)(t,x)\\
&\quad= \int_{-\infty}^t U(t,u)[f(u,T(u,t)x+\eta(u,T(u,t)x)) - f(u,T(u,t)x+\xi(u,T(u,t)x))]\,\mathrm{d}u\\
&\qquad+ \sum_{\tau_i<t} U(t,\tau_i^+)[g_i(T(\tau_i,t)x+\eta(\tau_i,T(\tau_i,t)x)) - g_i(T(\tau_i,t)x+\xi(\tau_i,T(\tau_i,t)x))]\\
&\qquad- \int_t^{\infty} V(t,u)[f(u,T(u,t)x+\eta(u,T(u,t)x)) - f(u,T(u,t)x+\xi(u,T(u,t)x))]\,\mathrm{d}u\\
&\qquad- \sum_{t\leqslant\tau_i} V(t,\tau_i^+)[g_i(T(\tau_i,t)x+\eta(\tau_i,T(\tau_i,t)x)) - g_i(T(\tau_i,t)x+\xi(\tau_i,T(\tau_i,t)x))].
\end{aligned}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\|f(u,T(u,t)x+\eta(u,T(u,t)x)) - f(u,T(u,t)x+\xi(u,T(u,t)x))\|\\
&\qquad\qquad\leqslant \delta\mathrm{e}^{-\varepsilon|u|}\|\eta(u,T(u,t)x) - \xi(u,T(u,t)x)\|,
\end{aligned}$$

and

$$\begin{aligned}
&\|g_i(T(\tau_i,t)x+\eta(\tau_i,T(\tau_i,t)x)) - g_i(T(\tau_i,t)x+\xi(\tau_i,T(\tau_i,t)x))\|\\
&\qquad\qquad\leqslant \delta\mathrm{e}^{-\varepsilon|\tau_i|}\|\eta(\tau_i,T(\tau_i,t)x) - \xi(\tau_i,T(\tau_i,t)x)\|.
\end{aligned}$$

Using (2.3) and (2.4), we thus obtain

$$\begin{aligned}
&\mathrm{e}^{\varepsilon|t|}\|F(\eta)(t,x) - F(\xi)(t,x)\|\\
&\quad\leqslant D\delta\int_{-\infty}^t \mathrm{e}^{(-a+\varepsilon)(t-u)}\mathrm{e}^{\varepsilon|u|}\|\eta(u,T(u,t)x) - \xi(u,T(u,t)x)\|\,\mathrm{d}u\\
&\qquad+ D\delta\sum_{\tau_i<t} \mathrm{e}^{(-a+\varepsilon)(t-\tau_i)}\mathrm{e}^{\varepsilon|\tau_i|}\|\eta(\tau_i,T(\tau_i,t)x) - \xi(\tau_i,T(\tau_i,t)x)\|
\end{aligned}$$

https://doi.org/10.1017/S0013091510000799 Published online by Cambridge University Press

$$+ D\delta \int_t^\infty \mathrm{e}^{(-b+\varepsilon)(u-t)} \mathrm{e}^{\varepsilon|u|} \|\eta(u, T(u,t)x) - \xi(u, T(u,t)x)\| \, \mathrm{d}u$$
$$+ D\delta \sum_{t \leqslant \tau_i} \mathrm{e}^{(-b+\varepsilon)(\tau_i - t)} \mathrm{e}^{\varepsilon|\tau_i|} \|\eta(\tau_i, T(\tau_i,t)x) - \xi(\tau_i, T(\tau_i,t)x)\|.$$

Taking the supremum in x yields

$$\begin{aligned}
&\mathrm{e}^{\varepsilon|t|} \sup_{x \in X} \|F(\eta)(t,x) - F(\xi)(t,x)\| \\
&\qquad \leqslant D\delta \int_{-\infty}^t \mathrm{e}^{(-a+\varepsilon)(t-u)} \mathrm{e}^{\varepsilon|u|} \|\eta_u - \xi_u\|_\infty \, \mathrm{d}u \\
&\qquad\quad + D\delta \sum_{\tau_i < t} \mathrm{e}^{(-a+\varepsilon)(t-\tau_i)} \mathrm{e}^{\varepsilon|\tau_i|} \|\eta_{\tau_i} - \xi_{\tau_i}\|_\infty \\
&\qquad\quad + D\delta \int_t^\infty \mathrm{e}^{(-b+\varepsilon)(u-t)} \mathrm{e}^{\varepsilon|u|} \|\eta_u - \xi_u\|_\infty \, \mathrm{d}u \\
&\qquad\quad + D\delta \sum_{t \leqslant \tau_i} \mathrm{e}^{(-b+\varepsilon)(\tau_i-t)} \mathrm{e}^{\varepsilon|\tau_i|} \|\eta_{\tau_i} - \xi_{\tau_i}\|_\infty \\
&\qquad \leqslant D\delta \biggl(\frac{1}{a-\varepsilon} + r_{a-\varepsilon} + \frac{1}{b-\varepsilon} + r_{b-\varepsilon} \biggr) \|\eta - \xi\|_\varepsilon.
\end{aligned}$$

Finally, taking the supremum in t, we obtain

$$\|F(\eta) - F(\xi)\|_\varepsilon \leqslant D\delta \biggl(\frac{1}{a-\varepsilon} + r_{a-\varepsilon} + \frac{1}{b-\varepsilon} + r_{b-\varepsilon} \biggr) \|\eta - \xi\|_\varepsilon.$$

Therefore, for δ sufficiently small, the operator F is a contraction in $\mathcal{X}$. □

It remains to show that the functions $h_t = \mathrm{Id} + \eta_t$ are homeomorphisms. This can be established following arguments in the proof of Theorem 1 in [**6**]. For completeness we sketch the argument.

We first note that for each $y \in X$ and $t \in \mathbb{R}$, the equation $h_t(x) = y$ has a solution of the form $x = y + z$ if and only if $z = -\eta_t(y + z)$. By the Brouwer Fixed-Point Theorem applied to the ball of radius $\|\eta_t\|_\infty$ there exists z satisfying $z = -\eta_t(y+z)$, and thus h_t is surjective. To show that h_t is injective, we observe that if $x, y \in X$ satisfy $h_t(x) = h_t(y)$ for some $t \in \mathbb{R}$, then

$$R(s,t)(h_t(x)) = R(s,t)(h_t(y))$$

for every $s \in \mathbb{R}$. By (3.4) we obtain $h_s(T(s,t)x) = h_s(T(s,t)y)$, and hence

$$T(s,t)(x-y) = -[\eta_s(T(s,t)x) - \eta_s(T(s,t)y)]. \tag{3.15}$$

The right-hand side is bounded in t and s. On the other hand, if $x - y \neq 0$, then either $P(t)(x-y) \neq 0$ or $Q(t)(x-y) \neq 0$, and thus the left-hand side of (3.15) is not bounded in t. This contradiction shows that $x - y = 0$, and hence h_t is injective for each $t \in \mathbb{R}$. Finally, since h_t is continuous and invertible, it follows from the Domain Invariance Theorem that h_t is a homeomorphism. This completes the proof of Theorem 3.1. □

4. Conjugacies under linear perturbations

This section is dedicated to the construction of conjugacies between the evolution operators defined by two linear differential equations. For this we need a stronger version of dichotomy. Namely, we say that (1.1) admits a *strong non-uniform exponential dichotomy* if there exist projections $P(t)$ for $t \in \mathbb{R}$ satisfying (2.2), and constants $c \geqslant a > 0$, $d \geqslant b > 0$, $D > 0$ and $\varepsilon \geqslant 0$ satisfying the inequalities (2.3) and (2.4), as well as

$$\|T(t,s)P(s)\| \leqslant D\mathrm{e}^{c(s-t)+\varepsilon|s|}, \quad t \leqslant s,$$
$$\|T(t,s)Q(s)\| \leqslant D\mathrm{e}^{d(t-s)+\varepsilon|s|}, \quad t \geqslant s.$$

Now we consider the linear impulsive differential equations (1.1) and

$$x' = \hat{A}(t)x, \quad t \neq \tau_i, \qquad \Delta x|_{t=\tau_i} = B_i x, \tag{4.1}$$

for some $m \times m$ matrices $\hat{A}(t)$ and B_i for each $t \in \mathbb{R}$ and $i \in \mathbb{Z}$. In addition to the hypotheses in § 2, we assume that $t \mapsto \hat{A}(t)$ has at most discontinuities of the first kind at the times τ_i. We shall always assume that (1.1) and (4.1) admit strong non-uniform exponential dichotomies in $\mathbb{R}$. Without loss of generality we take the same constants a, b, c, d, D and ε for the two dichotomies. Let also $T(t,s)$ and $\hat{T}(t,s)$ be evolution operators associated respectively to (1.1) and (4.1). We consider the corresponding projections $P(t)$ and $\hat{P}(t)$, as well as the corresponding stable and unstable subspaces

$$E(t) = P(t)X \quad \text{and} \quad F(t) = Q(t)X,$$

and

$$\hat{E}(t) = \hat{P}(t)X \quad \text{and} \quad \hat{F}(t) = \hat{Q}(t)X,$$

where $Q(t) = \mathrm{Id} - P(t)$ and $\hat{Q}(t) = \mathrm{Id} - \hat{P}(t)$ for each $t \in \mathbb{R}$. We note that $X = E(t) \oplus F(t)$ for each t.

Theorem 4.1. *Assume that (1.1) and (4.1) admit strong non-uniform exponential dichotomies, and that* $\dim E(t) = \dim \hat{E}(t)$ *for some* t *(and thus, for all* t*). If* $\min\{a,b\} > 2\varepsilon$*, then there exist homeomorphisms* $h_t \colon X \to X$ *for* $t \in \mathbb{R}$ *such that*

$$h_t \circ T(t,s) = \hat{T}(t,s) \circ h_s, \quad t,s \in \mathbb{R}. \tag{4.2}$$

Proof. We first explain how the result can be obtained by constructing separately conjugacies for the stable and unstable components. Namely, let us assume that we have constructed homeomorphisms $h_t^- : E(t) \to \hat{E}(t)$ for $t \in \mathbb{R}$ such that

$$h_t^- \circ T(t,s) = \hat{T}(t,s) \circ h_s^- \text{ on } E(s), \quad t,s \in \mathbb{R}, \tag{4.3}$$

and homeomorphisms $h_t^+ \colon F(-t) \to \hat{F}(-t)$ for $t \in \mathbb{R}$ such that

$$h_t^+ \circ S(t,s) = \hat{S}(t,s) \circ h_s^+ \text{ on } F(-s), \quad t,s \in \mathbb{R}. \tag{4.4}$$

We note that the homeomorphisms h_t^+ can be obtained repeating the construction of the homeomorphisms h_t^- replacing T and $\hat{T}$, respectively, by the evolution operators $S(t,s) =$

https://doi.org/10.1017/S0013091510000799 Published online by Cambridge University Press

$T(-t,-s)$ and $\hat{S}(t,s)=\hat{T}(-t,-s)$ (this corresponds to reverse the time direction, thus transforming the expansion along the unstable subspaces into contraction, as in the case of the homeomorphisms h_t^- defined on the stable subspaces). One can easily verify that for each $t\in\mathbb{R}$ the map $h_t\colon X\to X$ defined by

$$h_t(x,y)=h_t^-(x)+h_{-t}^+(y),\quad (x,y)\in E(t)\times F(t) \tag{4.5}$$

is a homeomorphism. Furthermore, the identities in (4.2) follow readily from (4.3) and (4.4).

Now we proceed with the construction of the homeomorphisms h_t^-. We follow closely arguments in [**4**], and thus we only sketch the construction (referring to that paper for details). For simplicity of the exposition we assume that $P(t)=\hat{P}(t)=\mathrm{Id}$ for each $t\in\mathbb{R}$, in which case $E(t)=\hat{E}(t)=X$ for each $t\in\mathbb{R}$. This corresponds to the assumption that the dichotomy only exhibits contraction. The general case can be obtained with straightforward modifications. Again, for simplicity of the exposition, we shall write h_t instead of h_t^- (although when $E(t)=\hat{E}(t)=X$ it follows readily from (4.5) that indeed $h_t=h_t^-$).

For each $t\in\mathbb{R}$ and $x\in X$, we set

$$q(t,x)=\int_0^\infty \|T(t+\tau,t)x\|\,\mathrm{d}\tau.$$

It follows from (2.3) that the function q is well defined (recall that we are only considering the stable direction). Furthermore, we have

$$q(t,T(t,s)x)=\int_0^\infty \|T(t+\tau,s)x\|\,\mathrm{d}\tau=\int_t^\infty \|T(u,s)x\|\,\mathrm{d}u,$$

and thus, for $x\neq 0$ the function $t\mapsto q(t,T(t,s)x)$ is strictly decreasing. One can also show that

$$q(t,T(t,s)x)\to 0 \qquad \text{when } t\to+\infty$$

and that

$$q(t,T(t,s)x)\to +\infty \quad \text{when } t\to-\infty.$$

Therefore, for each $s\in\mathbb{R}$ and $x\in X\setminus\{0\}$ there exists a unique time $t=\tau_{s,x}\in\mathbb{R}$ such that $q(t,T(t,s)x)=1$.

For each $t\in\mathbb{R}$, we define a map $h_t\colon X\to X$ by

$$h_t(x)=\begin{cases}\dfrac{\hat{T}(t,\tau_{t,x})T(\tau_{t,x},t)x}{\hat{q}(\tau_{t,x},T(\tau_{t,x},t)x)} & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$$

where

$$\hat{q}(t,x)=\int_0^\infty \|\hat{T}(t+\tau,t)x\|\,\mathrm{d}\tau.$$

https://doi.org/10.1017/S0013091510000799 Published online by Cambridge University Press

These maps are precisely the desired conjugacies. Let us sketch the argument. For $y = T(t,s)x$ and $\tau = \tau_{t,y}$, we have

$$q(\tau, T(\tau,s)x) = q(\tau, T(\tau,t)y) = 1.$$

Therefore, $\tau_{s,x} = \tau_{t,y} = \tau_{t,T(t,s)x}$ (in view of the uniqueness of each of these numbers), and we obtain

$$\begin{aligned} h_t(T(t,s)x) &= \frac{\hat{T}(t, \tau_{t,T(t,s)x})T(\tau_{t,T(t,s)x}, t)T(t,s)x}{\hat{q}(\tau_{t,T(t,s)x}, T(\tau_{t,T(t,s)x}, s)x)} \\ &= \frac{\hat{T}(t,\tau_{s,x})T(\tau_{s,x}, s)x}{\hat{q}(\tau_{s,x}, T(\tau_{s,x}, s)x)} \\ &= \hat{T}(t,s)\frac{\hat{T}(s,\tau_{s,x})T(\tau_{s,x}, s)x}{\hat{q}(\tau_{s,x}, T(\tau_{s,x}, s)x)} \\ &= \hat{T}(t,s)h_s(x) \end{aligned}$$

for each $t, s \in \mathbb{R}$. This establishes (4.2).

Repeating arguments from [**4**] we can establish the following two lemmas.

Lemma 4.2. *For each $t \in \mathbb{R}$ the function h_t is continuous at* 0*.*

Lemma 4.3. *There exists a continuous function $K \colon \mathbb{R} \times (X \setminus \{0\})^2 \to \mathbb{R}$ such that, for every $t \in \mathbb{R}$ and $x, \bar{x} \in X \setminus \{0\}$,*

$$|\tau_{t,x} - \tau_{t,\bar{x}}| \leqslant K(t,x,\bar{x})\|x - \bar{x}\|.$$

By Lemma 4.3, the function $x \mapsto \tau_{t,x}$ is locally Lipschitz (and thus continuous) in $X \setminus \{0\}$ for each fixed t.

Lemma 4.4. *For each $t \in \mathbb{R}$, the function $x \mapsto \hat{q}(\tau_{t,x}, T(\tau_{t,x}, t)x)$ is continuous on $X \setminus \{0\}$.*

Proof of Lemma 4.4. Take $x, \bar{x} \in X \setminus \{0\}$ and write

$$\tau = \tau_{t,x} \quad \text{and} \quad \bar{\tau} = \tau_{t,\bar{x}}.$$

Let also

$$y = \hat{T}(t,\tau)T(\tau,t)x \quad \text{and} \quad \bar{y} = \hat{T}(t,\bar{\tau})T(\bar{\tau},t)\bar{x}.$$

We have

$$\hat{q}(\tau, T(\tau,t)x) = \hat{q}(\tau, \hat{T}(\tau,t)y) = \int_0^\infty \|\hat{T}(\tau+u,t)y\| \,\mathrm{d}u = \int_\tau^\infty \|\hat{T}(z,t)y\| \,\mathrm{d}z$$

and

$$\hat{q}(\bar{\tau}, T(\bar{\tau},t)\bar{x}) = \int_{\bar{\tau}}^\infty \|\hat{T}(z,t)\bar{y}\| \,\mathrm{d}z.$$

To prove that the function $x \mapsto \hat{q}(\tau, T(\tau,t)x)$ (with $\tau = \tau_{t,x}$) is continuous, it is sufficient to find continuous functions $K_4 = K_4(t,\tau)$ and $K_5 = K_5(t,x,\bar{x},y,\tau)$ such that

$$Z := |\hat{q}(\tau, T(\tau,t)x) - \hat{q}(\bar{\tau}, T(\bar{\tau},t)\bar{x})| \leqslant K_4\|x-\bar{x}\| + K_5\|y-\bar{y}\|. \tag{4.6}$$

Indeed, since the jumps in (1.1) and (4.1) are the same, the function $x \mapsto y = \hat{T}(t,\tau_{t,x})T(\tau_{t,x},t)x$ is continuous for each $t \in \mathbb{R}$. Therefore, the desired statement follows from (4.6). On the other hand, inequality (4.6) can be obtained by repeating arguments in [**4**]. □

The continuity of each map h_t follows immediately from the above lemmas.

The inverse of h_t is obtained as follows. For each $t \in \mathbb{R}$, we define a map $g_t\colon X \to X$ by

$$g_t(x) = \begin{cases} \dfrac{T(t,\hat{\tau}_{t,x})\hat{T}(\hat{\tau}_{t,x},t)x}{q(\hat{\tau}_{t,x},\hat{T}(\hat{\tau}_{t,x},t)x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $\hat{\tau}_{t,x}$ is the unique real number such that $\hat{q}(\hat{\tau}_{t,x},\hat{T}(\hat{\tau}_{t,x},t)x) = 1$. Then

$$h_t(g_t(x)) = \frac{\hat{T}(t,\tau)T(\tau,\hat{\tau})\hat{T}(\hat{\tau},t)x}{\hat{q}(\tau,T(\tau,t)g_t(x))q(\hat{\tau},\hat{T}(\hat{\tau},t)x)}, \tag{4.7}$$

where $\tau = \tau(t,g_t(x))$ and $\hat{\tau} = \hat{\tau}(t,x)$. Now we observe that

$$1 = q(\tau,T(\tau,t)g_t(x)) = q\biggl(\tau, T(\tau,t)\frac{T(t,\hat{\tau})\hat{T}(\hat{\tau},t)x}{q(\hat{\tau},\hat{T}(\hat{\tau},t)x)}\biggr) = \frac{q(\tau,T(\tau,\hat{\tau})\hat{T}(\hat{\tau},t)x)}{q(\hat{\tau},\hat{T}(\hat{\tau},t)x)}, \tag{4.8}$$

and thus

$$f(\tau) := q(\tau,T(\tau,\hat{\tau})\hat{T}(\hat{\tau},t)x) = q(\hat{\tau},\hat{T}(\hat{\tau},t)x).$$

Proceeding as for $\tau_{t,x}$, one can show that for each $\alpha > 0$ there exists a unique real number τ such that $f(\tau) = \alpha$. Since $f(\hat{\tau}) = q(\hat{\tau},\hat{T}(\hat{\tau},t)x)$, it follows from uniqueness that $\tau = \hat{\tau}$. Proceeding in a similar manner to that in (4.8) and using the fact that $\tau = \hat{\tau}$, we obtain

$$\hat{q}(\tau,T(\tau,t)g_t(x)) = \frac{\hat{q}(\tau,\hat{T}(\tau,t)x)}{q(\tau,\hat{T}(\tau,t)x)} = \frac{\hat{q}(\hat{\tau},\hat{T}(\hat{\tau},t)x)}{q(\tau,\hat{T}(\tau,t)x)} = \frac{1}{q(\tau,\hat{T}(\tau,t)x)},$$

and it follows from (4.7) that

$$h_t(g_t(x)) = \frac{\hat{T}(t,\tau)\hat{T}(\tau,t)x}{\hat{q}(\tau,T(\tau,t)g_t(x))q(\tau,\hat{T}(\tau,t)x)} = x.$$

This shows that g_t is the inverse of h_t, which completes the proof of the theorem. □

Acknowledgements. This research was partly supported by the Fundação para a Ciência e a Tecnologia through CAMGSD, Lisbon.

References

1. D. Bainov, S. Kostadinov, N. Van Minh and P. Zabreiko, Topological equivalence and exponential dichotomy of linear impulsive equations, *Int. J. Theor. Phys.* **33** (1994), 1581–1597.
2. D. Bainov, S. Kostadinov, N. Van Minh and P. Zabreiko, Topological classification of impulsive differential equations, *Int. J. Theor. Phys.* **34** (1995), 1147–1162.
3. L. Barreira and Ya. Pesin, *Nonuniform hyperbolicity*, Encyclopedia of Mathematics and Its Applications, Volume 115 (Cambridge University Press, 2007).
4. L. Barreira and C. Valls, Conjugacies for linear and nonlinear perturbations of nonuniform behavior, *J. Funct. Analysis* **253** (2007), 324–358.
5. L. Barreira and C. Valls, *Stability of nonautonomous differential equations*, Lecture Notes in Mathematics, Volume 1926 (Springer, 2008).
6. L. Barreira and C. Valls, A simple proof of the Grobman–Hartman theorem for nonuniformly hyperbolic flows, preprint.
7. G. Belickiĭ, Functional equations, and conjugacy of local diffeomorphisms of finite smoothness class, *Funct. Analysis Applic.* **7** (1973), 268–277.
8. G. Belickiĭ, Equivalence and normal forms of germs of smooth mappings, *Russ. Math. Surv.* **33** (1978), 107–177.
9. C. Chicone and R. Swanson, Linearization via the Lie derivative, *Electron. J. Diff. Eqns*, Monograph 02 (2000) (available at http://www.emis.de/journals/EJDE/Monographs/02/chicone.pdf).
10. D. Grobman, Homeomorphism of systems of differential equations, *Dokl. Akad. Nauk SSSR* **128** (1959), 880–881.
11. D. Grobman, Topological classification of neighborhoods of a singularity in n-space, *Mat. Sb.* **56**(98) (1962), 77–94.
12. P. Hartman, A lemma in the theory of structural stability of differential equations, *Proc. Am. Math. Soc.* **11** (1960), 610–620.
13. P. Hartman, On the local linearization of differential equations, *Proc. Am. Math. Soc.* **14** (1963), 568–573.
14. V. Lakshmikantham, D. Bainov and P. Simeonov, *Theory of impulsive differential equations*, Series in Modern Applied Mathematics, Volume 6 (World Scientific, 1989).
15. P. McSwiggen, A geometric characterization of smooth linearizability, *Michigan Math. J.* **43** (1996), 321–335.
16. J. Moser, On a theorem of Anosov, *J. Diff. Eqns* **5** (1969), 411–440.
17. J. Palis, On the local structure of hyperbolic points in Banach spaces, *Annais Acad. Bras. Cienc.* **40** (1968), 263–266.
18. K. Palmer, A generalization of Hartman's linearization theorem, *J. Math. Analysis Applic.* **41** (1973), 753–758.
19. K. Palmer, The structurally stable linear systems on the half-line are those with exponential dichotomies, *J. Diff. Eqns* **33** (1979), 16–25.
20. O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, *Math. Z.* **32** (1930), 703–728.
21. C. Pugh, On a theorem of P. Hartman, *Am. J. Math.* **91** (1969), 363–367.
22. A. Samoilenko and N. Perestyuk, *Impulsive differential equations*, Nonlinear Science Series A: Monographs and Treatises, Volume 14 (World Scientific, 1995).
23. G. Sell, Smooth linearization near a fixed point, *Am. J. Math.* **107** (1985), 1035–1091.
24. S. Sternberg, Local contractions and a theorem of Poincaré, *Am. J. Math.* **79** (1957), 809–824.
25. S. Sternberg, On the structure of local homeomorphisms of Euclidean n-space, II, *Am. J. Math.* **80** (1958), 623–631.