Bull. Aust. Math. Soc. **85** (2012), 79–83
doi:10.1017/S0004972711002747

SUMSETS AND DIFFERENCE SETS CONTAINING A COMMON TERM OF A SEQUENCE

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(Received 28 April 2011)

Abstract

Let $\beta > 1$ be a real number, and let $\{a_k\}$ be an unbounded sequence of positive integers such that $a_{k+1}/a_k \le \beta$ for all $k \ge 1$. The following result is proved: if n is an integer with $n > (1 + 1/(2\beta))a_1$ and A is a subset of $\{0, 1, \ldots, n\}$ with $|A| \ge (1 - 1/(2\beta + 1))n + \frac{1}{2}$, then $(A + A) \cap (A - A)$ contains a term of $\{a_k\}$. The lower bound for $|A|$ is optimal. Beyond these, we also prove that if $n \ge 3$ is an integer and A is a subset of $\{0, 1, \ldots, n\}$ with $|A| > \frac{4}{5}n$, then $(A + A) \cap (A - A)$ contains a power of 2. Furthermore, $\frac{4}{5}$ cannot be improved.

2010 *Mathematics subject classification*: primary 11B13; secondary 11B83, 05D05.
Keywords and phrases: sumsets, difference sets, sequences.

1. Introduction

Erdős and Freiman [3] proved a conjecture of Erdős and Freud: if $A \subseteq [1, n]$ with $|A| > n/3$, then some power of 2 is the sum of elements of A. Later, Nathanson and Sárközy [7] proved that 3504 summands are enough. Freiman [4] reduced 3504 to 6. Finally, Lev [6] reduced 6 to 4. Here 4 is best possible (see Alon [2]). The key to Lev's proof is the following lemma: let $A \subseteq [0, n]$ with $|A| \ge \frac{1}{2}n + 1$. Then either A contains a power of 2, or there exist two distinct elements of A whose sum is a power of 2. Abe [1] and Pan [8] extended this result to the powers of an integer m.

For a set A of integers, let $A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$ and $A - A = \{a_1 - a_2 : a_1, a_2 \in A\}$. Recently, Kapoor [5] extended the above results to general sequences. He proved that for an unbounded sequence $\{a_k\}$ of positive integers with $a_{k+1}/a_k \to \alpha$ as $k \to \infty$, and $\beta > \max(\alpha, 2)$, if $A \subset [0, x]$ is a set of integers with $0 \in A$ and

$$|A| \ge \left(1 - \frac{1}{\beta}\right)x,$$

then $A + A$ contains a term of the sequence $\{a_k\}$.

This work was supported by the National Natural Science Foundation of China, Grant No. 11071121.

In this paper, the following result is proved.

Theorem 1.1. *Let $\beta > 1$ be a real number, and let $\{a_k\}$ be an unbounded sequence of positive integers such that $a_{k+1}/a_k \le \beta$ for all $k \ge 1$. Suppose that n is an integer with $n > (1 + 1/(2\beta))a_1$ and A is a subset of $\{0, 1, \dots, n\}$.*

(i) *If all a_k are even, and*

$$|A| > \Big(1 - \frac{1}{2\beta + 1}\Big)n,$$

then $(A + A) \cap (A - A)$ contains a term of $\{a_k\}$.

(ii) *If a_k are not all even, and*

$$|A| \ge \Big(1 - \frac{1}{2\beta + 1}\Big)n + \frac{1}{2},$$

then $(A + A) \cap (A - A)$ contains a term of $\{a_k\}$.

Furthermore (i) and (ii) are sharp.

From Theorem 1.1, by taking $a_k = 2^k$ and $\beta = 2$, we have the following corollary.

Corollary 1.2. *Let $n \ge 3$ be an integer and A be a subset of $\{0, 1, \dots, n\}$ such that $|A| > \frac{4}{5}n$. Then $(A + A) \cap (A - A)$ contains a power of 2. Furthermore, $\frac{4}{5}$ cannot be improved.*

2. Proof of the theorem

Let k be the least integer such that

$$n < \Big(1 + \frac{1}{2\beta}\Big)a_{k+1}.$$

Then $k \ge 1$ and

$$n \ge \Big(1 + \frac{1}{2\beta}\Big)a_k.$$

We will show that $a_k \in (A + A) \cap (A - A)$.

First we prove that $a_k \in A - A$. Suppose that $a_k \notin A - A$.

Case 2.1. $(1 + (1/2\beta))a_k \le n < 2a_k$.

Since $a_k = (a_k + i) - i$ for $i = 0, 1, \dots, n - a_k$, we have $|A \cap \{a_k + i, i\}| \le 1$ for each $i \in \{0, 1, \dots, n - a_k\}$. Hence

$$|A| \le n + 1 - (n - a_k + 1) = a_k \le \Big(1 - \frac{1}{2\beta + 1}\Big)n,$$

a contradiction.

Case 2.2. $2a_k \le n < (1 + (1/2\beta))a_{k+1}$.

Since $a_k = (a_k + i) - i$ for $i = 0, 1, \ldots, a_k - 1$, we have $|A \cap \{a_k + i, i\}| \le 1$ for each $i \in \{0, 1, \ldots, a_k - 1\}$. Hence $|A| \le n + 1 - a_k$.

If $a_k = 1$ (this is a special case of (ii)), then

$$n < \Big(1 + \frac{1}{2\beta}\Big)a_{k+1} \le \Big(1 + \frac{1}{2\beta}\Big)\beta = \beta + \frac{1}{2}.$$

Thus

$$|A| \le n + 1 - a_k = n < \Big(1 - \frac{1}{2\beta + 1}\Big)n + \frac{1}{2},$$

a contradiction.

If $a_k \ge 2$, then by

$$n < \Big(1 + \frac{1}{2\beta}\Big)a_{k+1} \le (2\beta + 1)\frac{1}{2}a_k \le (2\beta + 1)(a_k - 1),$$

we have

$$a_k - 1 > \frac{1}{2\beta + 1}n.$$

Thus

$$|A| \le n + 1 - a_k < n - \frac{1}{2\beta + 1}n = \Big(1 - \frac{1}{2\beta + 1}\Big)n,$$

a contradiction.

Now we prove that $a_k \in A + A$. Suppose that $a_k \notin A + A$.

Case 2.3. *a_k is even.*

Since $a_k = (\frac{1}{2}a_k - i) + (\frac{1}{2}a_k + i)$ for $i = 0, 1, \ldots, \frac{1}{2}a_k$, we have $\frac{1}{2}a_k \notin A$ and so $|A \cap \{\frac{1}{2}a_k - i, \frac{1}{2}a_k + i\}| \le 1$ for $i = 1, 2, \ldots, \frac{1}{2}a_k$. Hence $|A| \le n + 1 - \frac{1}{2}a_k - 1 = n - \frac{1}{2}a_k$. By

$$n < \Big(1 + \frac{1}{2\beta}\Big)a_{k+1} \le (2\beta + 1)\frac{1}{2}a_k,$$

we have

$$|A| \le n - \frac{1}{2}a_k < \Big(1 - \frac{1}{2\beta + 1}\Big)n,$$

a contradiction.

Case 2.4. *a_k is odd.*

Since $a_k = (\frac{1}{2}(a_k - 1) - i) + (\frac{1}{2}(a_k + 1) + i)$ for $i = 0, 1, \ldots, \frac{1}{2}(a_k - 1)$, we have $|A \cap \{\frac{1}{2}(a_k - 1) - i, \frac{1}{2}(a_k + 1) + i\}| \le 1$ for $i = 0, 1, \ldots, \frac{1}{2}(a_k - 1)$. Hence $|A| \le n + 1 - \frac{1}{2}(a_k + 1) = n - \frac{1}{2}a_k + \frac{1}{2}$. By

$$n < \Big(1 + \frac{1}{2\beta}\Big)a_{k+1} \le (2\beta + 1)\frac{1}{2}a_k,$$

we have

$$|A| \le n - \frac{1}{2}a_k + \frac{1}{2} < \Big(1 - \frac{1}{2\beta + 1}\Big)n + \frac{1}{2},$$

a contradiction. We have proved (i) and (ii).

Next we show that Theorem 1.1 is sharp by taking $a_k = m^k$ for a fixed integer $m \ge 2$. In this case $\beta = m$. It is clear that

$$|A| \ge \Big(1 - \frac{1}{2m + 1}\Big)n + \frac{1}{2}$$

is equivalent to

$$|A| > \Big(1 - \frac{1}{2m + 1}\Big)n + \frac{1}{2} - \frac{1}{4m + 2}. \tag{2.1}$$

Let $\mathbb{N}$ denote the set of nonnegative integers. For odd $m \ge 3$, let $n = (2m + 1)m^k/2 - 1/2$ and $A = [m^k/2 + 1/2, (2m + 1)m^k/2 - 1/2] \cap \mathbb{N}$, where k is a nonnegative integer. Then

$$|A| = m^{k+1} = \Big(1 - \frac{1}{2m + 1}\Big)\Big(\frac{(2m + 1)m^k}{2} - \frac{1}{2}\Big) + \frac{1}{2} - \frac{1}{4m + 2},$$

so that (2.1) does not hold. Since $(A - A) \cap \mathbb{N} \subseteq [0, m^{k+1} - 1]$ and $A + A \subseteq [m^k + 1, (2m + 1)m^k - 1]$, we have $(A + A) \cap (A - A) \subseteq [m^k + 1, m^{k+1} - 1]$, which contains no power of m.

For even $m \ge 2$, let $n = (2m + 1)m^k/2$ and $A = [m^k/2 + 1, (2m + 1)m^k/2] \cap \mathbb{N}$, where k is a nonnegative integer. Then

$$|A| = m^{k+1} = \Big(1 - \frac{1}{2m + 1}\Big)\frac{(2m + 1)m^k}{2},$$

so the bound in case (i) does not hold. It follows that $(A - A) \cap \mathbb{N} \subseteq [0, m^{k+1} - 1]$ and $A + A \subseteq [m^k + 2, (2m + 1)m^k]$. Hence $(A + A) \cap (A - A) \subseteq [m^k + 2, m^{k+1} - 1]$, which also contains no power of m. This completes the proof of Theorem 1.1.

Acknowledgement

We would like to thank the referee for his/her comments.

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