FINITE CO-DEDEKINDIAN GROUPS

by MARIAN DEACONESCU† and GHEORGHE SILBERBERG

(Received 18 October, 1994)

1. Introduction. A group G is called *Dedekindian* if every subgroup of G is normal in G.

The structure of the finite Dedekindian groups is well-known [**3**, Satz 7.12]. They are either abelian or direct products of the form $Q \times A \times B$, where Q is the quaternion group of order 8, A is abelian of odd order and $\exp(B) \leq 2$.

We may view a Dedekindian group G as a group satisfying the property that $\alpha(H) = H$ for every $H \leq G$ and for every $\alpha \in \mathrm{Inn}(G)$. This remark suggests the consideration of a new class of groups, called co-Dedekindian groups which are defined by a similar requirement. Although our definition makes sense for infinite groups we shall restrict here to the finite case.

DEFINITION. Let G be a group and let $\mathrm{Aut}_c(G)$ be its group of central automorphisms, so that $\mathrm{Aut}_c(G) = \{\alpha \in \mathrm{Aut}(G) \mid \alpha(x) \in xZ(G)$, for every $x \in G\}$. G is called a *co-Dedekindian group* (*$\mathscr{C}$-group* for short) if $\alpha(H) = H$ for every $H \leq G$ and for every $\alpha \in \mathrm{Aut}_c(G)$.

A first glance at the definition shows that the class of $\mathscr{C}$-groups is very large. If G is a group and if $Z(G) = 1$ or if $G' = G$, then $\mathrm{Aut}_c(G) = 1$ and G is a $\mathscr{C}$-group in an obvious manner. By a *trivial $\mathscr{C}$-group* we shall mean a group G with $\mathrm{Aut}_c(G) = 1$.

Since $Z(S_n) = 1$ for $n \geq 3$, it follows by Cayley's theorem that every finite group can be embedded into a trivial $\mathscr{C}$-group. This means that there is no hope for a compact description of the trivial $\mathscr{C}$-groups and turn the focus on nontrivial $\mathscr{C}$-groups.

The parallel with Dedekindian groups is clear. We may regard the abelian groups as trivial Dedekindian groups. A Dedekindian group is trivial if and only if $\mathrm{Inn}(G) = 1$. The nontrivial Dedekindian finite groups are the Hamiltonian groups whose structure was described above.

All groups in this paper are finite. The notation is standard and conforms to that of [**2**]. If G is a group and if $\alpha \in \mathrm{Aut}_c(G)$ we shall denote $F_\alpha = C_G(\alpha) = \{x \in G \mid \alpha(x) = x\}$, $K_\alpha = [G, \alpha] = \langle x^{-1}\alpha(x) \mid x \in G \rangle$. Also, $F = \cap\{F_\alpha \mid \alpha \in \mathrm{Aut}_c(G)\}$ and $K = \langle K_\alpha \mid \alpha \in \mathrm{Aut}_c(G) \rangle$.

Our first result is a Dedekind-like structure theorem. Unfortunately it holds only for $\mathscr{C}$-groups with trivial Frattini subgroup:

THEOREM 1. *Let G be a nontrivial $\mathscr{C}$-group such that $\Phi(G) = 1$. Then $G = F \times K$, $(|F|, |K|) = 1$, F is a trivial $\mathscr{C}$-group and K is a cyclic group of odd square-free order.*

The nilpotent $\mathscr{C}$-groups are good candidates for nontrivial $\mathscr{C}$-groups and we may

† The research of this author was supported by Grant SM082 from the Research Council, University of Kuwait.

expect that their structure is quite restricted. The following result shows that this is indeed the case under certain additional assumptions.

THEOREM 2. *Let G be a p-group. If G is a nonabelian $\mathscr{C}$-group, then $Z_2(G)$ is a Dedekindian group. If $Z_2(G)$ is nonabelian, then $G \simeq Q_8$. If $Z_2(G)$ is cyclic, then $G \simeq Q_{2^n}$, $n \geq 4$, where Q_{2^n} is the generalized quaternion group of order 2^n.*

2. Nontrivial $\mathscr{C}$-groups with trivial Frattini subgroup. In order to prove Theorem 1, we need first a number of results about arbitrary $\mathscr{C}$-groups. The first lemma is well-known (see [**1**]).

(2.1) *Let G be a group and let $\alpha \in \mathrm{Aut}_c(G)$.*

(i) *The function $\phi_\alpha : G \to G$, defined by $\phi_\alpha(x) = x^{-1}\alpha(x)$ for all $x \in G$ is an endomorphism of G, $\mathrm{Ker}\,\phi_\alpha = F_\alpha$, $\phi_\alpha(G) = K_\alpha$ and $|G| = |F_\alpha| \,.\, |K_\alpha|$. If, moreover, $(|\alpha|, |G|) = 1$ then $G = F_\alpha \times K_\alpha$.*

(ii) *$G' \leq F$ and $K \leq Z(G)$, so in particular F, K, F_α, K_α are normal subgroups in G.*

The following elementary consequence of (2.1)(i) will be used in the sequel:

(2.2) *Let G be a $\mathscr{C}$-group, let $\alpha \in \mathrm{Aut}_c(G)$ and let $H \leq G$. Then*

$$|H| = |H \cap F_\alpha| \,.\, |\phi_\alpha(H)|.$$

If G is a $\mathscr{C}$-group and if $\alpha \in \mathrm{Aut}_c(G)$, then F_α and K_α play a special role in the lattice of all subgroups.

(2.3) *Let G be a $\mathscr{C}$-group, let $\alpha \in \mathrm{Aut}_c(G)$ and let $H \leq G$. Then*
(i) *$H \cap F_\alpha = 1 \Rightarrow H \leq K_\alpha$,*
(ii) *$H \cap K_\alpha = 1 \Rightarrow H \leq F_\alpha$,*
(iii) *$G = HF_\alpha \Rightarrow K_\alpha \leq H$,*
(iv) *$G = HK_\alpha \Rightarrow F_\alpha \leq H$.*

Proof. Since the proofs are similar, we shall prove only (iv). Let $G = HK_\alpha$, so that $|G| = |H| \,.\, |K_\alpha| / |H \cap K_\alpha|$. By (2.1)(i), $|H| = |H \cap K_\alpha| \,.\, |F_\alpha|$. By (2.2), $|H \cap F_\alpha| \,.\, |\phi_\alpha(H)| = |H \cap K_\alpha| \,.\, |F_\alpha|$. Then $|(H \cap K_\alpha) : \phi_\alpha(H)| \,.\, |F_\alpha : (H \cap F_\alpha)| = 1$, forcing $F_\alpha \leq H$.

Now we can prove the following result.

(2.4) *Let G be a $\mathscr{C}$-group and let $\alpha \in \mathrm{Aut}_c(G)$. Then*
(i) *$F_\alpha \cap K_\alpha \leq \Phi(G)$*
(ii) *$F \cap K \leq \Phi(G)$.*

Proof. It is sufficient to prove only (ii). We may assume that $F \nleq \Phi(G)$. Choose a maximal subgroup M of G such that $F \nleq M$. Then $G = FM$, so $G = F_\alpha M$ for all $\alpha \in \mathrm{Aut}_c(G)$. By (2.3)(iii) it follows that $K_\alpha \leq M$ for all $\alpha \in \mathrm{Aut}_c(G)$, whence $K \leq M$. We have thus proved that if M is a maximal subgroup of G and if $F \nleq M$, then $K \leq M$. Now $F \cap K \leq \bigcap\{M \mid M \text{ is maximal in } G \text{ and } F \leq M\} \cap \bigcap\{M \mid M \text{ is maximal in } G \text{ and } F \nleq M\} = \Phi(G)$.

The next result shows that the elements of prime order are "separated" by F_α and K_α.

(2.5) *Let G be a $\mathscr{C}$-group, let $\alpha \in \mathrm{Aut}_c(G)$ and let $p \in \pi(G)$. If T_p is the set of all elements of order p of G, then $T_p \subseteq F_\alpha$ or $T_p \subseteq K_\alpha$.*

Proof. Let $x \in T_p$. If $\langle x \rangle \cap F_\alpha = 1$, then by (2.3)(i) $\langle x \rangle \leq K_\alpha$. This shows that

https://doi.org/10.1017/S0017089500031396 Published online by Cambridge University Press

$T_p \subseteq F_\alpha \cup K_\alpha$. Assume now that $x, y \in T_p$ such that $x \in F_\alpha - K_\alpha$ and $y \in K_\alpha - F_\alpha$. Since $y \in K_\alpha \leq Z(G)$, $[x, y] = 1$, so $xy \in T_p$. But clearly $xy \notin F_\alpha \cup K_\alpha$, a contradiction. Thus $T_p \subseteq F_\alpha$ or $T_p \subseteq K_\alpha$, as asserted.

As a corollary we have the following result.

(2.6) *Let G be a* $\mathscr{C}$*-group and let* $\alpha \in \mathrm{Aut}_c(G)$, *such that* $F_\alpha \cap K_\alpha = 1$. *Then* $G = F_\alpha \times K_\alpha$ *and* $(|F_\alpha|, |K_\alpha|) = 1$.

Proof. That $G = F_\alpha \times K_\alpha$ follows from hypothesis and (2.1)(i), while $(|F_\alpha|, |K_\alpha|) = 1$ follows by (2.5).

We are now in position to give a

Proof of Theorem 1. Let G be a nontrivial $\mathscr{C}$-group with $\Phi(G) = 1$. Then $F \cap K = 1$ by (2.4) and if $\alpha \in \mathrm{Aut}_c(G)$ we also have that $G = F_\alpha \times K_\alpha$, $(|F_\alpha|, |K_\alpha|) = 1$ by (2.4) and (2.6).

We prove first that $(|F|, |K|) = 1$.

Let $p \in \pi(K)$. Since K is abelian we can find $\alpha \in \mathrm{Aut}_c(G)$ such that $p \in \pi(K_\alpha)$. Then $(p, |F_\alpha|) = 1$ and since $F \leq F_\alpha$, $(p, |F|) = 1$.

Now we prove that $G = F \times K$. Since $F, K \trianglelefteq G$ and $F \cap K = 1$, it suffices to show that $G = FK$. Let x be a p-element of G. From (2.2) and (2.3) it follows that either $\langle x \rangle \leq F_\alpha$ or $\langle x \rangle \leq K_\alpha$ if $\alpha \in \mathrm{Aut}_c(G)$. It is easy to deduce that either $\langle x \rangle \leq F$ or $\langle x \rangle \leq K$, which shows that $G = FK = F \times K$.

Since $(|F|, |K|) = 1$, $\mathrm{Aut}_c(G) = \mathrm{Aut}_c(F) \times \mathrm{Aut}_c(K)$.

Since G is a $\mathscr{C}$-group, both F and K are $\mathscr{C}$-groups. Of course, F is a trivial $\mathscr{C}$-group, because $F = C_G(\mathrm{Aut}_c(G))$. K is a cyclic group because K is abelian and the condition of being a $\mathscr{C}$-group is equivalent to that of every subgroup of K being characteristic in K. Since $K \trianglelefteq G$, $\Phi(K) \leq \Phi(G) = 1$, so $|K|$ is square-free. Note that if $2 \in \pi(G)$, then $2 \in \pi(F)$. Indeed, if $\alpha \in \mathrm{Aut}_c(G)$, then $\alpha(x) = x$ for every involution of G. Hence $|K|$ is odd and the proof is complete.

3. Nilpotent $\mathscr{C}$-groups. If G is a nilpotent $\mathscr{C}$-group, then G is the direct product of its (characteristic) Sylow p-subgroups and every Sylow p-subgroup of G is also a $\mathscr{C}$-group. We will therefore focus here on p-groups which are nontrivial $\mathscr{C}$-groups. Note that an abelian p-group is a $\mathscr{C}$-group if and only if it is a cyclic p-group and that a cyclic p-group G is a trivial $\mathscr{C}$-group if and only if $|G| \leq 2$.

We shall now concentrate on nonabelian p-groups which are $\mathscr{C}$-groups. The following two lemmas are helpful:

(3.1) *Let G be a nonabelian p-group. If G is a* $\mathscr{C}$*-group, then* $\Omega_1(G) \leq F$.

Proof. Observe first that, since G is a $\mathscr{C}$-group, $\mathrm{Aut}_c(G)$ is an abelian group. This follows at once from the condition that $\alpha(x) \in \langle x \rangle$ for every $x \in G$ and every $\alpha \in \mathrm{Aut}_c(G)$. We may use now Corollary 2 of [**1**] to derive that $\mathrm{Aut}_c(G)$ is a p-group.

If $x \in T_p$, so that $|x| = p$, then by [**2**, Lemma 2.6.3], $x \in F$. This proves that $\Omega_1(G) \leq F$.

(3.2) *Let G be a nonabelian p-group. If G is a* $\mathscr{C}$*-group, then* $Z_2(G)$ *is a Dedekindian group.*

https://doi.org/10.1017/S0017089500031396 Published online by Cambridge University Press

Proof. Since G is nonabelian, $Z(G) < Z_2(G)$. Every element of $Z_2(G)$ induces by conjugation a central automorphism of G because $[Z_2(G), G] \leq Z(G)$. Since G is a $\mathscr{C}$-group, it follows that $[H, Z_2(G)] \leq H$ for every $H \leq G$. In particular, every subgroup of $Z_2(G)$ is normal in $Z_2(G)$, so $Z_2(G)$ is a Dedekindian group.

If G is a nonabelian p-group which is a $\mathscr{C}$-group then by (3.2) $Z_2(G)$ is either abelian or $Z_2(G)$ is the direct product of Q_8 by a group of exponent at most 2. It is a rather easy matter to determine those nonabelian p-groups G which are $\mathscr{C}$-groups and in which $Z_2(G)$ is a cyclic group.

(3.3) *Let G be a nonabelian p-group which is a $\mathscr{C}$-group. If $Z_2(G)$ is cyclic, then $p = 2$ and $G \simeq Q_{2^n}$, $n \geq 4$, where Q_{2^n} is the generalized quaternion group of order 2^n.*

Proof. Since $Z_2(G)$ is cyclic, it follows by [**3**, Satz 7.7] that $p = 2$ and G has a cyclic maximal subgroup. Then G must be isomorphic to one of the groups of the following list:
(i) $D_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, [a,b] = a^{-2} \rangle$,
(ii) $SD_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, [a,b] = a^{-2+2^{n-2}} \rangle$,
(iii) $\mathrm{Mod}_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2 = 1, [a,b] = a^{2^{n-2}} \rangle$,
(iv) $Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, [a,b] = a^{-2} \rangle$.

Since G is a $\mathscr{C}$-group, it follows that $Z_2(G) \leq C_G(\Omega_1(G))$ by (3.1). If $G \simeq D_{2^n}$, than $C_G(\Omega_1(G)) = C_G(G) = Z(G)$, a contradiction. If $G \simeq SD_{2^n}$, than $C_G(\Omega_1(G)) = C_G(\langle a^2, b \rangle) = Z(G)$, a contradiction. If $G \simeq \mathrm{Mod}_{2^n}$, then G has class 2, whence $Z_2(G) = G$ and G is not cyclic. The groups Q_{2^n} are co-Dedekindian groups for $n \geq 3$, but if $n = 3$, $Z_2(G) = G \simeq Q_8$ is not cyclic. Therefore $G \simeq Q_{2^n}$, $n \geq 4$, as asserted.

We are now in position to prove Theorem 2. Notice that by (3.2) and (3.3) we have only to tackle the case in which $Z_2(G)$ is nonabelian. We start by fixing some notation.

Let G be a nonabelian p-group which is a $\mathscr{C}$-group such that $Z_2(G)$ is nonabelian. Then, by (3.2), $Z_2(G)$ must be a nonabelian 2-group; in particular, $p = 2$ and $Z_2(G) = H \times S$, where $H = \langle a, b \mid a^4 = b^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \rangle \simeq Q_8$ and S is a group of exponent at most 2.

Throughout the rest of the proof we shall keep this notation fixed. We split the proof into a number of steps.

Step 1. $\Phi(G) \leq C_G(H)$.

Proof. Since $\Phi(G) = \langle \{x^2 \mid x \in G\} \rangle$, it suffices to prove that $x^2 \in C_G(H)$ for all $x \in G$. But $C_G(H) = C_G(a) \cap C_G(b)$ and, for symmetry reasons, it suffices to prove that $x^2 \in C_G(a)$ for every $x \in G$.

Let $x \in G - C_G(a)$. Since $a \in Z_2(G)$, $[a,x] \in Z(G)$ and $[a,x] \neq 1$. Then $[a,x]$ has order 2 because $\exp(Z(G)) = 2$ by hypothesis. Since G is a $\mathscr{C}$-group, $[a,x] \in \langle x \rangle$. Let $|x| = 2^k$, so that $[a,x] = x^{2^{k-1}}$. Then $a^{x^2} = 1$ and $x^2 \in C_G(a)$.

Step 2. $G = C_G(a) \cup C_G(b) \cup C_G(ab)$.

Proof. Let $x \in G - (C_G(a) \cup C_G(b))$ and let $|x| = 2^k$. Then, as in Step 1, we can write $[a,x] = x^{2^{k-1}}$, $[b,x] = x^{2^{k-1}}$. This yields $x^a = x^{2^{k-1}+1}$, $x^b = x^{2^{k-1}+1}$, whence $x^{ab} = x^{2^{2k-2}+1}$. Since $x \notin C_G(H)$ so $x \notin F = C_G(\mathrm{Aut}_c(G))$. Then $|x| > 2$ by (3.1), whence $|x| = 2^k \mid 2^{2k-2}$. As a consequence $x^{ab} = x$, whence $x \in C_G(ab)$.

https://doi.org/10.1017/S0017089500031396 Published online by Cambridge University Press

Step 3. *$C_G(a)$, $C_G(b)$ and $C_G(ab)$ are maximal subgroups of G.*

Proof. It follows from [**4**] and from the previous Step that G has a quotient K which is a Klein four group, and $C_G(a)$, $C_G(b)$, $C_G(ab)$ are the preimages of the three nontrivial subgroups of K, hence they are maximal subgroups of G.

Step 4. *$Z(G)=\langle a^2\rangle \simeq \mathbb{Z}_2$ and $Z(G)\leq \Phi(G)$.*

Proof. Let $z\in Z(G)$ be an involution. Define $\phi: G\to G$ by $\phi(x)=x$ if $x\in C_G(a)$ and $\phi(x)=zx$ if $x\notin C_G(a)$. Then $\phi\in \mathrm{Aut}_c(G)$ and since G is a $\mathscr{C}$-group $\phi(b)\in\langle b\rangle$, whence $zb\in\langle b\rangle$ and $z=b^2=a^2$. In particular, $Z(G)$ has a unique involution. This shows that $Z(G)$ is cyclic. Since $Z(G)\leq Z(Z_2(G))=\langle a^2\rangle\times S$, it follows that $Z(G)=\langle a^2\rangle\simeq\mathbb{Z}_2$.
It is now clear that $Z(G)\leq\Phi(G)$.

Step 5. *$\Omega_1(G)\leq\Phi(G)$ and $a^2\in\langle x\rangle$, for all $x\in G-\Phi(G)$.*

Proof. It is enough to show that there are no involutions in $G-\Phi(G)$.
Suppose that $x\in G-\Phi(G)$ is an involution and let M be a maximal subgroup of G such that $x\notin M$. Let $\phi: G\to G$ be defined by $\phi(g)=g$, if $g\in M$, and $\phi(g)=a^2g$, if $g\notin M$. Then $\phi\in\mathrm{Aut}_c(G)$ and in particular $\phi(x)\in\langle x\rangle$, since G is a $\mathscr{C}$-group. Then $a^2x\in\langle x\rangle$ and $a^2\in\langle x\rangle$. Since $|x|=2$, $a^2=x$. This contradicts Step 4.

Step 6. *$\Phi(G)$ is elementary abelian.*

Proof. Assume the contrary and let $x\in\Phi(G)$ with $|x|=4$. Then $ax\notin\Phi(G)$ because $a\notin C_G(b)$ and $C_G(b)$ is a maximal subgroup of G by Step 3. Since $\Phi(G)\leq C_G(H)$ by Step 1, $(ax)^4=a^4x^4=1$. In particular, $|ax|\leq 4$ and since $ax\notin\Phi(G)$ it follows by Step 5 that $|ax|=4$. Then $(ax)^2=a^2$ by Step 5. We get $x^2=1$, a contradiction.

Step 7. *$|\Phi(G)|=2$.*

Proof. By Step 6, if $x\in\Phi(G)$, then $|x|=2$. If $x\notin\Phi(G)$, then $x^2\in\Phi(G)=\langle\{g^2\mid g\in G\}\rangle$. Also, by Step 5, $a^2\in\langle x\rangle$ if $x\in G-\Phi(G)$. Then $x^2=a^2$ for every $x\in G-\Phi(G)$, whence

$$\Phi(G)=\langle\{g^2\mid g\in G\}\rangle=\{1,a^2\}.$$

Step 8. *$G=H\simeq Q_8$.*

Proof. By Steps 5 and 7, G contains a unique involution. Then $G\simeq Q_{2^n}$ for some $n\geq 3$, by [**3**, Satz 8.2]. Since if $n\geq 4$, $Z_2(G)\simeq\mathbb{Z}_4$, it follows that $n=3$ and $G\simeq Q_8$.
The proof of Theorem 2 is now complete.

Notice that by (3.2), (3.3) and by Theorem 2 the classification of all $\mathscr{C}$-groups of prime power order is reduced to that of nonabelian p-groups with abelian noncyclic second center.

4. Concluding remarks. (a) If G is a $\mathscr{C}$-group and $\Phi(G)\neq 1$ then Theorem 1 does not hold; for example, the cyclic group $\mathbb{Z}_4$ is a nontrivial $\mathscr{C}$-group which is indecomposable.
(b) The problem of deciding whether a group is a trivial $\mathscr{C}$-group is not a trivial one.

Of course, by (2.1)(ii), G is surely a trivial $\mathscr{C}$-group if $G = G'$, or if $Z(G) = 1$. But these conditions are not necessary: if $G = \mathbb{Z}_2 \times H$ and H is a nonabelian group of order 21, then G is a trivial $\mathscr{C}$-group but clearly $Z(G) \neq 1$ and $G \neq G'$. Theorem 1 gives a sufficient condition for a $\mathscr{C}$-group G to be a trivial $\mathscr{C}$-group: it suffices to have $\Phi(G) = 1$, G noncyclic and G purely nonabelian (i.e. G has no abelian direct factors).

(c) We remark here that nonabelian p-groups with abelian noncyclic second center and which are $\mathscr{C}$-groups do exist. An example is $G = \langle a, b \mid a^9 = 1,\ b^3 = a^6,\ [a,b]^3 = 1, [a,[a,b]] = a^3,\ [b,[a,b]] = 1 \rangle$ whose order is 81.

It can be shown that G is the unique group of order p^4 satisfying all these properties:

Let G be a nonabelian $\mathscr{C}$-group of order p^4, with $Z_2(G)$ abelian and noncyclic.

Then $1 < Z(G) < Z_2(G) < G$, $|Z(G)| = p$, $|Z_2(G)| = p^2$, $Z_2(G)$ is elementary abelian and G has class 3. From the relations $G' \leq Z_2(G)$, $G' \cap Z(G) \neq 1$, $G' \nleq Z(G)$, $G' \leq \Phi(G)$ we derive $G' = \Phi(G) = Z_2(G)$. Now let z be a generator of $Z(G)$.

If $x \notin \Phi(G)$ and M is a maximal subgroup of G such that $x \notin M$, then $\phi : G \to G$ defined by $\phi(x^i m) = x^i z^i m$, for every $m \in M$ and for every $i \in \{0, 1, \ldots, p-1\}$ is a central automorphism of G with $\phi(x) = xz$. Because G is a $\mathscr{C}$-group, $\alpha(x) \in \langle x \rangle$, that is $Z(G) \subset \langle x \rangle$ for every $x \in G - \Phi(G)$.

We obtain $|x| > p$ for every $x \in G - \Phi(G)$ and $\Omega_1(G) = Z_2(G)$. It is known that $G^p \leq \Phi(G)$, so that $\exp(G) = p^2$. Now if $x \in G - \Phi(G)$ then $|x| = p^2$ and $Z(G) \subset \langle x \rangle$, that is $Z(G) = \langle x^p \rangle$. It follows that $G^p = Z(G)$.

Finally we get

$$1 < Z(G) = G^p < Z_2(G) = \Phi(G) = \Omega_1(G) < G. \qquad (**)$$

Conversely, we will show that a group G of order p^4 satisfying (**) is a nonabelian $\mathscr{C}$-group with abelian noncyclic second center.

Let G be such a group. Obviously G has class 3, $|Z(G)| = p$ and $|Z_2(G)| = p^2$. Being generated by elements of order p, $Z_2(G)$ is elementary abelian.

Now let ϕ be a nontrivial central automorphism of G. Then $1 \neq K_\phi \leq Z(G)$, that is $K_\phi = Z(G)$. It follows that F_ϕ is a maximal subgroup of G. This implies $\phi(x) = x$, for every $x \in \Phi(G)$.

Let $x \in G - \Phi(G)$. From (**) we get $|x| > p$ and $x^p \in Z(G)$, that is $Z(G) = \langle x^p \rangle$.

Now $\phi(x) \in xZ(G) \subset \langle x \rangle$, hence G is a $\mathscr{C}$-group. We may observe that such a group G is nonregular because $|G^p| \,.\, |\Omega_1(G)| \neq |G|$ and the lack of regularity implies $p \leqslant 3$ (See [**3**], Satz 10.2 and Satz 10.7).

Moreover, $p \neq 2$ because $G^p \neq \Phi(G)$. We may conclude that the only nonabelian $\mathscr{C}$-groups G of order p^4 which have abelian noncyclic second center are the groups of order 81 satisfying (**).

There is a single group with these properties, namely $G = \langle a, b \mid a^9 = 1,\ b^3 = a^6, [a,b]^3 = 1,\ [a,[a,b]] = a^3,\ [b,[a,b]] = 1 >$ (See [**3**], Aufgabe 29 p. 349).

It is easy to see that $Z(G) = G^3 = \langle a^3 \rangle \simeq \mathbb{Z}_3$ and

$$Z_2(G) = \Phi(G) = \Omega_1(G) = \langle a^3, [a,b] \rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_3.$$

REFERENCES

1. M. J. Curran and D. J. McCaughan, Central automorphisms of finite groups, *Bull. Austral. Math. Soc.* **34** (1986), 191–198.

2. D. Gorenstein, *Finite groups* (Harper and Row, 1968).

3. B. Huppert, *Endliche Gruppen* I (Springer Verlag, 1967).

4. G. Scorza, I gruppi che possono pensarsi come somma di tre loro sottigruppi, *Boll. Un. Mat. Ital.* **5** (1926), 216–218.

Department of Mathematics
University of Timişoara
Bd. V. Pârvan nr.4
1900 Timişoara, Romania.

and

Department of Mathematics
University of Kuwait
P.O. Box 5969, Safat 13060
Kuwait.

Department of Mathematics
University of Timişoara
Bd. V. Pârvan nr.4
1900 Timişoara, Romania.