Canad. Math. Bull. Vol. **21** (3), 1978

APPLICATIONS OF VARIANTS OF THE HÖLDER INEQUALITY AND ITS INVERSES: EXTENSIONS OF BARNES, MARSHALL-OLKIN, AND NEHARI INEQUALITIES.

BY
CHUNG-LIE WANG*

I. **Introduction.** The primary aim of this paper is to extend Barnes [1], Marshall-Olkin [6], and Nehari [8] inequalities as applications of some results introduced in [10] by the author.

Since several results from various sources are adopted here, a unified notation is required in order to simplify our subsequent arguments. To this end, let $L_p = L_p(S, \sum, \mu)$, $p > 0$ (unless otherwise stated), be the space of all pth power non-negative integrable functions over a given finite measure space $(S, \sum, \mu)$ (where S may be regarded as a bounded subset of real numbers). For $f \in L_p$, we write

$$\|f\|_p = \left\{\int_S f^p \, d\mu\right\}^{1/p}.$$

If $S = \{1, \ldots, n\}$ for some positive integer $n \geq 1$, μ is chosen to be the counting measure on S (see [2]). In this case, every f of L_p is a finite sequence (or a vector) $\{a_j\}$ of non-negative real numbers a_j, $j = 1, \ldots, n$. However, we still use

$$\|f\|_p = \left\{\sum a_j^p\right\}^{1/p}.$$

Here and in what follows $\sum$ and $\prod$ are used to indicate $\sum_{j=1}^n$ and $\prod_{j=1}^n$ respectively, whenever confusion is unlikely to occur.

Two theorems from [10] are cited here for later use.

THEOREM 1.1. *If $f_1 \in L_p$ and $f_2 \in L_q$ with*

$$(1.1) \qquad (1/p)+(1/q) = 1/r, \qquad p, q, r > 0$$

then $f_1 f_2 \in L_r$, and

$$(1.2) \qquad \|f_1 f_2\|_r \leq \|f_1\|_p \, \|f_2\|_q.$$

THEOREM 1.2. *Under the assumptions of Theorem* 1.1,

$$(1.3) \qquad \|f_1\|_p \, \|f_2\|_q \leq C_{pq}^r \, \|f_1 f_2\|_r$$

Received by the editors April 4, 1977 and, in revised form, August 11, 1977.
* The author was supported (in part) by the N.R.C. of Canada (Grant No. A3116).

where C^r_{pq} is a positive constant depending on functions f_1 and f_2 (see below or [9, 10] *for details).*

II. **Barnes inequalities.** Let $f \in L_p[0, a]$, $g \in L_q[0, a]$. If $0 < \|f\|_p$, $\|g\|_q < \infty$, then the generalized Hölder inequality (1.2) may be written

$$H^r_{pq}(f, g) = \|fg\|_r / \|f\|_p \, \|g\|_q \le 1, \tag{2.1}$$

where p, q, r satisfy (1.1). The quantity $H^r_{pq}(f, g)$ defined by (2.1) is called the generalized Hölder quotient, while $H_{pq}(f, g) = H^1_{pq}(f, g)$ is called the Hölder quotient (see [1]). An inverse of the generalized Hölder inequality gives a lower bound on the generalized Hölder quotient for $p, q \ge r$ and an upper bound for $p, q \le r$. From (1.3), it follows that $H^r_{pq}(f, g) \ge (C^r_{pq})^{-1}$.

Nehari [8] has established some very general results, which will be discussed and extended later, by using the convex hull of a class of functions; while Barnes [1] has established results only in the case of two functions with the restriction of (1.1) (for $r = 1$) removed. In fact, their results are not quite comparable (see [1], p. 82). For establishing the remaining results in this section we also remove the restriction of (1.1).

As in [1], we use a certain partial order "$<$", defined on a class of functions, and say $f_0 < f$ or equivalently $f > f_0$ (see [1, 3]) if

$$\int_0^x f_0(t)\, dt \le \int_0^x f(t)\, dt, \qquad 0 \le x \le a$$

and

$$\int_0^a f_0(t)\, dt = \int_0^a f(t)\, dt.$$

We denote by f^- the rearrangement of f into decreasing order and define $f^+(x) = f^-(a - x)$ the rearrangement of f into increasing order (see [1, 5] for details of the properties of f^+, f^-). Now, noting

$$[H^r_{pq}(f, g)]^r = \|f^r g^r\|_1 / \|f^r\|_{p/r} \, \|g^r\|_{g/r'}$$

for any $p, q, r > 0$ we are ready to extend Theorems 1 and 2 given on pp. 83–84 of [1] as follows.

THEOREM 2.1. *Let f, g be non-negative functions on $[0, a]$ with $0 < \|f\|_p$, $\|g\|_q < \infty$. Then*

(i) *If f_0 is increasing with $f_0 < f^+$ and g_0 is decreasing with $g_0 > g^-$ then $H^r_{pq}(f, g) \ge H^r_{pq}(f_0, g_0)$ for $p, q \ge r$;*

(ii) *If f_0 and g_0 are both increasing functions with $f_0 < f^+$ and $g_0 < g^+$ then $H^r_{pq}(f, g) \le H^r_{pq}(f_0, g_0)$ for $p, q \le r$.*

THEOREM 2.2. *Let f, g be non-negative concave functions on $[0, a]$ with $0 < \|f\|_p$, $\|g\|_q < \infty$. Then*

(i) $H^r_{pq}(f, g) \ge \Gamma^2(r+1) F(a) / \Gamma(2r+2)$ for $p, q \ge r$.

Equality holds in case one of the functions f, g is αx and the other one is $\beta(a-x)$ with α, β positive constants.

(ii) $H^r_{pq}(f, g) \le F(a)/(2r+1)^{1/r}$ for $-r<p,\ q<r$.

Equality holds in case $f=\alpha x$, $g=\beta x$ or $f=\alpha(a-x)$, $g=\beta(a-x)$. Here and in what follows Γ is the usual gamma function and

$$F(a)=(1+p)^{1/p}(1+q)^{1/q}a^{(1/r)-(1/p)-(1/q)}$$

Note that if $\|f\|_p=\|g\|_q=a=1$, then

$$\|fg\|_r \ge \Gamma^2(r+1)(1+p)^{1/p}(1+q)^{1/q}/\Gamma(2r+1)$$

for $0<r\le 1$, which is an extension of a Bellman inequality (e.g. see [4, 9] for a detailed account).

Proofs of Theorems 2.1 and 2.2 are similar to those of [1] and thus omitted.

III. **Marshall-Olkin inequalities.** Theorem 1.1 is rewritten explicitly in the form of finite sums.

THEOREM 3.1. *If $a_j, b_j \ge 0$ for $j=1,\ldots,n$ with (1.1), then*

$$\left(\sum a_j^r b_j^r\right)^{1/r} \le \left(\sum a_j^p\right)^{1/p}\left(\sum b_j^q\right)^{1/q}. \tag{3.1}$$

From (3.1), for $p=r(u-w)/(u-v)$, $q=r(u-w)/(v-w)(u>v>w>0)$, $a_j^p=p_jx_j^w$, $b_j^q=p_jx_j^u(p_j, x_j\ge 0, j=1,\ldots,n)$, follows

$$\left(\sum p_jx_j^v\right)^{u-w} \le \left(\sum p_jx_j^w\right)^{u-v}\left(\sum p_jx_j^u\right)^{v-w}. \tag{3.2}$$

This is called Lyapunov inequality (see [5, 7]). For our purpose, $\sum p_j=1$ is assumed. Since the p_j are probabilities, we may consider a random variable X with the distribution $P\{X=x_j\}=p_j$, $j=1,\ldots,n$ (see [6] for more details).

In order to extend Theorem 3.1 given on page 506 of [6], two lemmas given on pp. 504–505 of [6] are listed here as Lemmas 3.1 and 3.2 with a slight change of symbols to suit our notation.

LEMMA 3.1. *If X is a random variable satisfying $P\{m\le X\le M\}=1$, with $m>0$, and Z is a positive random variable, then*

$$s[EZX^s-aEZX^t-bEZ]\ge 0, \quad \text{for} \quad s<t, \quad st\ne 0, \tag{3.3}$$

where

$$a=\frac{M^s-m^s}{M^t-m^{t'}} \qquad b=\frac{M^tm^s-M^sm^t}{M^t-m^t}$$

and EZ (etc.) is the expectation of Z. Equality holds if and only if $P\{X=m\}+P\{X=M\}=1$. (Note that $a>0$ if and only if $st>0$, *and $b>0$ if and only if $t>0$*).

https://doi.org/10.4153/CMB-1978-060-3 Published online by Cambridge University Press

LEMMA 3.2. *Under the assumptions of Lemma* 3.1 *with* $P\{Z>0\}=1$, *for* $s<t$

$$\frac{(EZX^t)^{1/t}}{(EZX^s)^{1/s}}\le k(EZ)^{(1/t)-(1/s)}, \tag{3.4}$$

where

$$k=\left[\frac{s(\delta^t-\delta^s)}{(t-s)(\delta^s-1)}\right]^{1/t}\left[\frac{t(\delta^t-\delta^s)}{(t-s)(\delta^t-1)}\right]^{-1/s}$$

and $\delta=M/m$. Equality holds if and only if $P\{X=m\}+P\{X=M\}=1$ and $EZX^t=sb[a(t-s)]^{-1}EZ$.

When $(X,Z)=(VU^{-1}, V^{-s}U^t)$ are chosen, we obtain from (3.3) and (3.4) that if $P\{m\le VU^{-1}\le M\}=1$, with $m>0$, then for $s<t$

$$s[EU^{t-s}-aEV^{t-s}-bEU^tV^{-s}]\ge 0, \tag{3.5}$$

$$(EU^{t-s})^{-1/s}(EV^{t-s})^{1/t}\le k(EU^tV^{-s})^{(1/t)-(1/s)}. \tag{3.6}$$

To obtain (1.3) (or an extension of Theorem 3.1 of [6]), let $f_1^p=U^{t-s}$, $f_2^q=V^{t-s}$, $f_1^r=U^t$, $f_2^r=V^{-s}$, $p=(t-s)r/t$, $q=(t-s)r/-s$, $t<0$, and $\theta=\delta^{(t-s)r}$. A direct substitution in (3.5) and (3.6) yields

THEOREM 3.1. *If* $f_1\in L_p$ *and* $f_2\in L_q$ *with* (1.1) *and* $\ell<f_1^{-p}f_2^q\le\ell\theta$, *then* $f_1f_2\in L_r$,

$$\ell^{1/q}\theta^{1/q}(\theta^{1/p}-1)\,\|f_1\|_p^p+\ell^{-1/p}(\theta^{1/q}-1)\,\|f_2\|_q^q\le(\theta^{1/r}-1)\,\|f_1f_2\|_r^r, \tag{3.7}$$

and

$$\|f_1f_2\|_r\ge C_{pq}^r\,\|f_1\|_p\,\|f_2\|_q, \tag{3.8}$$

where

$$C_{pq}^r=\frac{p^{1/p}q^{1/q}\theta^{1/pq}(\theta^{1/p}-1)^{1/p}(\theta^{1/q}-1)^{1/q}}{r^{1/r}(\theta^{1/r}-1)^{1/r}}. \tag{3.9}$$

REMARK 3.1. Inequalities (3.7) and (3.8) are reversed for $0<p\le r$. It is also evident that (3.8) can be obtained by applying the geometric-arithmetic inequality to (3.7) (e.g. see [10]).

IV. **Nehari inequalities.** In order to extend several results of [8], an elementary inequality is listed here as Lemma 4.1. Its proof is omitted, since it can be easily derived from a theorem given on page 76 of [5] and the fact that every negative function of a concave function is convex.

LEMMA 4.1. *For* $x_j, q_j\ge 0$, $j=1,\ldots,n$ *with* $\sum q_j=1$,

$$\sum q_jx_j^\alpha\le\left(\sum q_jx_j\right)^\alpha,\qquad 0\le\alpha\le 1. \tag{4.1}$$

We now modify two lemmas given on page 407 *of* [8].

https://doi.org/10.4153/CMB-1978-060-3 Published online by Cambridge University Press

LEMMA 4.2 *Let $\phi_j(t)(j=1,\ldots,n)$ be non-negative continuous convex functions for $t\geq 0$. Let H_j be a set of non-negative functions f_j such that $\phi_j(f_j)\in L_1$ and let $C(H_j)$ denote the convex hull of H_j. If the inequality*

$$(4.2)_n \qquad \sum A_j \int_S \phi_j(f_j)\,d\mu \leq C_n \int_S \prod f_j^r\,d\mu, \qquad 0\leq r\leq 1$$

(A_j, C_n positive constants) holds for all $f_j\in H_j(j=1,\ldots,n)$, then it also holds for $f_j\in C(H_j)$.

LEMMA 4.3. *Let $H_j(j=1,\ldots,n)$ be a subset of L_{p_j}, where $p_j\geq 1$, $0<r\leq 1$, $1/r=\sum 1/p_j$. If the inequality*

$$(4.3)_n \qquad \prod \|f_j\|_{p_j} \leq D_n \|\prod f_j\|_r$$

holds for functions $f_j\in H_j$, then it also holds for functions $f_j\in C(H_j)$.

To simplify the writing, we prove these two lemmas for $n=2$. The extensions of the arguments to general n are routine. Establishing Lemma 4.2, it is sufficient to show that $(4.2)_2$ holds for all convex combinations

$$(4.4)\quad \begin{aligned} F_1 &= \alpha_1^{(1)}f_1^{(1)}+\cdots+\alpha_{m_1}^{(1)}f_1^{(m_1)}(\alpha_{k_1}^{(1)}>0,\ \alpha_1^{(1)}+\cdots+\alpha_{m_1}^{(1)}=1,\ f_1^{(k_1)}\in H_1)\\ F_2 &= \alpha_1^{(2)}f_2^{(1)}+\cdots+\alpha_{m_2}^{(2)}f_2^{(m_2)}(\alpha_{k_2}^{(2)}>0,\ \alpha_1^{(2)}+\cdots+\alpha_{m_2}^{(2)}=1,\ f_2^{(k_2)}\in H_2)\end{aligned}$$

provided $(4.2)_2$ holds for $f_j=f_j^{(k_j)}$, $k_j=1,\ldots,m_j$, $j=1,2$.

By (4.4) and Jensen's inequality, we have

$$(4.5) \qquad \phi_j(F_j)\leq \sum_{k_j=1}^{m} \alpha_{k_j}^{(j)}\phi_j(f_j^{(k_j)}), \qquad j=1,2.$$

Hence, from $(4.2)_2$, (4.4), (4.5) and Lemma 4.1 with $\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2}\alpha_{k_1}^{(1)}\alpha_{k_2}^{(2)}=1$, follows

$$\begin{aligned} A_1\int_S \phi_1(F_1)\,d\mu + A_2\int_S \phi_2(F_2)\,d\mu &\leq \sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2}\alpha_{k_1}^{(1)}\alpha_{k_2}^{(2)}\left\{A_1\int_S \phi_1(f_1^{(k_1)})\,d\mu + A_2\int_S \phi_2(f_2^{(k_2)})\,d\mu\right\}\\ &\leq C_2\sum_{k_1}\sum_{k_2}\alpha_{k_1}^{(1)}\alpha_{k_2}^{(2)}\int_S (f_1^{(k_1)}f_2^{(k_2)})^r\,d\mu\\ &\leq C_2\int_S\left(\sum_{k_2}\sum_{k_2}\alpha_{k_1}^{(1)}\alpha_{k_2}^{(2)}f_1^{(k_1)}f_2^{(k_2)}\right)^r d\mu\\ &= C_2\int_S F_1^rF_2^r\,d\mu.\end{aligned}$$

This completes the proof of Lemma 4.2.

The proof of Lemma 4.3 is similar, except that Jensen's inequality and

Lemma 4.1 have been replaced by forward and backward Minkowski inequalities. By (4.4) and forward Minkowski inequality, we have, for $p_j \geq 1$,

$$\|F_j\|_{p_j} \leq \sum_{k_j=1}^{m_j} \alpha_{k_j}^{(j)} \|f_j^{(k_j)}\|_{p_j}, \qquad j=1,2 \tag{4.6}$$

Hence, from $(4.3)_2$, (4.4), (4.6) and backward Minkowski inequality, follows

$$\begin{aligned}\|F_1\|_{p_1} \|F_2\|_{p_2} &\leq \sum_{k_1} \sum_{k_2} \alpha_{k_1}^{(1)} \alpha_{k_2}^{(2)} \|f_1^{(k_1)}\|_{p_1} \|f_2^{(k_2)}\|_{p_2} \\ &\leq D_2 \sum_{k_1} \sum_{k_2} \alpha_{k_1}^{(1)} \alpha_{k_2}^{(2)} \|f_1^{(k_1)} f_2^{(k_2)}\|_r \\ &\leq D_2 \left\| \sum_{k_1} \sum_{k_2} \alpha_{k_1}^{(1)} \alpha_{k_2}^{(2)} f_1^{(k_1)} f_2^{(k_2)} \right\|_r \\ &= D_2 \|F_1 F_2\|_r, \qquad (0<r\leq 1)\end{aligned}$$

and Lemma 4.3 is established.

In view of Lemmas 4.2 and 4.3, we can state, without proofs, Theorems **4.1**, **4.2** and 4.3, which are extensions of Theorems 3.1, 5.1, 6.1 and 6.2 given on pp. 410–417 of [8]. The proofs are similar to those of [8].

THEOREM 4.1. *Let $\phi_j(t)(j=1,2)$ denote a function which vanishes for $t=0$ and is continuous non-decreasing and convex for $t \geq 0$. If*

$$0<m_j \leq f_j \leq M_j < \infty, \qquad j=1,2, \tag{4.7}$$

and $\phi_j(f_j) \in L_1$, then, for any two positive constants A_1, A_2 we have the inequality

$$A_1 \int_S \phi_1(f_1)\, d\mu + A_2 \int_S \phi_2(f_2)\, d\mu \leq C_2 \int_S f_1^r f_2^r\, d\mu, \qquad 0<r\leq 1,$$

where

$$C_2 = \max\{\delta(m_1, m_2), \delta(M_1, M_2), \delta(M_1, m_2), \delta(m_1, M_2)\}$$

and

$$\delta(x, y) = [A_1\phi_1(x) + A_2\phi_2(y)]/x^r y^r.$$

THEOREM 4.2. *Let $f_1 \in L_p$, $f_2 \in L_q$, where $1/r = (1/p) + (1/q)$, $p, q \geq 0$, $0<r\leq 1$ If* (4.7) *is assumed, and if $\eta_1, \eta_2 (0 \leq \eta_1, \eta_2 \leq 1)$ are defined by*

$$\int_S f_j\, d\mu = [m_j + \eta_j(M_j - m_j)]\mu(S), \qquad j=1,2,$$

then

$$\|f_1\|_p \|f_2\|_q \leq D_2 \|f_1 f_2\|_r$$

where

$$D_2 = \frac{\{m_1^p + (M_1^p - m_1^p)\eta_1\}^{1/p}\{m_2^q + (M_2^q - m_2^q)\eta_2\}^{1/q}}{\{m_1^r m_2^r + m_1^r(M_2^r - m_2^r)\eta_2 + m_2^r(M_1^r - m_1^r)\eta_1 + \gamma(M_1^r - m_1^r)(M_2^r - m_2^r)\}^{1/r}}$$

and

$$\gamma = \max\{0, \eta_1 + \eta_2 - 1\}.$$

THEOREM 4.3. *Let $f_1, \ldots, f_n$ be continuous non-negative concave functions in* $[0, 1]$, *and let*

$$\int_0^1 f_j\, dx = 1/2, \qquad j = 1, \ldots, 2.$$

If

$$1/r = \sum 1/p_j,\ p_j > 0, \qquad j = 1, \ldots, n, \qquad 0 < r \le 1,$$

then

$$\sum (1 + 1/p_j)\, \|f_j\|_{p_j}^{p_j} \le C_n \left\|\prod f_j\right\|_r^r \tag{4.8}$$

and

$$\prod \|f_j\|_{p_j} \le D_n \left\|\prod f_j\right\|_{r'} \tag{4.9}$$

where

$$C_n = \frac{\Gamma(nr+2)}{([nr/2]!)^2} = \frac{\prod (1+p_j)^{1/p_j}}{r^{1/r}} D_n.$$

There will be equalities in (4.8) *and* (4.9) *respectively if $f_j = x$ for* $[nr/2]$ *of the subscripts j, and $f_j = 1 - x$ in the other cases.*

REMARK 4.1. Theorem 4.3 is an extension of Theorem 6.1 and 6.2 given on pp. 414–417 of [8]. It is obvious also that (4.9) can be derived from (4.8) (see Remark 3.1 above).

V. **Concluding remarks.** 5.1 In view of the above results, we may conclude that most of remarks made in [1, 6, 8] can be carried over to the cases considered here. However, we refer them to the aforementioned papers for details.

5.2 Based upon all possible transliterations (say $p < 0$, $q < 0$, $r > 0$, etc) of the standard normalized Hölder inequality introduced in [10], corresponding variants of the inequalities of Barnes [1], Marshall-Olkin [6] and Nehari [8] could be established without any difficulty (perhaps with tedious notation). However, those will not be further exploited here.

5.3 The above investigation convincingly reveals that variants of the Hölder inequality provide means for wider applications than the standard one.

References

1. D. C. Barnes, *Some complements of Hölder's inequality*. J. Math. Anal. Appl. **26** (1969), 82–87.
2. R. G. Bartle, *The elements of integration*. John Wiley and Sons, New York, 1966.
3. E. F. Beckenbach and R. Bellman, *Inequalities*. Springer-Verlag, Berlin, 1961.
4. R. Bellman, *Converses of Schwarz's inequality*. Duke Math. J. **23** (1956), 429–434.
5. G. A. Hardy, J. E. Littlewood and G. Polya, *Inequalities*. 2nd ed. Cambridge Univ. Press, Cambridge, 1959.
6. A. W. Marshall and I. Olkin, *Reversal of the Lyapunov, Hölder and Minkowski inequalities and other extensions of the Kantorovich inequality*. J. Math. Anal. Appl. **8** (1964), 503–514.
7. D. S. Mitrinovic, *Analytic inequalities*. Springer-Verlag, Berlin, 1970.
8. Z. Nehari, *Inverse Hölder inequalities*. J. Math. Anal. Appl. **21** (1968), 405–420.
9. Chung-lie Wang, *An extension of a Bellman inequality*, Utilitas Mathematica, **8** (1975), 251–256.
10. Chung-lie Wang. *Variants of the Hölder inequality and its inverses*, Can. Math. Bull., **20** (1977), 377–384.

Dept. of Mathematics
University of Regina
Regina, Saskatchewan S4S 0A2

https://doi.org/10.4153/CMB-1978-060-3 Published online by Cambridge University Press