BULL. AUSTRAL. MATH. SOC. 45N05
VOL. 36 (1987) 353-360

ON THE KNESER-HUKUHARA PROPERTY FOR INTEGRAL EQUATIONS IN LOCALLY CONVEX SPACES

STANISLAW SZUFLA

This paper contains a Hukuhara - type theorem for nonlinear Volterra integral equations in locally convex spaces.

1. Introduction

Let $T = \{(t,s): 0 \le s \le t \le a\}$ and let W be an open subset of a quasicomplete locally convex topological vector space E. In this paper we consider the integral equation

$$(1) \qquad x(t) = q(t) + \int_0^t f(t,s,x(s))ds ,$$

where f is a bounded continuous function from $T \times W$ into E and q is a continuous function from $[0,a]$ into W. We shall give sufficient conditions for the existence of a continuous solution of (1). These conditions are formulated in terms of measures of noncompactness (see [9]). In particular, they hold in the case when $f = f_1 + f_2$, where f_1 is completely continuous and f_2 satisfies a Kamke condition (see [5]). Moreover, we shall show that the set of all continuous solutions of (1) is a continuum in the corresponding space of continuous functions. For the case when E is a Banach space, similar problems were investigated in several papers (see [10], [8], [11], [13]). Our considerations are based on the Lemma from Section 2 which gives a topological characterization of sets of fixed points of a certain class of nonlinear operators,

Received 9 October 1986.

namely the so-called Volterra operators introduced in [14]. Section 2 is a complement to the papers [12] and [3] (see also [14]), but our Lemma does not follow from these papers.

2. The basic lemma

Let K be a bounded convex subset of a normed space and let Y be a Hausdorff topological vector space. Denote by $C = C(K,Y)$ the space of continuous functions $K \to Y$ with the topology of uniform convergence.

LEMMA. *Assume that* $F: C \to C$ *is a continuous mapping such that*

1. *the set* $F(C)$ *is equiuniformly continuous;*
2. *there exist* $t_0 \in K$ *and* $x_0 \in Y$ *such that*

$$F(x)(t_0) = x_0 \quad \text{for all} \quad x \in C\ ;$$

3. *for every* $\varepsilon > 0$ *and* $x,y \in C$

$$x \,|\, K_\varepsilon = y \,|\, K_\varepsilon \implies F(x)|\,K_\varepsilon = F(y)\,|K_\varepsilon\ ,$$

where $K_\varepsilon = \{t \in K\colon \|t - t_0\| \le \varepsilon\}$;

4. *every sequence* (x_n) *in* C *such that* $\lim_{n\to\infty} (x_n - F(x_n)) = 0$ *has a limit point.*

Then the set $S_F = \{x \in C\colon x = F(x)\}$ *is nonempty and connected whenever it is compact.*

Proof. By Lemma 1 of [12] there exists a sequence (F_n) such that $I - F_n$ is a homeomorphism $C \to C$ and $\lim_{n\to\infty} F_n(x) = F(x)$ uniformly in $x \in C$. From this and condition 4 it is clear that $S_F \neq \emptyset$. Suppose that S_F is compact and not connected. Thus there are nonempty compact sets S_0, S_1 such that $S_F = S_0 \cup S_1$ and $S_0 \cap S_1 = \emptyset$. As C is a completely regular space, this implies (see [7], Chapter 41, II, Remark 3) that there exists a continuous function $w\colon C \to [0,1]$ such that $w(x) = 0$ for $x \in S_0$ and $w(x) = 1$ for $x \in S_1$. Fix $u_0 \in S_0$, $u_1 \in S_1$ and a positive integer n. Put

$$e_n(r) = (1 - r)(u_0 - F_n(u_0)) + r\big(u_1 - F_n(u_1)\big) \qquad (0 \le r \le 1)$$

and $u_{nr} = (I - F_n)^{-1}\big(e_n(r)\big)$. Since $e_n(r)$ depends continuously on r

https://doi.org/10.1017/S0004972700003646 Published online by Cambridge University Press

and $I - F_n$ is a homeomorphism, the mapping $r \to w(u_{nr})$ is continuous on $[0,1]$. Moreover, $u_{n0} = u_0$ and $u_{n1} = u_1$, so that $w(u_{n0}) = 0$ and $w(u_{n1}) = 1$. From this we deduce that there exists $r_n \in [0,1]$ such that

$$w(u_{nr_n}) = 1/2 . \tag{2}$$

For convenience put $v_n = u_{nr_n}$. As $\lim_{n\to\infty} e_n(r) = 0$ uniformly in r, we get

$$\lim_{n\to\infty} \left(v_n - F(v_n)\right) = \lim_{n\to\infty} \left(F_n(v_n) - F(v_n) + e_n(r_n)\right) = 0 . \tag{3}$$

Consequently, by condition 4, the sequence (v_n) has a limit point v. In view of (3) it is clear that $v \in S_F$, so that $w(v) = 0$ or $w(v) = 1$. On the other hand, (2) implies that $w(v) = 1/2$, which yields a contradiction.

3. Volterra integral equations in locally convex spaces

Now we return to the equation (1). Let P be a family of continuous seminorms generating the topology of E. For any $p \in P$ and for any bounded subset X of E denote by $\beta_p(X)$ the infimum of all $\varepsilon > 0$ for which there exists a finite subset Z of E such that $X \subset Z + B_p(\varepsilon)$, where $B_p(\varepsilon) = \{x \in E: p(x) \le \varepsilon\}$. The family $\left(\beta_p(X)\right)_{p\in P}$ is called the Hausdorff measure of noncompactness of X (for properties see [9]).

Let us recall that a function $h: T \times R_+ \to R_+$ is called a Kamke function if h satisfies the Caratheodory conditions and, for any $0 < d \le a$, the function $u = 0$ is the unique nonnegative continuous solution of the inequality $u(t) \le \int_0^t h\left(t,s,u(s)\right)ds$ on $[0,d]$.

THEOREM 1. *Assume that for any* $p \in P$ *there exists a Kamke function* $(t,s,u) \to h_p(t,u)$ *such that* h_p *is nondecreasing in* u *and*

$$\beta_p\left(f(t,[0,t] \times X)\right) \le h_p\left(t,\beta_p(X)\right) \tag{4}$$

for each $t \in [0,a]$ *and for each bounded subset* X *of* E.

https://doi.org/10.1017/S0004972700003646 Published online by Cambridge University Press

Then there exists an interval $J = [0,d]$ such that the set of all continuous solutions $x: J \to E$ of (1), considered as a subset of $C(J,E)$, is nonempty, compact and connected.

Proof. As W is open and q is continuous, we can choose a set B of the form $B = \{x \in E: p_i(x) \le b\ ,\ i = 1,\dots,m\}$, where $p_1,\dots,p_m \in P$, such that $q(t) + B \subset W$ for all $t \in [0,a]$.
Let $M = \sup\{p_i(f(t,s,x)) : i = 1,\dots,m,\ (t,s) \in T,\ x \in W\}$, $d = \min(a,b/M)$ and $J = [0,d]$. Put

$$r(x) = \begin{cases} x & \text{for} \quad x \in B \\ x/Q(x) & \text{for} \quad x \in E \setminus B\ , \end{cases}$$

where Q is the Minkowski functional of B, and put

$$g(t,s,x) = f(t,s,q(s) + r(x - q(s))) \quad \text{for } (t,s) \in T \text{ and } x \in E\ .$$

From the known properties of Minkowski's functional it follows that r is a continuous function from E into B and $r(X) \subset \bigcup_{0 \le \lambda \le 1} \lambda X$ for each subset X of E. Hence

$$\beta_p(\{q(s) + r(x - q(s)) : s \in [0,a], x \in X\}) \le \beta_p(X)$$

for any $p \in P$ and for any bounded subset X of E. This shows that g is a bounded continuous function from $T \times E$ into E and g satisfies (4).

We introduce a mapping F defined by

$$F(x)(t) = q(t) + \int_0^t g(t,s,x(s))ds \quad \text{for} \quad t \in J \quad \text{and} \quad x \in C\ ,$$

where $C = C(J,E)$. It can be easily verified that F is a continuous mapping $C \to C$, the set $F(C)$ is bounded and equicontinuous, and (1) is equivalent to the equation $x = F(x)$. Now we shall show that F satisfies condition 4. Let (x_n) be a sequence such that

(5) $$\lim_{n\to\infty} (x_n - F(x_n)) = 0\ .$$

Put $V = \{x_n: n = 1,2,\dots\}$ and $V(t) = \{u(t): u \in V\}$. As the set $F(C)$ is equicontinuous, it follows from (5) that V is also equicontinuous. Therefore for any $p \in P$ the function $t \to v(t) = \beta_p(V(t))$ is

https://doi.org/10.1017/S0004972700003646 Published online by Cambridge University Press

continuous on J. For a given $t \in J$ we divide the interval $[0,t]$ into n parts $0 = t_0 < t_1 < \dots < t_n = t$ in such a way that $\Delta t_i = t_i - t_{i-1} = t/n$ for $i = 1,\dots, n$.

Let $V_i = \{u(s) : u \in V,\ t_{i-1} \le s \le t_i\}$. Then

$$F(V)(t) \subset q(t) + \sum_{i=1}^{n} \Delta t_i \ \overline{\text{conv}}\ g(t,[0,t] \times V_i) .$$

Moreover, for any i, $1 \le i \le n$, there exists $s_i \in [t_{i-1},t_i]$ such that

$$\beta_p(V_i) = \sup \{\beta_p\big(V(s)\big) : t_{i-1} \le s \le t_i\} = v(s_i)$$

(see [1], Theorem 2.2). Hence, by (5), (4) and corresponding properties of β_p, we obtain

$$v(t) \le \beta_p\big(F(V)(t)\big) \le \sum_{i=1}^{n} \Delta t_i \beta_p\big(g(t,[0,t] \times V_i)\big) \le \sum_{i=1}^{n} \Delta t_i h_p(t,\beta_p(V_i))$$
$$= \sum_{i=1}^{n} \Delta t_i h_p\big(t,v(s_i)\big) .$$

But if $n \to \infty$, then $\sum_{i=1}^{n} \Delta t_i h_p\big(t,v(s_i)\big) \to \int_0^t h_p\big(t,v(s)\big)\, ds$. Thus

$$v(t) \le \int_0^t h_p\big(t,v(s)\big)ds \quad \text{for} \quad t \in J .$$

As h_p is a Kamke function, this implies that

$$\beta_p\big(V(t)\big) = 0 \quad \text{for all} \quad t \in J .$$

Since this equality holds for every $p \in P$, it follows that for any $t \in J$ the set $V(t)$ is relatively compact in E. By Ascoli's theorem [6 ; Theorem 7.17] from this we deduce that V is relatively compact in C. Hence the sequence (x_n) has a limit point. On the other hand, as $S_F = F(S_F)$, by repeating the above argument we infer that S_F is compact. Applying now the Lemma, we conclude that S_F is nonempty and connected.

4. Kneser's theorem for weak solutions of the Cauchy problem

In this section we shall present another application of the Lemma. Let E be a (sequentially) weakly complete Banach space, $B = \{x \in E : \|x - x_0\| \le b\}$, and let $f: [0,a] \times B \to E$ be a bounded weakly- weakly continuous function. Let $M = \sup \{ \|f(t,x)\| : 0 \le t \le a,\ x \in B\}$,

https://doi.org/10.1017/S0004972700003646 Published online by Cambridge University Press

$d = \min(a, b/M)$ and $J = [0,d]$. Denote by E_w the space E provided with the weak topology.

THEOREM 2. *Assume that*

$$\beta\big(f(J \times X)\big) \le h\big(\beta(X)\big) \quad \textit{for each subset } X \textit{ of } B ,$$

where h *is a nondecreasing Kamke function and* β *is the measure of weak noncompactness (see* [2]*).*

Then the set of all weak solutions of the Cauchy problem

$$(6) \qquad x' = f(t,x) , \quad x(0) = x_0 ,$$

defined on J *, is nonempty, compact and connected in* $C(J,E_w)$.

Proof. Put

$$r(x) = \begin{cases} x & \text{if } x \in B \\ x_0 + b(x - x_0)/\|x - x_0\| & \text{if } x \in E \setminus B \end{cases}$$

and

$$F(x)(t) = x_0 + \int_0^t f\big(s, r(x(s))\big)ds \qquad (t \in J,\ x \in C) ,$$

where $C = C(J,E_w)$.

It is known that F is a continuous mapping $C \to C$, the set $F(C)$ is bounded and equiuniformly (strongly) continuous, and the function $x \in C$ is a weak solution of (6) if and only if $x = F(x)$. Arguing similarly as in the proof of Theorem 1, we can show that F satisfies condition 4 and the set S_F is nonempty and compact. By the Lemma it follows from this that S_F is connected.

Remark. The assumptions of Theorem 1 or 2 guarantee that the corresponding operator F satisfies condition 4, but it is an open question whether it satisfies the stronger condition:
for every net (x_α)

$$\lim \big(x_\alpha - F(x_\alpha)\big) = 0 \implies (x_\alpha) \text{ has a limit point.}$$

Therefore our theorems do not follow from the results of [3] and [12].

References

[1] A. Ambrosetti, "Un teorema di esistenza per le equazioni differenziali negli spazi di Banach", *Rend. Sem. Mat. Univ. Padova,* 39 (1967), 349 - 369.

[2] E. Cramer, V. Lakshmikantham and A. Mitchell, "On the existence of weak solutions of differential equations in nonreflexive Banach spaces", *Nonlinear Anal.* 2 (1978), 169-177.

[3] J. Dubois and P. Morales, "On the Hukuhara - Kneser property for some Cauchy problems in locally convex topological vector spaces", *Lecture Notes in Math.* 964 (1982), 162-170.

[4] M. Hukuhara, "Sur l'application qui fait correspondre a un point un continu bicompact", *Proc. Japan Acad.* 31 (1955), 5-7.

[5] M. Hukuhara, "Théorems fondamentaux de la théorie des équations différentielles ordinaires dans l'espace vectoriel topologique", *J. Fac. Sci. Univ. Tokyo, Sec.I.,* 8 (1959), 111-138.

[6] J.L. Kelley, *General topology,* (Toronto - New York - London 1957).

[7] K. Kuratowski, *Topology, Vol.II,* (New York - London - Warszawa 1968).

[8] V. Lakshmikantham, "Existence and comparison results for Volterra integral equations in a Banach space", *Lecture Notes in Math.* 737 (1979), 120-126.

[9] B.N. Sadovskii, "Limit-compact and condensing operators", *Russian Math. Surveys,* 27 (1972), 85-155.

[10] S. Szufla, "On the existence of solutions of Volterra integral equations in Banach spaces", *Bull. Acad. Polon. Sci. Ser. Sci. Math.* 22 (1974), 1209-1213.

[11] S. Szufla, "On Volterra integral equations in Banach spaces", *Funkcial. Ekvac.* 20 (1977), 247-258.

[12] S. Szufla, "Sets of fixed points of nonlinear mappings in function spaces", *Funkcial. Ekvac.* 22 (1979), 121-126.

[13] R.L. Vaughn, "Criteria for the existence and comparison of solutions to nonlinear Volterra integral equations in Banach spaces", *Nonlinear Equations in Abstract Spaces,* ed. V. Lakshmikantham (New York 1978), 463-468.

[14] G. Vidossich, "A fixed point theorem for function spaces", *J. Math. Anal. Appl.* 36 (1971), 581-587.

A. Mickiewicz University
Poznań
Poland