CONDENSOR PRINCIPLE AND THE UNIT CONTRACTION

MASAYUKI ITÔ

To Professor Kiyoshi Noshiro on the occasion of his 60th birthday.

Introduction

Deny introduced in [4] the notion of functional spaces by generalizing Dirichlet spaces. In this paper, we shall give the following necessary and sufficient conditions for a functional space to be a real Dirichlet space.

Let $\mathscr{X}$ be a regular functional space with respect to a locally compact Hausdorff space X and a positive measure ξ in X. The following four conditions are equivalent.

(1) The unit contraction operates on $\mathscr{X}$.

(2) $\mathscr{X}$ satisfies the condensor principle.

(3) $\mathscr{X}$ satisfies the strong complete maximum principles.

(4) $\mathscr{X}$ is a real Dirichlet space.

Furthermore for an invariant functional space $\mathscr{X}$ on a locally compact abelian group X, we shall show the following equivalence without assuming the regularity.

$\mathscr{X}$ is special Dirichlet space if and only if $\mathscr{X}$ satisfies the condensor principle.

1. Preliminaries on regular functional spaces

Let X be a locally compact Hausdorff space and ξ be a positive measure in X which is everywhere dense in X (i.e., $\xi(\omega)>0$ for any non-empty open set ω in X). According to Deny [4], we give the definition of a functional space.

Definition 1. A functional space $\mathscr{X}=\mathscr{X}(X, \xi)$ with respect to X and ξ is a Hilbert space of real valued functions $u(x)$ which is locally summable for ξ, the following condition being satisfied: (i) For any compact subset K in X, there existes a positive number $A(K)$ such that

Received May 6, 1966.

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

$$\int_K |u(x)|\,d\xi(x) \leq A(K)\|u\|$$

for any u in $\mathscr{X}$.

Two functions which are equal locally almost everywhere for ξ represent the same element in $\mathscr{X}$. The norm in $\mathscr{X}$ is denoted by $\|u\|$, the associated scalar product by (u, v). Let C_K be the space of finite continuous functions with compact support provided with the topology of uniform convergence.

DEFINITION 2. A functional space $\mathscr{X}=\mathscr{X}(X, \xi)$ is said to be regular if $\mathscr{X}\cap C_K$ is dense both in $\mathscr{X}$ and in C_K.

By the condition (i), for any bounded measurable function f with compact support, there exists an element u_f in a functional space $\mathscr{X}$ such that

$$(u_f, u)=\int fu\,d\xi$$

for any u in $\mathscr{X}$. Such an element u_f is said to be the potential generated by f. More generally we define potentials as follows.

DEFINITION 3.[1] Let $\mathscr{X}$ be a regular functional space. The element u is called a potential if there exists a real Radon measure μ such that

$$(u, f)=\int f\,d\mu$$

for any f in $\mathscr{X}\cap C_K$. Such an element u is denoted by u_μ. Especially if μ is positive, u_μ is said to be a pure potential.

According to Beurling and Deny [2], we define the capacity of an open set is defined as follows:

$$Cap(\omega)=inf\{\|u\|^2;\ u\varepsilon\mathscr{X}, u(x)\geq 1\ p.p.\ in\ \omega\}.$$

If there are no such functions, $Cap\,(\omega)=+\infty$.

LEMMA 1. *Let $\mathscr{X}$ be a regular functional space and f be a function in $\mathscr{X}\cap C_K$. Then for each positive number ε,*

$$Cap(\{x\varepsilon X;\ f(x)>\varepsilon\})\leq\frac{\|f\|^2}{\varepsilon^2}.$$

By the definition of the capacity, this is evident.

LEMMA 2. *For a relatively compact open set ω in X, put*

$$E_\omega=\{\overline{u_\mu\varepsilon\mathscr{X};\ S_\mu\subset\omega,\ \mu\geq 0}\}.\ [2]$$

---

1) Cf. [2], p. 209.

2) S_μ is the support of μ.

Then there exists a unique element u_γ *which minimizes*

$$I(u_\mu)=||u_\mu||^2-2\int d\mu$$

in E_ω *and for which*

$$Cap(\omega)=||u_\gamma||^2=\int d\gamma.$$

Proof. Obviously E_ω is a closed convex cone in $\mathscr{X}$. Since ω is a relatively compact set, there exists a function f in $\mathscr{X}\cap C_K$ such that $f(x)\geq 1$ in ω. Then

$$I(u_\mu)\geq||u_\mu||^2-2\int f\,d\mu=||u_\mu-f||^2-||f||^2.$$

Hence $I(u_\mu)$ is bounded from below in E_ω. Therefore there exists a unique pure potential u_γ such that

$$I(u_\gamma)\leq I(u_\mu)$$

for any u_μ in E_ω. Then

$$\int d\mu\leq(u_\gamma, u_\mu) \tag{1}$$

and

$$\int d\gamma=||u_\gamma||^2. \tag{2}$$

By (1), $u_\gamma(x)\geq 1$ *p.p.* in ω. Hence

$$||u_\gamma||^2\geq Cap(\omega).$$

On the other hand it is known that there exists a sequence (u_{f_n}) of pure potentials such that $u_{f_n}\to u_\gamma$ strongly in $\mathscr{X}$, where f_n is a positive bounded measurable function with support in ω.[3] For any u in $\mathscr{X}$ such that $u(x)\geq 1$ *p.p.* in ω,

$$(u_{f_n}, u)=\int f_n u\,d\xi\leq\int f_n\,d\xi.$$

Since the measure f_n converges vaguely to γ and ω is relatively compact,

$$\lim_{n\to\infty}\int f_n\,d\xi=\int d\gamma.$$

---

3) Cf. [4], p. 3 and [6].

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

Hence

$$(u_\gamma, u) \geq \int d\gamma = \|u_\gamma\|^2,$$

i.e., $\|u\| \geq \|u_\gamma\|$. Consequently

$$Cap(\omega) = \|u_\gamma\|^2 = \int d\gamma.$$

Lemma 3. *Let $\mathscr{X}$ be a regular functional space on X and ω be an open set in X. For any increasing net $(\omega_\alpha)_{\alpha \in I}$ of relatively compact open sets exhausting ω,*

$$\lim_{\alpha \in I} Cap(\omega_\alpha) = Cap(\omega).$$

Proof. Obviously $Cap(\omega_\alpha)$ increases with α. First we suppose that $Cap(\omega) < +\infty$. Then $Cap(\omega_\alpha)$ is bounded. Let u_{γ_α} be the pure potential such that $Cap(\omega_\alpha) = \|u_{\gamma_\alpha}\|^2$. Suppose that $\alpha \leq \beta$. Then

$$\begin{aligned} \|u_{\gamma_\alpha} - u_{\gamma_\alpha}\|^2 &= \|u_{\gamma_\alpha}\|^2 - 2(u_{\gamma_\alpha}, u_{\gamma_\beta}) + \|u_{\gamma_\beta}\|^2 \\ &\leq \|u_{\gamma_\beta}\|^2 - \|u_{\gamma_\alpha}\|^2. \end{aligned}$$

Hence (u_{γ_β}) is a fundamental net in $\mathscr{X}$. There exists an element u in $\mathscr{X}$ such that $u_{\gamma_\alpha} \to u$ strongly in $\mathscr{X}$. For any positive bounded measurable function f with compact support such that $S_f \subset \omega$, there exists α_0 in I such that

$$(u_f, u_{\gamma_\alpha}) = \int u_{\gamma_\alpha} f d\xi \geq \int f\, d\xi$$

for any $\alpha \geq \alpha_0$. Therefore

$$(u_f, u) \geq \int f\, d\xi,$$

i.e., $u(x) \geq 1$ *p.p.* in ω. Hence

$$Cap(\omega) \leq \|u\|^2.$$

Consequently

$$\lim_{\alpha \in I} Cap(\omega_\alpha) = Cap(\omega).$$

In the case that $Cap(\omega) = +\infty$, it is evident that

$$\lim_{\alpha \in I} Cap(\omega_\alpha) = +\infty$$

by the above proof.

LEMMA 4. Let ω_n be an open set in X $(n=1, 2, \ldots\ldots)$.
Put

$$\omega = \bigcup_{n=1} \omega_n$$

Then

$$Cap(\omega) \leq \sum_{n=1}^{\infty} Cap(\omega_n).$$

By Lemmas 2 and 3, we can prove in the same manner as Deny [5].[4]

RPOPOSITION 1.[5] *Let $\mathscr{X}$ be a regular functional space on X. For any u in $\mathscr{X}$, there exists a function u^* with the following properties.*

(1.1) *$u(x)=u^*(x)$ p.p. in X and $u^*(x)=0$ outside some σ-compact set.*

(1.2) *There exists a decreasing sequence (ω_n) of open sets such that*

$$\lim_{n\to\infty} Cap(\omega_n)=0$$

and $u^*(x)$ is continuous on $\mathscr{C}\omega_n$ for each n.

(1.3) *For any pure potential u in $\mathscr{X}$, u^* is μ-measurable and*

$$(u, u_\mu) = \int u^* \, d\mu.$$

By Lemmas 1, 2, 3, and 4, we can prove in the same manner as Deny [5].

We say that u^* is *the refinement of* u. Furthermore we have

LEMMA 5. *For any u in $\mathscr{X}$, u^* is μ-measurable for any u_μ in $\mathscr{X}$ such that S_μ^+ is compact and*

$$S_\mu^+ \cap S_\mu^- = \phi.$$

Proof. S_μ^+ being compact, we can take an open set ω in X such that $\omega \supset S_\mu^+$ and

$$S_\mu^- \cap \overline{\omega} = \phi.$$

Put

$$\mathscr{X}_\omega = \{\overline{u \varepsilon C_K \cap \mathscr{X};\ S_u \subset \omega}\}.$$

Then $\mathscr{X}_\omega$ is a regular functional space on ω. We take another open set $\omega^{(1)}$

[4] Cf. [5], p. 136.
[5] Cf. [2], p. 209.

such that

$$S_\mu^+ \subset \omega^{(1)} \subset \overline{\omega}^{(1)} \subset \omega.$$

Let (ω_n) be the sequence in Proposition 1. Put

$$\omega_n' = \omega^{(1)} \cap \omega_n.$$

Let $Cap'(\omega_n')$ be the capacity of ω_n' relative to the functional space $\mathscr{X}_\omega$. Obviously

$$\lim_{n\to\infty} Cap'(\omega_n') = 0.$$

Let u'_{γ_n} be the pure potential in $\mathscr{X}_\omega$ such that

$$Cap'(\omega_n') = ||u'_{\gamma_n}||^2.$$

Then

$$\int_{\omega_n'} d\mu^+ \leq (u_\mu, u'_{\gamma_n}) \leq ||u_\mu|| \, ||u'_{\gamma_n}|| \to 0$$

as $n \to +\infty$. Therefore u^* is μ^+-measurable. Similarly u^* is μ^-–measurable.

2. The unit contraction and Condensor principle

First we define the unit contraction on 1–dimensional Euclidean space R.

Definition 5. We call the projection T of R to the closed interval $[0, 1]$ the unit contraction on R.

Let $\mathscr{X}$ be a regular functional space with respect to X and ξ.

Definition 6. We say that the unit contraction T operates on $\mathscr{X}$ if for any u in $\mathscr{X}$, Tu is in $\mathscr{X}$ and $||Tu|| \leq ||u||$.

Definition 7. We say that $\mathscr{X}$ satisfies the condensor principle if for any couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being relatively compact, there exists a potential u_μ such that

(C. 1) $0 \leq u_\mu(x) \leq 1$ *p.p.* in X,

(C. 2) $u_\mu(x) = 1$ *p.p.* in ω_1 and $u_\mu(x) = 0$ *p.p.* in ω_0,

(C. 3) $u_\mu \varepsilon \overline{E_{\omega_1} - E_{\omega_0}}$, where E_{ω_1} and E_{ω_0} are the sets which we defined in Lemma 2.

We shall call the above potential u_μ the condensor potential with respect to ω_1 and ω_0.

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

LEMMA 6. *Suppose that* $\mathscr{X}$ *satisfies the condensor principle. For any couple of open sets* ω_1 *and* ω_0 *with disjoint closures,* ω_1 *being relatively compact, put*

$$A_{1,0}=\{u\in\mathscr{X};\ u(x)\geq 1\ p.p.\ in\ \omega_1\ and\ u(x)\leq 0\ p.p.\ in.\ \omega_0\}.$$

Then there exists a unique element in $\mathscr{X}$ *whose norm is minimum in* $A_{1,0}$ *and it is equal to the condensor potential with respect to* ω_1 *and* ω_0.

Proof. Obviously $A_{1,0}$ is non-empty closed convex set in $\mathscr{X}$. Hence there exists a unique element $u_{1,0}$ in $A_{1,0}$ such that $||u_{1,0}||\leq||u||$ for any u in $A_{1,0}$. Let u_μ be the condensor potential with respect to ω_1 and ω_0. Since u_μ is in $A_{1,0}$, $||u_\mu||\geq||u_{1,0}||$. On ther other hand there exists a sequence $(u_{\mu_{1,n}}-u_{\mu_{0,n}})$ such that $u_{\mu_{1,n}}$ and $u_{\mu_{0,n}}$ are pure potentials,

$$S_{\mu_{1,n}}\subset\omega_1,\ S_{\mu_{0,n}}\subset\omega_0$$

and $u_{\mu_{1,n}}-u_{\mu_{0,n}}$ converges strongly to u_μ in $\mathscr{X}$ as $n\to+\infty$. For any u in $A_{1,0}$,

$$(u,\ u_{\mu_{1,n}}-u_{\mu_{0,n}})=\int u^*\,d\mu_{1,n}-\int u^*\,d\mu_{0,n}\geq(u_\mu,\ u_{\mu_{1,n}}-u_{\mu_{0,n}}),$$

because $u^*(x)\geq 1$ $p.p.p.$ in ω_1 and $u^*(x)\leq 0$ $p.p.p.$ in ω_0.[6] Hence

$$||u||\cdot||u_\mu||\geq(u,u_\mu)\geq||u_\mu||^2,$$

i.e., $||u||\geq||u_\mu||$. Consequently $u_{1,0}=u_\mu$.

LEMMA 7. *Let* $\mathscr{X}$ *be a regular functional space. Each element in* $\overline{E_{\omega_1}-E_{\omega_0}}$ *is a potential in* $\mathscr{X}$.

Proof. For any u in $\overline{E_{\omega_1}-E_{\omega_0}}$, there exists a sequence $(u_{\mu_n}-u_{\nu_n})$ of $E_{\omega_1}-E_{\omega_0}$ tending strongly to u in $\mathscr{X}$. Since

$$\bar{\omega}_0\cap\bar{\omega}_1=\phi$$

and $C_K\cap\mathscr{X}$ is dense in C_K, (μ_n) and (ν_n) are vaguely bounded. Hence we may assume that there exist positive measures μ and ν such that $\mu_n\to\mu$ and $\nu_n\to\nu$ vaguely as $n\to+\infty$. Therefore

$$(u,f)=\int f\,d(\mu-\nu)$$

for any f in $C_K\cap\mathscr{X}$. Consequently

$$u=u_{\mu-\nu}.$$

[6] Cf. [6], Lemma 2. A property is said to hold $p.p.p.$ on a subset E in X if the property holds μ-$p.p.$ for any pure potential u_μ in E such that $S_\mu\subset E$.

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

By Lemma 7, we obtain the following lemma.

LEMMA 8. *Let $\mathscr{X}$ be a regular functional space. Let $A_{1,0}$ be the same as in Lemma 6. The element u' whose norm is minimum $A_{1,0}$ is contained in $\overline{E_{\omega_1}-E_{\omega_0}}$.*

Proof. By Lemma 7, we can consider the following valuation:

$$I'(u_{\mu_1}-u_{\mu_0})=\|u_{\mu_1}-u_{\mu_0}\|^2-2\int d\mu_1$$

for any $u_{\mu_1}-u_{\mu_0}$ in $\overline{E_{\omega_1}-E_{\omega_0}}$. Similarly as in Lemma 2, $I'(u_{\mu_1}-u_{\mu_0})$ is bounded from below on $\overline{E_{\omega_1}-E_{\omega_0}}$. Since $\overline{E_{\omega_1}-E_{\omega_0}}$ is a non-empty closed convex set in $\mathscr{X}$, there exists a unique element $u_{\gamma_1}-u_{\gamma_0}$ in $\overline{E_{\omega_1}-E_{\omega_0}}$ such that

$$I'(u_{\gamma_1}-u_{\gamma_0})\leq I'(u_{\mu_1}-u_{\mu_0})$$

for any $u_{\mu_1}-u_{\mu_0}$ in $\overline{E_{\omega_1}-E_{\omega_0}}$. Similarly is as the proof of Lemma 2,

$$u'=u_{\gamma_1-\gamma_0}.$$

Now we remark that the regular functional space $\mathscr{X}$ satisfies the equilibrium principle if $\mathscr{X}$ satisfies the condensor principle. That is, for any relatively compact open set ω, there exists a pure potential u_μ such that

(E. 1) $$0\leq u_\mu(x)\leq 1 \quad p.p. \text{ in } X,$$

(E. 2) $$u_\mu(x)=1 \; p.p. \text{ in } \omega,$$

(E. 3) $$u_\mu \text{ is contained in } E_\omega.$$

Such element u_μ is called an equilibrium potential of ω.

LEMMA 9. *Let $\mathscr{X}$ be the regular functional space which satisfies the condensor principle. For any couple of open sets ω_1 and ω_0 with disjoing closures, ω_1 being relatively compact, let u_μ be the condensor potential with respect to ω_1 and ω_0. Then*

$$\int d\mu\geq 0.$$

Proof. We take a relatively compact open set ω such that $\omega\supset\overline{\omega_1}$. Let u_ν be the equilibrium potential of ω. Since by Lemma 5,

$$u_\nu^*(x)=1 \quad p.p.p. \text{ in } \omega,$$
$$0\leq u_\nu^*(x)\leq 1 \quad p.p.p. \text{ in } X,$$

we have

$$(u_\mu, u_\nu)=\int u_\nu{}^* d\mu^+ - \int u_\nu{}^* d\mu^- \leq \int d\mu^+ - \int_\omega d\mu^-.$$

On the other hand since we have

$$u_\mu{}^*(x) \geq 0 \quad p.p.p. \quad \text{in } X,$$

$$(u_\mu, u_\nu) = \int u_\mu{}^* dv \geq 0.$$

Hence

$$\int d\mu^+ \geq \int_\omega d\mu^-.$$

ω being arbitrary, we obtain that the total mass of μ is non-negative.

LEMMA 10. *Let $\mathscr{H}$ be the same as above. Let F_1 be a compact and F_0 be a closed set such that*

$$F_1 \cap F_0 = \phi.$$

Then there exists a potential u_μ in $\mathscr{H}$ such that

(C'. 1) $\quad 0 \leq u_\mu{}^*(x) \leq 1 \quad p.p. \quad X,$

(C' 2) $\quad u_\mu{}^*(x)=1 \quad p.p.p.$ in F_1, $u_\mu{}^*(x)=0 \quad p.p.p.$ *in* F_0,

(C' 3) $\quad S_\mu + \subset F_1, S_\mu - \subset F_0,$

(C' 4) $\quad \int d\mu \geq 0.$

Proof. We take two decreasing nets $(\omega_{1,\alpha})_{\alpha\varepsilon I}$ and $(\omega_{0,\alpha})_{\alpha\varepsilon I}$ of open sets converging to F_1, F_0 such that $\omega_{1,\alpha}$ is relatively compact for any $\alpha\varepsilon I$,

$$\omega_{1,\alpha} \supset F_1, \ \omega_{0,\alpha} \supset F_0$$

and for any, $\alpha < \beta$,

$$\overline{\omega_{1,\alpha}} \subset \omega_{1,\beta}, \ \overline{\omega_{0,\alpha}} \subset \omega_{0,\beta}.$$

Let u_{μ_α} be the condensor potential with respect to $\omega_{1,\alpha}$ and $\omega_{0,\alpha}$. Since $u^*_{\mu_\alpha}(x)$ is bounded in X, by Lemma 5,

$$(u_{\mu_\alpha}, u_{\mu_\beta})=\int u^*_{\mu_\alpha} d\mu^+_\beta - \int u^*_{\mu_\beta} d\mu^-_\beta = ||u_{\mu_\beta}||^2$$

for any $\alpha \leq \beta$. Hence $||u_{\mu_\alpha}|| \geq ||u_{\mu_\beta}||$ for any $\alpha \leq \beta$, i.e., $\{||u_{\mu_\alpha}||\}$ is convergent. Furthermore we have

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

$$\|u_{\mu_\alpha}-u_{\mu_\beta}\|^2=\|u_{\mu_\alpha}\|^2-2(u_{\mu_\alpha}, u_{\mu_\beta})+\|u_{\mu_\beta}\|^2=\|u_{\mu_\alpha}\|^2-\|u_{u_\beta}\|^2.$$

Therefore there exists an element u in $\mathscr{X}$ such that $u_{\mu_\alpha}\to u$ strongly in $\mathscr{X}$. Obviously the sets $(\mu_\alpha^+)_{\alpha\varepsilon I}$ and $(\mu_\alpha^-)_{\alpha\varepsilon I}$ are vaguely bounded, and hence we may assume that there exist two positive measures μ_1 and μ_0 such that $(\mu_\alpha^-)_{\alpha\varepsilon I}$ and (μ_α^-) converge vaguely to μ_1 and, μ_0, respectively. By the definition of a potential in $\mathscr{X}$,

$$u=u_{\mu_1-\mu_0}.$$

We shall show that this element u is the required element. Evidently

$$S_{\mu_1}\subset F_1, S_{\mu_0}\subset F_0.$$

Since we have

$$u^*_{\mu_\alpha}=1 \text{ } p.p.p. \text{ in } \omega_{1,\alpha} \text{ and } u^*_{\mu_\alpha}=0 \text{ } p.p.p. \text{ in } \omega_{0,\alpha},$$
$$u^*=1 \text{ } p.p.p. \text{ in } F_1 \text{ and } u^*=0 \text{ } p.p.p. \text{ in } F_0.$$

It is evident that u satisfies the condition (C′. 1). Finally we prove that u satisfies the condition (C'. 4). $S_{\mu_\alpha^\pm}$ being in a fixed compact set,

$$\lim_{\alpha\varepsilon I}\int d\mu_\alpha^+=\int d\mu_1.$$

On the other hand

$$\lim_{\alpha\varepsilon I}\int d\mu_\alpha^-\geq\int d\mu_0.$$

By Lemma 9, we obtain the inequality

$$\int d\mu_1\geq\int d\mu_0.$$

We call such a potential u_μ the condensor potential with respect to F_1 and F_0. Now we consider the strong complete maximum principle.

Definition 7.[6] We say that a regular functional space $\mathscr{X}$ satisfies the strong complete maximum principle if the following condition is fulfiled. For a potential u_f, f being locally summable for ξ, and a pure potential u_ν in $\mathscr{X}$ and a non-negative constant c, suppose that

$$u_f^*(x)\leq u_\nu^*(x)+c$$

$p.p.p.$ on K_{f^+}. Then

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

$$u_f(x) \leq u_\nu(x) + c$$

p.p. in X.

In this definition, K_{f^+} is a set whose complement is of f^+-measure zero.

By the above lemmas, we obtain the following theorem.

THEOREM 1. *If a regular funtiocnal space $\mathscr{X}$ satisfies the condensor principles, then $\mathscr{X}$ satisfies the strong complete maximum principle.*

Proof. Let u_f, u_ν and c be the same as in Definition 7. Suppose that there exists a compact set K_1 in $\mathscr{C}K_{f^+}$ such that $\xi(K_1) > 0$ and

$$u_f(x) > u_\nu(x) + c$$

on K_1. Since

$$u_f^*(x) = u_f(x) \text{ } p.p. \text{ in } X \text{ and } u_\nu^*(x) = u_\nu(x) \text{ } p.p. \text{ in } X, \text{ } u_f^*(x) > u_\nu^*(x) + c$$

p.p. on K_1. Therefore there exists a compact set K_2 in K_1 such that $\xi(K_2) > 0$ and

$$u_f^*(x) > u_\nu^*(x) + c$$

on K_2. By Proposition 1, there exists a decreasing sequence (ω_n) of open sets such that

$$\lim_{n \to \infty} Cap(\omega_n) = 0,$$

$u_f^*(x)$ and $u_\nu^*(x)$ are continuous on $\mathscr{C}\omega_n$. Since $\xi(\omega_n) \searrow 0$ as $n \to +\infty$, there exists a number n such that

$$\xi(K_2 \cap \mathscr{C}\omega_n) > 0.$$

We take a compact set K such that

$$K \subset K_2 \cap \mathscr{C}\omega_n \text{ and } \xi(K) > 0.$$

Then $u_f^*(x)$ and $u_\nu^*(x)$ are continuous and $u_f^*(x) > u_\nu^*(x) + c$ on K, and hence there exists a positive number a such that

$$u_f^*(x) - u_\nu^*(x) - c > a$$

on K. Since f is locally summable for ξ, there exists an open set G such that $G \supset K$ and

$$\int_G f^+(x) d\xi(x) < \frac{1}{2} a \cdot Cap(K),$$

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

where

$$Cap(K)=\inf_{k\subset\omega} Cap(\omega),$$

because we have

$$\int_K f^+(x)d\xi(x)=0 \text{ and } Cap(K)>0.$$

Put

$$K'_{f^+}=K_{f^+}\cap\mathscr{C}G.$$

By the measurablity of f, there exists an increasing sequence (F_n) of compact sets such that $F_n\subset K'_{f^+}$ and

$$\lim_{n\to\infty}\xi(F_n\cap F)=\xi(K'_{f^+}\cap F)$$

for any compact set F. Let u_{μ_n} be the condensor potential with respect to K and F_n. Similarly as the proof of Lemma 10, there exists a potential u_μ such that $u_{\mu_n}\to u_\mu$ strongly in $\mathscr{X}$ and $S_{\mu^+}\subset K$. By Lemmas 9 and 10,

$$(u_\mu, u_\nu)=\int(u_f^*(x)-u_\nu^*(x))\,d\mu\geq(a+c)\int d\mu^+-c\int d\mu^-\geq a\int d\mu^+=a\|u_\mu\|^2\geq a\cdot\mathrm{Cap}(K).$$

Let $(G_\alpha)_{\alpha\in I}$ be an increasing net of relatively compact open sets such that $G_\alpha\supset G$ and $G_\alpha\nearrow X$. Similarly as the above, we can take the condensor potential u_{μ_α} with respect to K and $K'_{f^+}\cup\mathscr{C}G_\alpha$. Since u_{μ_α} is a bounded measurable function with compact support, u_{μ_α} is f-integrable and

$$(u_{\mu_\alpha}, u_f-u_\nu)=\int u_{\mu_\alpha}(x)f^+(x)d\xi(x)$$

$$-\Big(\int u_{\mu_\alpha}(x)f^-(x)d\xi(x)+\int u_{\mu_\alpha}^*(x)d\nu(x)\Big)$$

$$\leq\int_G u_{\mu_\alpha}(x)f^+(x)d\xi(x)\leq\int_G f^+(x)d\xi(x)\leq\frac{1}{2}a\cdot Cap(K).$$

Now since $(u_{\mu_\alpha})_{\alpha\in I}$ converges strongly to u_μ in $\mathscr{X}$,

$$(u_\mu, u_f-u_\nu)\leq\frac{1}{2}a\cdot Cap(K).$$

This is a contradiction and the proof is completed.

3. Main theorems

First we consider the resolvent operator on a regular functional space $\mathscr{X}$

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

or $L^2=L^2(\xi)$.

LEMMA 11.[7] *Let f be in L^2 or in $\mathscr{X}$. For each positive number λ, there exists a unique element $R_\lambda f$ in $\mathscr{X}$ which minimizes the following quadratic form:*

$$F(u)=\|u\|^2+\int|u(x)-f(x)|^2 d\xi(x)$$

in the set

$$A_f=\{u\varepsilon\mathscr{X};\ u-f\varepsilon L^2\}.$$

$R_\lambda f$ is also the only element u in $\mathscr{X}$ such that $u-f$ is in L^2 and

$$\lambda(u, v)+\int(u-f)v\, d\xi=0$$

for any v in $L^2\cap\mathscr{X}$.

This is obtained by Beurling and Deny [2] for the case when $\mathscr{X}$ is a Dirichlet space. For the case when $\mathscr{X}$ is a regular functional space, this is proved in the same way. We call such an operator R_λ the resolvent operator. Before we prove the main theorem, we prepare the following lemma.

LEMMA 12. *Let $\mathscr{X}$ be a regular functional space on X. Suppose that $\mathscr{X}$ satisfies the strong complete maximum principle. Then for any positive bounded function f with compact support,*

$$0\leq R_\lambda f(x)\leq M$$

p.p. in X, where

$$M=\operatorname*{ess.sup}_{x\varepsilon X} f(x).$$

Proof. First we shall prove that

$$R_\lambda f(x)\geq 0$$

p.p. in X. By the second part of Lemma 11, $R_\lambda f$ is the potential generated by $f-R_\lambda f$ in $\mathscr{X}$. Since the potential u_f generated by f is in $\mathscr{X}$, there exists a potential $u_{R_\lambda f}$ generated by $R_\lambda f$ in $\mathscr{X}$. Then

$$u_f-\lambda R_\lambda f=u_{R_\lambda f}\,.$$

Hence

$$u_f^*(x)-\lambda(R_\lambda f)^*(x)=u_{R_\lambda f}^*(x)$$

[7] Cf. [2], p. 211.

p.p.p. in X. Since

$$R_\lambda f(x)=(R_\lambda f)^*(x)$$

p.p. in X, we have

$$u_{R_\lambda f}=u_{(R_\lambda f)^*}.$$

Since

$$u_f^*(x)\geq u^*_{(R_\lambda f)^+}(x)$$

p.p.p. on $K_{(R_\lambda f)^{*+}}$, by Theorem 1,

$$u_f(x)\geq u_{(R_\lambda f)^*}(x)$$

p.p. in X. Therefore $R_\lambda f\geq 0$ *p.p.* in X.

Next we shall show that

$$R_\lambda f(x)\leq M$$

p.p. in X. There exists a function g in C_K such that $g(x)\geq f(x)$ *p.p.* in X and $g(x)\leq M$. Since by the above arguement, R_λ is a positive operator,

$$R_\lambda f(x)\leq R_\lambda g(x)$$

p.p. in X. Similarly as above,

$$(R_\lambda g)^*(x)=u^*_{g-(R_\lambda g)^+}(x)$$

p.p.p. in X. Similarly as in the first part of this lemma,

$$M\geq g(x)\geq(R_\lambda g)^*(x)$$

p.p.p. in $K_{((g-R_\lambda g)^*)^+}$. Hence

$$M\geq u_{(g-(R_\lambda g)^*)}^*(x)$$

p.p.p. in $K_{(g-(R_\lambda g)^*)^+}$. By the strong complete maximum principle,

$$M\geq u_{g-(R_\lambda g)^*}(x)$$

p.p. in X. Consequently

$$R_\lambda f\leq R_\lambda g\leq M$$

p.p. in X. This completes the proof.

Now we shall show the following main theorem.

THEOREM 2. *Let $\mathscr{H}$ be a regular functional space with respect to X and ξ.*

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

Then the following four conditions are equivalent.

(1) *The unit contraction operates on $\mathscr{X}$.*

(2) *$\mathscr{X}$ satisfies the condensor principle.*

(3) *$\mathscr{X}$ satisfies the strong complete maximum principle.*

(4) *$\mathscr{X}$ is a real Dirichlet space with respect to X and ξ.*[8]

Proof. First we shall prove the implication (1) ⇨ (2). For any couple of open sets ω_1 and ω_0 with disjoint closures, ω_1 being relatively compact, let $A_{1,0}$, E_{ω_1} and E_{ω_0} be the same as defined before. Let $u_{1,0}$ be a unique element in $\mathscr{X}$ whose norm is minimum in $A_{1,0}$. Since the unit contraction T operates on $\mathscr{X}$, $Tu_{1,0}$ is in $A_{1,0}$ and

$$\|Tu_{1,0}\| \leq \|u_{1,0}\|.$$

Therefore $Tu_{1,0}=u_{1,0}$. By Lemma 8, $u_{1,0}$ belongs to $\overline{E_{\omega 1}-E_{\omega 0}}$ and hence it is the condensor potential with respect to ω_1 and ω_0.

The implication (2) ⇨ (3) was proved in Theorem 1.

Next we shall show the implication (3) ⇨ (4). For a positive number λ, let R_λ be a resolvent operator. For any f, g in $C_K \cap \mathscr{X}$,

$$(R_\lambda f, R_\lambda g)=\frac{1}{\lambda}\int(f-R_\lambda f)R_\lambda g\,d\xi=\frac{1}{\lambda}\int(g-R_\lambda g)R_\lambda f\,d\xi,$$

Hence

$$(R_\lambda f,\ g)=(R_\lambda g,\ f)$$

and

$$\int R_\lambda f g\,d\xi=\int R_\lambda g f\,d\xi.$$

Hence by Lemma 12, there exists a positive symmetric measure σ_λ on $X\times X$ such that

$$\int R_\lambda f(x)g(x)\,d\xi(x)=\iint f(x)g(y)\,d\sigma_\lambda(x,y)$$

for any f, g in C_K and σ_λ is sub-markovian, *i. e.*, the projection of σ_λ on X is less than or equal to ξ. Let m_λ be the density of the projection of σ_λ on X. By the second part of Lemma 11, for any f, g in $C_K \cap \mathscr{X}$,

---

[8] A real Dirichlet space with respect to X and ξ is a Dirichlet space with respect to X and ξ which consists of real functions. For Dirichlet spaces, see [2], p. 209.

$$(R_\lambda f,\ g)=\frac{1}{\lambda}\int(f-R_\lambda f)g\,d\xi$$
$$=\frac{1}{\lambda}\left\{\int(1-m_\lambda)fg\,d\xi+\frac{1}{2}\iint(f(x)-f(y))(g(x)-g(y))\,d\sigma_\lambda(x,y)\right\}$$

Now by the first part of Lemma 11, for any positive number λ,

$$\|R_\lambda f\|\leq\|f\|.$$

And by the second part of Lemma 11,

$$(R_\lambda f,\ R_\lambda f-f)=-\int|R_\lambda f-f|^2 d\xi.$$

Therefore $R_\lambda f\to f$ strongly in L^2, and hence $R_\lambda f\to f$ weakly in $\mathscr{X}$ as $\lambda\to 0$. Since

$$\lim_{\lambda\to 0}\|R_\lambda f\|\geq\|f\|\geq\|R_\lambda f\|$$

for any $\lambda>0$, $R_\lambda f\to f$ strongly in $\mathscr{X}$ as $\lambda\to 0$. Next we shall prove the following assertion: for a function f in C_K, suppose that

$$H_\lambda(f)=\frac{1}{\lambda}\left\{\int(1-m_\lambda)|f|^2\,d\xi+\frac{1}{2}\iint|f(x)-f(y)|^2\,d\sigma_\lambda(x,\ y)\right\}$$

is bounded with respect to λ. Then f is in $\mathscr{X}$ and $H_\lambda(f)\to\|f\|^2$ as $\lambda\to 0$. In fact,

$$H_\lambda(f)=\frac{1}{\lambda}\int(1-R_\lambda f)\,f\,d\xi\geq\frac{1}{\lambda}\int(f-R_\lambda f)\,R_\lambda f\,d\xi=\|R_\lambda f\,\|^2.$$

Hence $(R_\lambda f)$ is bounded with respect to λ, and we may assume that there exists an element u in $\mathscr{X}$ such that $R_\lambda f\to u$ weakly in $\mathscr{X}$ as $\lambda\to 0$. On the other hand by the second part of Lemma 11, $R_\lambda f\to f(x)$ *p.p.* in X. Consequently $u(x)=f(x)$ *p.p.* in X, *i, e.*, f is in $\mathscr{X}$ and $H_\lambda(f)\to\|f\|^2$ as $\lambda\to 0$. Thus we obtain:

For any f in $C_K\cap\mathscr{X}$ and any normal contraction T[9] on R, Tf is in $\mathscr{X}$ and $\|Tf\|\leq\|f\|$. Because Tf is in C_K and

$$H_\lambda(Tf)\leq H_\lambda(f)$$

for any λ.

Furthermore for any u in $\mathscr{X}$, there exists a sequence (f_n) in $C_K\cap\mathscr{X}$ converging to u. By the results that Tf_n is in $\mathscr{X}$, $\|Tf_n\|\leq\|f_n\|$ and $Tf_n(x)$ converges to $Tu(x)$ *p.p.* in X, Tu is in $\mathscr{X}$ and $\|Tu\|\leq\|u\|$. Consequently $\mathscr{X}$

is a real Dirichlet space.

The implication (4) $\Rightarrow$ (1) is evident. This completes the proof.

By the above main theorem, we obtain the following another characterization of a real Dirichlet space.

THEOREM 3. *A regular functional space $\mathscr{X}$ is a real Dirichlet space if and only if there exists number $M\neq 0$ such that u_M is in $\mathscr{X}$ and $\|u_M\|\leq\|u\|$ for any u in $\mathscr{X}$, where*

$$u_M(x)=\inf(u(x), M)$$

if $M>0$,

$$u_M(x)=\sup(u(x), M)$$

if $M<0$.

Proof. Suppose that there exists a number $M\neq 0$ such that u_M is in $\mathscr{X}$ and $\|u_M\|\leq\|u\|$. It is sufficient to prove the thoerem for the case $M>0$. Put

$$u_1(x)=\inf(u(x). 1)$$

for any u in $\mathscr{X}$. Then

$$u_1(x)=M^{-1}\inf(Mu(x), M),$$

and hence u_1 is in $\mathscr{X}$ and $\|u_1\|\leq\|u\|$. On the other hand for a sequence (a_n) of negative numbers tending to 0.,

$$u_{a_n}(x)=\sup(u(x),\ a_n)=\frac{a_n}{M}\inf\Big(\frac{M}{a_n}u(x), M\Big).$$

Hence u_{a_n} is in $\mathscr{X}$ and $\|u_{a_n}\|\leq\|u\|$. We may assume that there exists an element u' such that $u_{a_n}\to u'$ weakly in $\mathscr{X}$. Since $u_{a_n}(x)$ converges to $u'(x)$ *p.p.* in X, u^+ is in $\mathscr{X}$ and

$$\|u\|\geq\varliminf_{n\to\infty}\|u_{a_n}\|\geq\|u^+\|$$

Let T be the unit contraction on R. Then $Tu=u_1{}^+$. Consequently T operates on $\mathscr{X}$. By Theorem 2, $\mathscr{X}$ is a real Dirichlet space.

The converse is evident. This completes the proof.

DEFINITION 8. We say that the positive contraction on R operates on a regular functional space $\mathscr{X}$ if for any u in $\mathscr{X}$, u^+ is in $\mathscr{X}$ and $\|u^+\|\leq\|u\|$.

---

9) A normal contraction T is a transformation of R into itself such that $|Ta_1-Ta_2|\leq|a_1-a_2|$ for any couple a_1 and a_2 in R and $T(0)=0$. Cf. [2], p. 209.

Remark. *There exists a regular functional space on which the positive contraction operates and which is not a real Dirichlet space. We can construct such an example when X is a finite space.* (*Cf.* [1].)

Similarly as Theorem 2, we obtain the following theorem. First we give a definition.

DEFINITION 9.[10] We say that a regular functional space satisfies the balayage principle if the following condition is satisfied: for any pure potential u_μ and any open set ω in X, there exists a pure potential $u_{\mu'}$ such that

(B. 1) $u_\mu(x) \geq u_{\mu'}(x)$ *p.p.* in X,

(B. 2) $u_\mu(x) = u_{\mu'}(x)$ *p.p.* in, ω,

(B. 3) $u_{\mu'} \varepsilon E_\omega$.

THEOREM 4. *A regular functional space $\mathscr{X}$ satisfies the balayage principle if and only if the positive contraction operates on $\mathscr{X}$.*

We can prove in the same way as the proof of Theorem 2.

4. Special Dirichlet spaces

Let X be a locally compact abelian group and ξ be the Haar measure on X which we denote by dx.

DEFINITION 10.[11] A functional space $\mathscr{X}$ with respact to X and ξ is called an invariant functional space if for any x in X and any u in $\mathscr{X}$,

$$U_x u \in \mathscr{X} \text{ and } \|U_x u\| = \|u\|,$$

where $U_x u$ is a function obtained from u by the translation x (*i.e.*, $U_x u(y) = u(y-x)$).

DEFINITION 11.[12] An invariant functional space $\mathscr{X}$ is called a special Dirichlet space if $\mathscr{X}$ is a real Dirichlet space.

LEMMA 13. *For any u in an invariant functional space $\mathscr{X}$ and any bounded measurable function f with compact support, $u*f$ is in $\mathscr{X}$ and*

$$(u*f, v) = \int (U_{-x}u, v) f dx$$

for any v in $\mathscr{X}$.

10) Cf. [2], p. 210.

11) After Deny's terminology, this is the functional space which is invariant by the translation.

12) Cf. [2], p. 215.

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

For the proof, see [3] and [4].

Using Theorem 2, we obtain the following theorem.

THEOREM 5. *An invariant functional space $\mathscr{X}$ is a special Dirichlet space if and only if $\mathscr{X}$ satisfies the condensor principle.*[13]

Proof. It is well-known that a special Dirichlet space satisfies the condensor principle. It is sufficient to prove the "if" part. By Lemma 13 and the condensor principle, $C_K \cap \mathscr{X}$ is total in C_K.[14] We shall show that $C_K \cap \mathscr{X}$ is dense in $\mathscr{X}$. Put

$$\mathscr{X}' = \overline{C_K \cap \mathscr{X}},$$

Then by Theorem 2, $\mathscr{X}'$ is a special Dirichlet space on X. First we shall prove that for each u in $\mathscr{X}$ with compact support, u is in $\mathscr{X}'$. We take a net $(f_\alpha)_{\alpha \in I}$ of C_K such that

$$f_\alpha(x) \geq 0, \int f_\alpha(x)\,dx = 1$$

and $(f_\alpha)_{\alpha \in I}$ converges vaguely to the unit measure ε at 0 and (S_{f_α}) converges to $\{0\}$. Since the mapping: $x \to U_x u$ is strongly continuous for any u in $\mathscr{X}$, there exists α_0 in I such that

$$\|U_x u - u\| < \delta$$

for any $x \in -S_{f_\alpha}$, $\alpha \geq \alpha_0$, for a given positive number δ. Therefore

$$\|u * f_\alpha - u\|^2 = \|u * f_\alpha\|^2 - 2(u * f_\alpha, u) + \|u\|^2 < 4\|u\|\delta + \delta^2.$$

$u * f_\alpha$ is in $C_K \cap \mathscr{X}$, and hence u is in $\mathscr{X}'$. Let $(F_\alpha)_{\alpha \in J}$ be a net of compact sets such that $F_\alpha \to X$. Put

$$E_{\mathscr{C} F_\alpha} = \overline{\left\{ u_f \in \mathscr{X};\ f \text{ is a bounded measurable function with compact support} \atop S_f \subset \mathscr{C} F_\alpha \right\}}.$$

Then $E_{\mathscr{C} F_\alpha}$ is a closed subspace of $\mathscr{X}$. For any u in $\mathscr{X}$, let u_α be the projection of u to $E_{\mathscr{C} F_\alpha}$. Then $u(x) = u_\alpha(x)$ *p.p.* in $\mathscr{C} F_\alpha$. Hence by the above result, $u - u_\alpha$ is in $\mathscr{X}'$. On the other hand obviously (u_α) converges strongly

---

[13] Let ω be an open set in X and the notation E_ω be the same as in Lemma 2. Without the condition of regularity, we can only consider potentials generated by bounded measurable functions with compact support. Then $E_\omega = \overline{\{u_f \in \mathscr{X};\ S_f \subset \omega\}}$.

[14] Cf. [6].

https://doi.org/10.1017/S0027763000012319 Published online by Cambridge University Press

to 0 in $\mathscr{X}$, hence $(u-u_\alpha)$ converges strongly to u. That is, u is in $\mathscr{X}'$. Consequently $\mathscr{X}$ is a special Dirichlet space.

References

[1] A. Beurling & J. Deny: Espaces de Dirichlet, Le case élémentaire, Acta Math., **99** (1958), 103–124.
[2] ——————: Dirichlet spaces, Proc. Nat. Acad. Sc. U.S.A., **45** (1959), 208–215.
[3] J. Deny: Sur les espaces de Dirichlet, Sémin. théorie du potentiel, Paris, 1957.
[4] ————: Principe complet de mixamum et Contractions, Ann. Inst. Fourier, **15** (1965), 259–272.
[5] ————: Les potentiel d'énergie finie. Acta Math., **82** (1950) 107–182.
[6] M. Itô: Characterizations of supports of balayaged measures, Nagoya Math. J., 28 (1966), 203–230.
[7] G. Lion: Principle complet du maximum et semi-groupes sous-markoviens, Sémin. théorie du potentiel, Paris, 1964.

Mathematical Institute Nagoya University