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STRONG PSEUDO-CONVEXITY AND SYMMETRIC DUALITY IN NONLINEAR PROGRAMMING

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Abstract

In this note, the weak duality theorem of symmetric duality in nonlinear programming and some related results are established under weaker (strongly Pseudo-convex/strongly Pseudo-concave) assumptions. These results were obtained by Bazaraa and Goode [1] under (stronger) convex/concave assumptions on the function.

1. Introduction

We use the following notation and terminology throughout the paper. Let $\psi(x, y)$ be a real-valued twice-differentiable function, defined on an open set in R^{n+m} containing $C_1 \times C_2$, where C_1 and C_2 are closed convex cones with non-empty interiors in R^n and R^m respectively. Let C_1^* be the *polar* of C_1, that is

$$C_1^* = \{ z \cdot x'z \leqslant 0 \text{ for each } x \in C_1 \text{ where } x' \text{ represents the transpose of } x \}. \tag{1}$$

C_2^* is defined similarly. $\nabla_x \psi(x_0, y_0)$ denotes the gradient vector of ψ with respect to x at the point (x_0, y_0), $\nabla_y \psi(x_0, y_0)$ is defined similarly. $\nabla_{xx} \psi(x_0, y_0)$ denotes the matrix (Hessian) of second partial derivative with respect to x evaluated at (x_0, y_0). $\nabla_{xy} \psi(x_0, y_0)$, $\nabla_{yx} \psi(x_0, y_0)$ and $\nabla_{yy} \psi(x_0, y_0)$ are defined similarly.

DEFINITION 1. If f is a scalar-valued differentiable function on a convex set $\Gamma \subset R^n$, and $K(x, y)$ is an arbitrary positive scalar function satisfying

$$K(x, y)\{f(y) - f(x)\} \geqslant (y - x)' \nabla f(x), \tag{2}$$

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then we say that f is *strongly Pseudo-convex with respect to* $K(x, y)$ (see [2], [5]). If $K(x, y) = 1$ then (2) reduces to the definition of convex function.

DEFINITION 2. If f is a scalar-valued differentiable function on a convex set $\Gamma \subset R^n$ and $K(x, y)$ is an arbitrary positive scalar function satisfying

$$K(x, y)\{f(y) - f(x)\} \leqslant (y - x)^t \nabla f(x), \tag{3}$$

then we say that f is *strongly Pseudo-concave with respect to* $K(x, y)$. If $K(x, y) = 1$ then (3) reduces to the definition of concave function.

It may be noted that strong Pseudo-convexity is weaker than convexity and stronger than Pseudo-convexity.

It may be remarked here that strong Pseudo-convexity is not a modification of the usual pseudoconvexity, but rather is a special case of *invex*, as mentioned by Mond [7]. Thus

$$f(y) - f(x) \geqslant [h(x, y)]^t \nabla f(x),$$

with $h(x, y) = (y - x)/K(x, y)$, shows the *invex* property.

We say that ψ is *strongly Pseudo-convex/strongly Pseudo-concave* on $C_1 \times C_2$ if and only if $\psi(\cdot, y)$ is strongly Pseudo-convex with respect to a positive scalar function K_1 on C_1 for each given $y \in C_2$ and $\psi(x, \cdot)$ is strongly Pseudo-concave with respect to a positive scalar function K_2 on C_2 for each given $x \in C_1$.

Let us consider a pair of nonlinear programs, as follows.

P_0 (Primal): Minimize $\{f(x, y) = \psi(x, y) - y^t \nabla_y \psi(x, y)\}$
subject to $(x, y) \in C_1 \times C_2$, $\nabla_y \psi(x, y) \in C_2^*$.

D_0 (Dual): Maximize $\{g(x, y) = \psi(x, y) - x^t \nabla_x \psi(x, y)\}$
subject to $(x, y) \in C_1 \times C_2$, $-\nabla_x \psi(x, y) \in C_1^*$.

For notational convenience, the sets of feasible solutions of P_0 and D_0 are denoted by P and D respectively, that is

$$P = \{(x, y) \in C_1 \times C_2 \colon \nabla_y \psi(x, y) \in C_2^*\}$$

and

$$D = \{(x, y) \in C_1 \times C_2 \colon -\nabla_x \psi(x, y) \in C_1^*\}.$$

2. Main results

THEOREM 1. *Let ψ be strongly Pseudo-convex/strongly Pseudo-concave on $C_1 \times C_2$ with respect to scalar-valued functions $K_1 \geqslant 1$ and $K_2 \geqslant 1$ respectively. Then*

$$\operatorname*{Inf}_{(x, y) \in P} f(x, y) \geqslant \operatorname*{Sup}_{(x, y) \in D} g(x, y).$$

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PROOF. Let $(x, y) \in P$ and $(u, v) \in D$. It is sufficient to prove that $f(x, y) \geqslant g(u, v)$.

Since ψ is strongly Pseudo-convex/strongly Pseudo-concave on $C_1 \times C_2$ with respect to the scalar-valued functions $K_1 \geqslant 1$ and $K_2 \geqslant 1$ respectively, the following two inequalities hold.

$$K_1(u, x)\{\psi(x, v) - \psi(u, v)\} \geqslant (x - u)^t \nabla_u \psi(u, v)$$

or

$$\psi(x, v) - \psi(u, v) \geqslant \frac{(x - u)^t}{K_1(u, x)} \nabla_u \psi(u, v), \tag{4}$$

$$K_2(y, v)\{\psi(x, v) - \psi(x, y)\} \leqslant (v - y)^t \nabla_y \psi(x, y)$$

or

$$\psi(x, v) - \psi(x, y) \leqslant \frac{(v - y)^t}{K_2(y, v)} \nabla_y \psi(x, y). \tag{5}$$

By multiplying by -1 in (5) and adding it to (4), we get

$$0 \geqslant \psi(u, v) + \frac{(x - u)^t \nabla_u \psi(u, v)}{K_1(u, x)} - \psi(x, y) - \frac{(v - y)^t \nabla_y \psi(x, y)}{K_2(y, v)} \tag{6}$$

$$= \psi(u, v) + \frac{x^t \nabla_u \psi(u, v)}{K_1(u, x)} - \frac{u^t \nabla_u \psi(u, v)}{K_1(u, x)}$$

$$- \psi(x, y) - \frac{v^t \nabla_y \psi(x, y)}{K_2(y, v)} + \frac{y^t \nabla_y \psi(x, y)}{K_2(y, v)}.$$

Since $u \in C_1$ and $-\nabla_u \psi(u, v) \in C_1^* \Rightarrow -u^t \nabla_u \psi(u, v) \leqslant 0$, by the definition of polar, we have

$$\frac{-u^t \nabla_u \psi(u, v)}{K_1(u, x)} \geqslant -u^t \nabla_u \psi(u, v) \quad \text{as } K_1(u, x) \geqslant 1. \tag{7}$$

Similarly $y \in C_2$ and $\nabla_y \psi(x, y) \in C_2^* \Rightarrow y^t \nabla_y \psi(x, y) \leqslant 0$. So we have

$$\frac{y^t \nabla_y \psi(x, y)}{K_2(y, v)} \geqslant y^t \nabla_y \psi(x, y) \quad \text{as } K_2(y, v) \geqslant 1, \tag{8}$$

$$\frac{x^t \nabla_u \psi(u, v)}{K_1(u, x)} \geqslant 0 \quad \text{as } -x^t \nabla_u \psi(u, v) \leqslant 0 \quad \text{and} \quad K_1(u, x) \geqslant 1, \tag{9}$$

$$\frac{-v^t \nabla_y \psi(x, y)}{K_2(y, v)} \geqslant 0 \quad \text{as } v^t \nabla_y \psi(x, y) \leqslant 0 \quad \text{and} \quad K_2(y, v) \geqslant 1. \tag{10}$$

Using (7), (8), (9) and (10) in (6), we get

$$0 \geqslant \psi(u, v) - u^t \nabla_u \psi(u, v) - \{\psi(x, y) - y^t \nabla_y \psi(x, y)\}$$
$$= g(u, v) - f(x, y) \Rightarrow f(x, y) \geqslant g(u, v).$$

This completes the proof.

Theorem 1 was motivated by the works of Bazaraa and Goode [1] and Dantzig et al. [4], who proved the same result under stronger assumptions on the cone and the function. In [4], the cone was taken to be non-negative orthant and the function convex/concave. Bazaraa and Goode [1] generalized the results of [4] to arbitrary cones. In Theorem 1 we assume the function to be strongly Pseudo-convex/strongly Pseudo-concave, which is weaker than convex/concave.

It may be noted here that the result does not hold only under Pseudo-convexity/Pseudo-concavity assumptions, and this follows from the following example: Let $n = m = 1$. Let $C_1 = \{x: x \geqslant 0\}$, $C_2 = \{y: y \geqslant 0\}$. Let $\psi(x, y) = \exp(x - y^2)$. Then it is easy to check that ψ is Pseudo-convex/Pseudo-concave on $C_1 \times C_2$. But in this case

$$f(0,2) = 9e^{-4} \leqslant g(0,0) = 1,$$

which contradicts Theorem 1. However, the conclusion of Theorem 1 holds under Pseudo-convexity/Pseudo-concavity, provided we make use of an additional feasibility assumption. This has been discussed in [6].

The following results are also true under weaker assumptions on the function. Since the proofs use ideas similar to those used in [1], we state the theorems without proofs.

Theorem 2. *Suppose that (x_0, y_0) solves P_0, and suppose that $\nabla_{yy}\psi(x_0, y_0)$ is negative definite. Then $(x_0, y_0) \in D$ and $f(x_0, y_0) = g(x_0, y_0)$. Further, if ψ is strongly Pseudo-convex/strongly Pseudo-concave with respect to scalar-valued functions $K_1 \geqslant 1$ and $K_2 \geqslant 1$, then (x_0, y_0) is an optimal solution of problem D_0.*

Theorem 3. *Suppose that (x_0, y_0) solves D_0, and $\nabla_{xx}\psi(x_0, y_0)$ is positive definite. Then $(x_0, y_0) \in P$ and $f(x_0, y_0) = g(x_0, y_0) = \psi(x_0, y_0)$. Further, if ψ is strongly Pseudo-convex/strongly Pseudo-concave with respect to scalar-valued functions $K_1 \geqslant 1$ and $K_2 \geqslant 1$, then (x_0, y_0) is an optimal solution of problem P_0.*

3. Special case

We now consider a special case of the symmetric dual programs, namely the case when the vector y and the corresponding cone C_2 are deleted from the formulation. Denoting $\psi(x, y)$ by $f(x)$ and C_1 by C, these two problems arise as special cases of P_0 and D_0.

P_1 (Primal): Minimize $f(x)$ subject to $x \in C$.

D_1 (Dual): Maximize $f(x) - x'\nabla f(x)$ subject to $x \in C$ and $-\nabla f(x) \in C^*$.

Theorem 1 holds, that is $x \in C$, $u \in C$ with $-\nabla f(u) \in C^*$ when f is strongly Pseudo-convex with respect to a scalar function $K \geqslant 1$. To prove this, observe that f is strongly Pseudo-convex with respect to scalar function $K \geqslant 1$. So we have

$$K(u, x)\{f(x) - f(u)\} \geqslant (x - u)'\nabla f(u) = x'\nabla f(u) - u'\nabla f(u)$$
$$\geqslant -u'\nabla f(u) \quad \text{as } x'\nabla f(u) \geqslant 0,$$

that is

$$f(x) - f(u) \geqslant \frac{-u'\nabla f(u)}{K(u, x)} \geqslant -u'\nabla f(u) \quad \text{as } K(u, x) \geqslant 1,$$

that is $f(x) \geqslant f(u) - u'\nabla f(u)$, and this completes the proof.

It may be noted that since y is deleted from the problem, a direct application of Theorem 2 does not hold. However, the theorem is indeed true, that is, if x_0 solves P_1 then it solves D_1. In order to show this we need the following Lemma.

LEMMA. *Consider the problem: minimize $f(x)$ subject to $x \in C$, where C is a closed convex cone. If x_0 solves the problem, then $-\nabla f(x_0) \in C^*$ and $x_0'\nabla f(x_0) = 0$. If f is strongly Pseudo-convex with respect to an arbitrary positive scalar function K, then conditions are sufficient for x_0 to solve the problem.*

PROOF. The first part of the proof is same as that of Lemma ([1], page 7) where no strong Pseudo-convexity is required. The second part of the proof is as follows.

If x_0 solves the problem then $-\nabla f(x_0) \in C^*$ and $x_0'\nabla f(x_0) = 0$. Now assume that f is strongly Pseudo-convex with respect to an arbitrary positive scalar function K, and $x_0 \in C$ with $-\nabla f(x_0) \in C^*$ and $x_0'\nabla f(x_0) = 0$. Then, for each $x \in C$, we have

$$K(x_0, x)\{f(x) - f(x_0)\} \geqslant (x - x_0)'\nabla f(x_0) = x'\nabla f(x_0) - x_0'\nabla f(x_0)$$
$$= x'\nabla f(x_0) \quad \text{as } x_0'\nabla f(x_0) = 0$$
$$\geqslant 0 \quad \text{as } x \in C \quad \text{and} \quad -\nabla f(x_0) \in C^*$$
$$\Rightarrow f(x) - f(x_0) \geqslant 0 \quad \text{as } K > 0$$
$$\Rightarrow f(x) \geqslant f(x_0).$$

This completes the proof.

It may be noted that if x_0 is an optimal solution of the primal problem P_1 then $-\nabla f(x_0) \in C^*$, x_0 is indeed a feasible solution of the dual D_1. In other words the optimality of P_1 ensures the feasibility of D_1. The following theorem gives a parallel of Theorem 2.

THEOREM 4. *Suppose that f is strongly Pseudo-convex with respect to a scalar function $K \geqslant 1$, and x_0 solves the problem P_1. Then x_0 solves the problem D_1.*

PROOF. Let x be a feasible solution of D_1, that is $x \in C$ and $-\nabla f(x) \in C^*$. Since x_0 solves the problem P_1 then by the above lemma $-\nabla f(x_0) \in C^*$ and $x_0^t \nabla f(x_0) = 0$. Since f is strongly Pseudo-convex with respect to a scalar function $K \geqslant 1$, we have

$$K(x, x_0)\{f(x_0) - f(x)\} \geqslant (x_0 - x)^t \nabla f(x) = x_0^t \nabla f(x) - x^t \nabla f(x)$$
$$\geqslant -x^t \nabla f(x) \quad \text{as } x_0^t \nabla f(x) \geqslant 1,$$

that is

$$f(x_0) - f(x) \geqslant \frac{-x^t \nabla f(u)}{K(x, x_0)} \geqslant -x^t \nabla f(x),$$

as $-x^t \nabla f(x) \leqslant 0$ and $K(x, x_0) \geqslant 1$, that is

$$f(x_0) \geqslant f(x) - x^t \nabla f(x),$$

that is

$$f(x_0) - x_0^t \nabla f(x_0) \geqslant f(x) - x^t \nabla f(x),$$

as $x_0^t \nabla f(x) = 0$.

This shows that x_0 solves D_1. The converse of this theorem can be obtained as a special case of Theorem 3, as long as $\nabla_{xx} f(x_0)$ is positive definite.

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