MODULAR REPRESENTATIONS OF ABELIAN GROUPS
WITH REGULAR RINGS OF INVARIANTS

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§ 1. Introduction

Let $k$ be a field of characteristic $p$ and $G$ a finite subgroup of $GL(V)$ where $V$ is a finite dimensional vector space over $k$. Then $G$ acts naturally on the symmetric algebra $k[V]$ of $V$. We denote by $k[V]^\sigma$ the subring of $k[V]$ consisting of all invariant polynomials under this action of $G$. The following theorem is well known.

**Theorem 1.1** (Chevalley-Serre, cf. [1, 2, 3]). Assume that $p = 0$ or $(|G|, p) = 1$. Then $k[V]^\sigma$ is a polynomial ring if and only if $G$ is generated by pseudo-reflections in $GL(V)$.

Now we suppose that $|G|$ is divisible by the characteristic $p(> 0)$. Serre gave a necessary condition for $k[V]^\sigma$ to be a polynomial ring as follows.

**Theorem 1.2** (Serre, cf. [1, 3]). If $k[V]^\sigma$ is a polynomial ring, then $G$ is generated by pseudo-reflections in $GL(V)$.

But the ring $k[V]^\sigma$ of invariants is not always a polynomial ring, when $G$ is generated by pseudo-reflections in $GL(V)$ (cf. [1, 3]).

In this paper we shall completely determine abelian groups $G$ such that $F_p[V]^\sigma$ are polynomial rings ($F_p$ is the field of $p$ elements). Our main result is

**Theorem 1.3.** Let $V$ be a vector space over $F_p$ and $G$ an abelian group generated by pseudo-reflections in $GL(V)$. Let $G_p$ denote the $p$-part of $G$ and assume that $G_p \not\cong \{1\}$. Then the following statements on $G$ are equivalent:

1) $F_p[V]^\sigma$ is a polynomial ring.

2) The natural $F_pG_p$-module $V$ defines a couple $(V, G_p)$ which decomposes to one dimensional subcouples (for definitions, see § 2).

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The computation of invariants of elementary abelian $p$-groups $G$ plays an essential role in the proof of this theorem. Therefore we need to study the structure of $F_pG$-modules $V$ such that $F_p[V]^g$ are polynomial rings under some additional hypothesis (see § 3). In § 4 our main result shall be reduced to (3.2).

Hereafter $k$ stands for the prime field of characteristic $p > 0$ and without specifying we assume that all vector spaces are defined over $k$.

§ 2. Preliminaries

An element $\sigma$ of $GL(V)$ is said to be a pseudo-reflection if $\dim(1-\sigma)V \leq 1$. We say that a graded ring $R = \oplus_{n \geq 0} R_n$ is defined over a field $K$, when $R_n = K$ and $R$ is a finitely generated $K$-algebra. It is well known that $R$ is a polynomial ring over $K$ if $R$ is regular at the homogeneous maximal ideal $\oplus_{n > 0} R_n$. For a subset $A$ of a ring $R$, $\langle A \rangle_R$ denotes the ideal of $R$ generated by $A$. To simplify our notation we put $\langle A \rangle = \langle A \rangle_{k[V]}$ if $A$ is a subset of the fixed $k$-space $V$ (for a subset $B$ of a group, $\langle B \rangle$ means the subgroup generated by $B$).

**Proposition 2.1.** Let $G$ be an abelian group generated by pseudo-reflections in $GL(V)$ and let $G_p$ denote the $p$-part of $G$. Then $k[V]^G$ is a polynomial ring if and only if $k[V]^G_p$ is a polynomial ring.

**Proof.** Let $k$ be the algebraic closure of $k$ and let $G_p'$ be the $p'$-part of $G$. Since $G$ is an abelian group generated by pseudo-reflections in $GL(k \otimes_k V)$, we can immediately find a $kG_p'$-submodule $V_p$ and a $kG_p'$-submodule $V_p'$ such that $V_p \subseteq (k \otimes_k V)^{G_p}$, $V_p' \subseteq (k \otimes_k V)^{G_p}$ and $k \otimes_k V = V_p \oplus V_p'$. Therefore

$$k \otimes_k k[V]^G \cong k[V]^G \cong k[V_p]^G_p \otimes_k k[V_p'^{G_p}].$$

and $k[V_p']^{G_p}$ is a polynomial ring. The assertion follows from these facts, because $k[V]^G$ and $k[V_p]^G_p$ are graded algebras defined over fields.

**Proposition 2.2.** If $G$ is an abelian $p$-group generated by pseudo-reflections in $GL(V)$, then $V/V^G$ is a trivial $kG$-module (i.e. $G$ acts trivially on $V/V^G$).

**Proof.** Let $\sigma \in G - \{1\}$ be a pseudo-reflection and choose $Z \subseteq V$ to satisfy $(1-\sigma)V = kZ$. Clearly it suffices to prove that $Z \subseteq V^G$. Since $G$
is abelian, \( \tau(kZ) = (1 - \sigma)\tau(V) = kZ \) for any element \( \tau \) of \( G \). Hence the map \( \chi: G \to \ker \tau \) defined by

\[
\tau \mapsto \frac{\tau^{-1}(Z)}{Z}
\]

is a group homomorphism, where \( \ker \tau \) is the unit group of \( k \). But we have \( \Hom(G, \ker \tau) = \{1\} \), as \( G \) is a \( p \)-group. This implies that \( Z \in V^\sigma \).

\((V, G)\), which is called a couple, stands for a pair of a group \( G \) and a \( G \)-faithful \( kG \)-module \( V \) such that \( V/V^\sigma \) is a nonzero trivial \( kG \)-module (in this case \( G \) is an elementary abelian \( p \)-group). The dimension of \( (V, G) \) is defined to be \( \dim V/V^\sigma \). We say \( (U, H) \) is a subcouple of \( (V, G) \) if \( H \) is a subgroup of \( G \) and \( U \) is a \( kH \)-submodule of \( V \).

Let us associate \( (V, G) \) with the subspace

\[
\mathcal{A}(V, G) = \sum_{\sigma \in G} (1 - \sigma)V
\]

of \( V^\sigma \) and the subring \( \mathcal{A}(V, G) \) which is the image of the canonical ring homomorphism

\[
k[V]/\langle V^\sigma \rangle \to k[V/V^\sigma] .
\]

**Lemma 2.3.** For any couple \((V, G)\) the \( k \)-algebra \( \mathcal{A}(V, G) \) is a polynomial ring.

**Proof.** Putting

\[
R = \overline{k}[k \otimes_k V]/[(\overline{k} \otimes_k V^\sigma)_{k(k \otimes_k V)}^G ,
\]

we see that

\[
R \cong \overline{k} \otimes_k \mathcal{A}(V, G)
\]

as graded algebras defined over \( \overline{k} \). Let \( \mathfrak{M}_i \) \((i = 1, 2)\) be maximal ideals of \( \overline{k}[\overline{k} \otimes_k V] \) which contain the ideal \( \langle \overline{k} \otimes_k V^\sigma \rangle_{k(k \otimes_k V)} \). Then, by the definition of a couple, we can select a coordinate transform

\[
\rho: \overline{k}[\overline{k} \otimes_k V] \to \overline{k}[\overline{k} \otimes_k V]
\]

sending \( \mathfrak{M}_i \) to \( \mathfrak{M}_2 \) which commutes with the action of \( G \). The contractions of \( \mathfrak{M}_i \) \((i = 1, 2)\) to \( \overline{k}[\overline{k} \otimes_k V]^\sigma \) define maximal ideals \( \mathfrak{M}_i \) of \( R \) respectively and the transform \( \varphi \) induces \( R_{\mathfrak{M}_i} \cong R_{\mathfrak{M}_2} \). Hence we conclude that \( R \) is regular, because it is an affine domain. From this \( \mathcal{A}(V, G) \) is a polynomial ring.
We say that \((V, G)\) decomposes to subcouples \((V, G_i)\) \((1 \leq i \leq m)\) if \(G = \bigoplus_{1 \leq i \leq m} G_i, V^g \subseteq V_i \subseteq V^{g_j}\) for all \(1 \leq i, j \leq m\) with \(i \neq j\) and
\[
\frac{V}{V^g} = \bigoplus_{1 \leq i \leq m} \frac{V_i}{V^g}.
\]
The set consisting of these subcouples is called a decomposition of \((V, G)\). Further \((V, G)\) is defined to be decomposable, when it has a decomposition \(\{(V, G_i) : 1 \leq i \leq m\}\) with \(m \geq 2\).

**Proposition 2.4.** Let \((V, G)\) be a couple which decomposes to subcouples \((V, G_i)\) \((1 \leq i \leq m)\). Then the following conditions are equivalent:

1. \(k[V]^g\) is a polynomial ring.
2. \(k[V_i]^g\) \((1 \leq i \leq m)\) are polynomial rings.

**Proof.** Suppose that \(k[V]^g\) is a polynomial ring. Since \(k[V]^g\) contains \(k[V_i]^g\), the canonical \(kG_i\)-epimorphism \(V \to V_i\) induces a graded epimorphism
\[
\psi_i : k[V]^g \longrightarrow k[V_i]^g.
\]
Clearly \(V^g = V_i^g\) and \(\psi_i(\langle V^g \rangle^g) = \langle V_i^g \rangle^g\). Hence \(\langle V^g \rangle^g = \langle V_i^g \rangle^g\) implies
\[
\langle \langle V_i^g \rangle^g \rangle^g = \langle \langle V_i^g \rangle^g \rangle^g.
\]
By (2.3) we see that \(\mathcal{P}(V, G_i)\) are polynomial rings and therefore \(k[V_i]^g\) \((1 \leq i \leq m)\) are also polynomial rings. Conversely we assume the condition (2). Denote by \(n_i\) the dimension of \((V, G_i)\) \((1 \leq i \leq m)\) and let \(f_{ij}\) \((1 \leq j \leq n_i)\) be homogeneous polynomials in \(k[V_i]\) such that \(k[V_i]^g = k[V_i^g][f_{ij}, \ldots, f_{in_i}]\) \((1 \leq i \leq m)\). Then it follows easily that \(k[V]^g = k[V^g][f_{ij} : 1 \leq i \leq m, 1 \leq j \leq n_i]\).

For a one dimensional couple \((V^g \oplus kX, G)\) we call
\[
F(X) = \prod_{x \in G} a(X)
\]
the canonical \((V^g \oplus kX, G)\)-invariant on \(X\). \(F(X)\) satisfies the identity
\[
F(Y_1 + Y_2) = F(Y_1) + F(Y_2).
\]
Clearly we must have \(k[V^g \oplus kX]^g = k[V^g][F(X)]\) and hence

**Corollary 2.5.** If a couple \((V, G)\) decomposes to one dimensional subcouples, then \(k[V]^g\) is a polynomial ring.

**Proposition 2.6.** Let \(G\) be a subgroup of \(GL(V)\) and let \(H\) be the
inertia group of a prime ideal $\mathfrak{p}$ of $k[V]$ under the natural action of $G$. If $k[V]^G$ is a polynomial ring, then $k[V]^G$ is also a polynomial ring.

This proposition is almost evident.

**Lemma 2.7.** Let $(V, G)$ be a couple with $\dim V^G = 1$ and suppose that 
\[ \{X_i: 0 \leq i \leq m\} \] 
is a $k$-basis of $V$ with $V^G = kX_0$. Further, for non-negative integers $t(i)$ $(1 \leq i \leq m)$, let $R$ be the graded polynomial subalgebra $k[X_0, X_i^{p(i)}, \ldots, X_m^{p(m)}]$ of $k[V]$. Then $R^G$ is a polynomial ring and we can effectively determine a regular system of homogeneous parameters of $R^G$.

**Proof.** We prove this by induction on $|G|$ and may assume that 
\[ t(1) = \cdots < \cdots < t(m_{i-1}) < t(m_{i-1} + 1) = t(m_{i-1} + 2) \cdots = t(m_i) < \cdots < \cdots = t(m_n) \]
where $m_n$ is equal to $m$. Let us put 
\[ U_t = \bigoplus_{0 \leq j \leq m_1} kX_j^{p(m_j)} \]
and 
\[ U'_t = U_t \bigoplus \bigoplus_{m_{i-1} < j \leq m_i} kX_j^{p(m_j)} \]
respectively and moreover define $G_t$ to be the stabilizer of $G$ at $U_t$. Then there is a subgroup $G_2$ such that $G = G_t \oplus G_2$. Because $U_t$ is a $G_2$-faithful $kG_2$-module with $(G_2 - 1)U_t = k[X_0^{p(m_1)}]$, we deduce that the natural short exact sequence 
\[ 0 \rightarrow U_t \rightarrow U'_t \rightarrow \bigoplus_{m_{i-1} < j \leq m_i} kX_j^{p(m_j)} \mod U_t \rightarrow 0 \]
of $kG$-modules is $G_2$-split. Therefore we may suppose that $X_j^{p(m_j)}$ $(2 \leq i \leq n; m_{i-1} < j \leq m_i)$ are invariants of $G_t$. On the other hand we can effectively determine homogeneous polynomials $f_i$ $(1 \leq i \leq m_i)$ which satisfy $k[U_t]^G = k[X_0^{p(m_1)}, f_1, \ldots, f_{m_1}]$. Hence it follows that $R^G = S^{G_2}[f_1, \ldots, f_{m_1}]$ where $S = k[X_0][X_i^{p(i)}]: 2 \leq i \leq n, m_{i-1} < j \leq m_i]$. Then the assertion is shown from the induction hypothesis.

When $W$ is a $kH$-submodule of $U$ for a subgroup $H$ of $GL(U)$, we denote by $H(W)$ the kernel of the canonical homomorphism $H \rightarrow GL(U/W)$.

**Proposition 2.8.** Let $(V, G)$ be a couple such that $k[V]^G$ is a polynomial ring. Then we can effectively determine a regular system of homogeneous parameters of $k[V, G]$. 

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Proof. Let 

\[ 0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_d = V^o \]

be an ascending chain of subspaces with \( \dim W_i/W_{i-1} = 1 \). Put \( R_0 = k[V] \) and define

\[ R_i = R^o_{i+1}/W_i R^o_{i-1} \quad (1 \leq i \leq d) \]

inductively where \( G_i \) denotes \( G(W_i) \). Then obviously the natural map

\[ \mathcal{D}(V, G) \longrightarrow R_d \]

is an isomorphism, because, by (2.6), \( k[V]^G \) are polynomial rings. Hence this proposition follows from (2.7).

**Lemma 2.9.** Let \((V, G)\) be a one dimensional couple and suppose that \(\{X, T_1, \cdots, T_d\}\) is a \(k\)-basis of \(V\) with \(V^o = \bigoplus_{i \leq d} kT_i\). Further let \(F(X)\) denote the canonical \((V, G)\)-invariant on \(X\). If \( \bigoplus_{i \leq d} kT_i \supseteq \mathcal{A}(V, G) \) and \( \bigoplus_{i = 2} kT_i \supseteq \mathcal{A}(V, G) \), then we have \(F(T_i) \in \langle T_2, T_3, \cdots, T_d \rangle\) and

\[ F(X) \equiv X^{p^a} - T_1^{p^a-p^a-1}X^{p^a-1} \pmod{\langle T_3, T_4, \cdots, T_d \rangle} \]

where \( p^a = |G| \).

**Proof.** Choose a \(k\)-basis \(\{Z_j: 1 \leq j \leq u\}\) of \(\mathcal{A}(V, G)\) such that \(Z_i \equiv T_i \mod \bigoplus_{i \neq j} kT_i\) and \( \bigoplus_{i \leq d} kT_i \supseteq \{Z_1, Z_2, \cdots, Z_u\} \). Putting \(F_i(X) = X^{p^a} - Z_i^{-1}X\), we inductively define

\[ F_{i+1}(X) = F_i(X)^p - F_i(Z_{i-1})^{p^{-1}}F_i(X) \quad (i < u) . \]

Then there exist elements \(\sigma_i\) \((1 \leq i \leq u)\) in \(G\) which satisfy \((\sigma_i - 1)X = Z_i\) and therefore we must have \(F(X) = F_u(X)\). From this we deduce that

\[ F(T_i) = F_{u-1}(T_i)^p - F_{u-1}(Z_i)^{p^{-1}}F_{u-1}(T_i) \equiv 0 \pmod{\langle T_2, T_3, \cdots, T_d \rangle} \]

and

\[ F(X) = F_{u-1}(X)^p - F_{u-1}(Z_i)^{p^{-1}}F_{u-1}(X) \equiv X^{p^a} - T_1^{p^a-p^a-1}X^{p^a-1} \pmod{\langle T_3, T_4, \cdots, T_d \rangle} , \]

since \(Z_i \equiv T_i \mod \bigoplus_{i \leq d} kT_i\) and \( F_{u-1}(X) \equiv X^{p^a-1} \pmod{\langle T_3, T_4, \cdots, T_d \rangle} \).

Let \(\mathcal{D} = \{(V^o \oplus W_i, G_i): 1 \leq i \leq m\}\) be a decomposition of \((V, G)\) and put \(\text{supp}_\theta L = \{i_0: V^o \oplus \bigoplus_{i \neq i_0} W_i \not\supsetneq L\}\) for a subset \(L\) of \(V\). Let us consider an element \(\theta\) of \(GL(V)\) with the property that \(V^{(\theta)} \supseteq V^o\). We say \(\theta\) is
$\mathcal{D}$-admissible if $G$ contains some subgroups $G_i$ ($1 \leq i \leq m$) which give another decomposition $\mathcal{D}' = \{(V^\alpha \oplus \theta(W_i), G_i): 1 \leq i \leq m\}$ of $(V, G)$. In the case of $\dim W_i = 1$ the transform $\theta$ is characterized by

**Proposition 2.10.** If $W_i = kX_i$ ($1 \leq i \leq m$) then the following conditions are equivalent:

1. $\theta$ is $\mathcal{D}$-admissible.
2. There is a permutation $\pi$ on $\{1, 2, \ldots, m\}$ such that $|G_i| = |G_{\pi(i)}|$, $\mathcal{A}(V^\alpha \oplus W_{\pi(i)}, G_{\pi(i)}) \supseteq \mathcal{A}(V^\alpha \oplus W_j, G_j)$ ($j \in \text{supp}_\theta \theta(W_i)$) and $\pi(i) \in \text{supp}_\theta \theta(W_i)$ for $1 \leq i \leq m$.

**Proof.** Suppose that the condition (2) is satisfied and let $G_{i_0}$ be

$$\{\tau \in \text{GL}(V): V^\tau \supseteq V^\alpha \oplus \theta(W_{i_0}) \text{ and } \mathcal{A}(V^\alpha \oplus W_{\pi(i_0)}, G_{\pi(i_0)}) \supseteq (1-\tau)V\}$$

for $1 \leq i_0 \leq m$. Furthermore set

$$J = \{i: \mathcal{A}(V^\alpha \oplus W_i, G_i) \supseteq \mathcal{A}(V^\alpha \oplus W_{\pi(i_0)}, G_{\pi(i_0)})\}$$

and

$$J' = \{i: \mathcal{A}(V^\alpha \oplus W_i, G_i) = \mathcal{A}(V^\alpha \oplus W_{\pi(i_0)}, G_{\pi(i_0)})\}.$$  

Since $G_{i_0} \cong \{1\}$, we pick up any element $\sigma$ from $G_{i_0} - \{1\}$. Then, for each $j \in J$, we can choose $\tau_j \in G_j$ with $(1 - \tau_j)V = (1 - \sigma)V$. Clearly there are integers $0 \leq \mu(j) < p$ ($j \in J'$) such that

$$\left(1 - \prod_{j \in J'} \tau_j^{(j)}\right)\theta(X_i) = (1 - \sigma)\theta(X_i)$$

for $\pi(i) \in J'$. Further let us define integers $0 \leq \mu(j) < p$ ($j \in J - J'$) to satisfy

$$\prod_{j \in J} \tau_j^{(j)}\theta(X_i) = \theta(X_i) \quad (\pi(i) \in J - J').$$

Consequently we see that

$$\left(1 - \prod_{j \in J} \tau_j^{(j)}\right)\theta(X_i) = (1 - \sigma)\theta(X_i) \quad (1 \leq i \leq m),$$

which yields

$$\sigma = \prod_{j \in J} \tau_j^{(j)}.$$ 

Thus the couple $(V, G)$ decomposes to $(V^\alpha \oplus \theta(W_i), G_i')$ ($1 \leq i \leq m$) since $G \supseteq G_i'$ and $|G_i| = |G_i'|$ ($1 \leq i \leq m$).
Conversely assume that \((V, G)\) has another decomposition \(\mathcal{D}' = \{(V^a \oplus \theta(W_i), G_i): 1 \leq i \leq m\}\) and let \(f_i(\theta(X_i))\) be the canonical \((V^a \oplus \theta(W_i), G_i)\)-invariant on \(\theta(X_i)\). If
\[
\theta(X_i) = \sum_{1 \leq j \leq m} a_{ij} X_j
\]
for some \(a_{ij} \in k\), we have
\[
f_i(\theta(X_i)) = \sum_{1 \leq j \leq m} a_{ij} f_j(X_j).
\]
Select a subgroup \(H_{ij}\) of \(GL(V^a \oplus W_j)\) such that \(k[V^a \oplus W_j]^{H_{ij}} = k[V^a][f_j(X_j)]\). Then the natural \(kH_{ij}\)-module \(V^a \oplus W_j\) defines a couple which satisfies that \(\mathcal{A}(V^a \oplus W_j, H_{ij}) = \mathcal{A}(V^a \oplus \theta(X_i), G_i)\). On the other hand \(f_i(\theta(X_i))\) can be expressed as
\[
f_i(\theta(X_i)) = \sum_{1 \leq j \leq m} a_{ij} h_{ij} + g_i
\]
for \(g_i \in \langle V^a \rangle_{k[V^a]}\) and \(h_{ij} \in k[V^a \oplus W_j]^{H_{ij}}\) where each \(h_{ij}\) is monic as a polynomial of \(X_j\). Therefore the canonical \((V^a \oplus W_j, G_j)\)-invariant \(F_j(X_j)\) on \(X_j\) divides \(f_i(X_i)\) in \(k[V^a \oplus W_j]\) \((j \in \text{supp}_G \theta(X_i))\). From this we must have \(\mathcal{A}(V^a \oplus \theta(W_i), G_i) \supseteq \mathcal{A}(V^a \oplus W_j, G_j)\) \((j \in \text{supp}_G \theta(X_i))\) for \(1 \leq i \leq m\). The remainder of (2) follows directly from the equality
\[
k[V^a][F_1(X_1), \ldots, F_m(X_m)] = k[V^a][f_1(\theta(X_1)), \ldots, f_m(\theta(X_m))].
\]
We say that \((V, G)\) is homogeneous when \(\mathcal{A}(V, G)\) is homogeneous concerning the natural graduation induced from that of \(k[V]\) (i.e. \(\mathcal{A}(V, G)\) is generated by some homogeneous part as a \(k\)-algebra). A couple \((V, G)\) is defined to be quasi-homogeneous if there is a subspace \(W\) of \(V^a\) with \(\text{codim}_V W = 1\) such that \(G(W) = \{1\}\) or \((V, G(W))\) is a homogeneous subcouple which satisfies \(\dim(V, G) = \dim(V, G(W))\).

\section{Computation of invariants}
Let \((V^a \oplus kX_i, H_i)\) \((1 \leq i \leq m)\) be subcouples of \((V, G)\) with
\[
\dim(V^a + \sum_{1 \leq i \leq m} kX_i) = m + \dim V^a
\]
such that \(V^{H_j} \cap X_i = \{0\}\) and \(G(W) = \bigoplus_{1 \leq i \leq m} H_i\) for a subspace \(W\) of \(V^a\) with \(\text{codim}_V W = 1\). We define \(Z, T_i\) and \(W_j\) to satisfy \(V^a = W \oplus kZ, W = \bigoplus_{1 \leq i \leq m} kT_i\) and \(kX_j = W_j\) \((1 \leq j \leq m)\) respectively. \(F_i = F_i(X_i)\) denotes the canonical \((V^a \oplus W_i, H_i)\)-invariant on \(X_i\). For any \(n\) and \(c = (c_1, \ldots, c_n) \in \mathbb{Z}^n\), let \(\|c\|\) denote the sum \(\sum_{1 \leq i \leq n} c_i\) and \(\{e_i: 1 \leq i \leq n\}\) be the standard
basis of $\mathbb{Z}^n$ ($\mathbb{Z}$ is the set of all integers). Further we suppose that there are pseudo-reflections $\sigma_j \in G - G(W)$ ($1 \leq j \leq m$) with $[\lambda_{ij}] \in GL_n(k)$ where

$$\lambda_{ij} = \frac{(\sigma_j - 1)X_i \mod W}{Z \mod W}.$$ 

**Lemma 3.1.** Let $R$ be a subalgebra of $k[V]^G$ which contains $k[V]^G$. Assume that $F_1^m, F_2^m, \ldots, F_m^m$ ($0 \leq c_i < p$) are linearly independent over $R$ and let $g_i$ be an element of the $R$-module

$$\bigoplus_{c \in \Gamma} RF_1^c F_2^c \cdots F_m^c$$

where $\Gamma = \{c = (c_1, \ldots, c_m) \in \mathbb{Z}^m : 0 \leq c_i < p \text{ and } \|c\| > 1\}$. Then $g_i = 0$ if $g_i g_j \in k[V]^G$ for a polynomial $g_j \in k[V]$ with $(\sigma_j - 1)g_j \in R$ ($1 \leq j \leq m$).

**Proof.** For $\tau = (\tau_1, \ldots, \tau_m) \in \mathbb{Z}^m$ with $0 \leq \tau_i < p$ let

$$\mathcal{P}_\tau : \bigoplus_{0 \leq c_i < p} RF_1^c F_2^c \cdots F_m^c \rightarrow RF_1^c F_2^c \cdots F_m^c$$

denote the canonical projection. Choose an element $\xi = (\xi_1, \ldots, \xi_m) \in \Gamma$ such that $\mathcal{P}_\tau(g_i) = 0$ at each $\tau \in \Gamma$ with $\|\tau\| > \|\xi\|$. We may assume that $\xi_i > 0$. Besides we define $\eta = (\eta_1, \ldots, \eta_m)$ as $\xi - e_i$ and put $\partial_i \eta = \eta + e_i$ ($1 \leq i \leq m$). Then clearly

$$\mathcal{P}_\tau((\sigma_j - 1)g_i) = \mathcal{P}_\tau((1 - \sigma_j)g_i) = 0,$$

because $(\sigma_j - 1)g_i \in R$ and $\eta \neq 0$. Further, as

$$(\sigma_j - 1)F_i(X_i) = F_i((\sigma_j - 1)X_i) \in k[V]^G$$

and $k[V]^G \cong R$, we have

$$(0 \Rightarrow) \mathcal{P}_\tau((\sigma_j - 1)g_i) = \sum_{1 \leq i \leq m} \mathcal{P}_\tau((\sigma_j - 1)\mathcal{P}_\tau(g_i))$$

$$= \sum_{1 \leq i \leq m} \mathcal{P}_\tau((\sigma_j - 1)\mathcal{P}_{\tau_i}(g_i))$$

$$= \sum_{\tau_i < p-1} (\eta_i + 1)F_i((\sigma_j - 1)X_i)\mathcal{P}_{\tau_i}(g_i)F_i(X_i)^{-1}$$

for all $1 \leq j \leq m$. On the other hand the polynomials

$$F_i((\sigma_j - 1)X_i) - \lambda_{ij} F_i(Z) \quad (1 \leq i, j \leq m)$$

are contained in $k[W]$ and hence the terms of $\mathcal{P}_\tau((\sigma_j - 1)g_i)$ with variables $Z, T_i, X_i$ whose degrees are maximal on $Z$ are also terms of

$$\sum_{\tau_i < p-1} \lambda_{ij}(\eta_i + 1)F_i(Z)\mathcal{P}_{\tau_i}(g_i)F_i(X_i)^{-1},$$
where $X_j$ ($j > m$) are defined such that $\{Z, T_i, X_j\}$ is a $k$-basis of $V$. This implies that

$$\mathcal{P}_{\delta X_j}(g_i) = \mathcal{P}_{\delta T_i}(g_i) = 0.$$  

Now let us study a decomposition of $(V, G)$ in the case where $m \geq 2$, $V = V^\alpha \oplus \bigoplus_{1 \leq i \leq m} W_i$, $G(W) = \bigoplus_{1 \leq i \leq m} H_i$ and $|H_i| = p^i$ ($1 \leq i \leq m$) (observe that $(V, G)$ is quasi-homogeneous). The rest of this section is devoted to the proof of the following proposition.

**Proposition 3.2.** If $k[V]^\alpha$ is a polynomial ring, then $(V, G)$ is decomposable.

$I$, $(1 \leq s \leq \nu)$ stand for equivalence classes of $I = \{1, 2, \ldots, m\}$ with respect to the relation $\sim$ induced by $i \sim j$ when $\mathcal{A}(V^\alpha \oplus W_i, H_i) = \mathcal{A}(V^\alpha \oplus W_j, H_j)$. For each $I_s$, there is a subset $J_s$ of $I$ with $|I_s| = |J_s|$ such that the submatrix $[\lambda_{ij}]_{I_s \times J_s}$ ($1 \leq s \leq \nu$) is non-singular ($J_s$ ($1 \leq s \leq \nu$) are not always disjoint). We may assume that $[\lambda_{ij}]_{I_s \times J_s}$ ($1 \leq s \leq \nu$) are monomial matrices, replacing a decomposition of $(V, H)$ consisting of one dimensional subcouples by the use of an admissible transform.

Moreover suppose that $k[V]^\alpha$ is a polynomial ring over $k$. Since

$$\mathcal{D}(V, G) \subseteq \left( k[V]^\alpha \langle W \rangle^\alpha / \langle W \rangle^\alpha G(\langle W \rangle^\alpha) \right) / \left( k[V]^\alpha \langle W \rangle^\alpha / \langle W \rangle^\alpha G(\langle W \rangle^\alpha) \right)$$

we have $k[V]^\alpha = k[V^\alpha][f_1, \ldots, f_m]$ for homogeneous polynomials $f_i \in k[V]$ with $f_i \equiv F_i \bmod \langle V^\alpha \rangle G(\langle W \rangle^\alpha)$. Then it follows from (3.1) that

$$F_i = F_i^\alpha + \sum_{1 \leq j \leq m} F_j h_{ij} \quad (1 \leq i \leq m)$$

where $h_{ij}$ are homogeneous in $k[V^\alpha]$.

We wish to claim $h_{ij} = 0$ ($i \not\sim j$) and show this only for the case of $i = 1$. Suppose that $T_i$ ($1 \leq i \leq t$) span the subspace $\mathcal{A}(V^\alpha \oplus W_i, H_i)$ of $V^\alpha$ and set

$$Z_j = Z + \sum_{1 \leq s \leq d} b_{js} T_s \in (\sigma_j - 1)V$$

where $b_{js} \in k$. For $c = (c_0, \ldots, c_{\nu}) \in \mathbb{N}^\nu$ and $g \in k[V^\alpha]_{(p^t+1)}$, $\Phi(g) \in k$ is defined to be the coefficient of

$$T_i^c_{t^1} \cdots T_{c^d} \cdots Z_{c^1}^{p^{t+1-\|c\|}}$$

in $g$ which is regarded as a polynomial of $T_i$ ($1 \leq i \leq d$) and $Z$ ($N$ is the set of all non-negative integers). Especially we denote by $a_i(c)$ the value $\Phi(Z^{p^t} h_{i1})$.  

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\[\text{https://doi.org/10.1017/S0027763000019875}\]
**Lemma 3.3.** Let $c$ be an element of $N^d$ such that $\|c\| < p^i$. Then we have

$$a_i(c) = \begin{cases} -1 & \text{if } i = 1 \text{ and } c = 0 \\ 0 & \text{otherwise} \end{cases}.$$ 

**Proof.** Suppose that an element $c \in N^d$ satisfies $\|c\| < p^i$. Then

$$\Phi_c(F_i(Z)^p) = \begin{cases} 1 & (c = 0) \\ 0 & (c \neq 0) \end{cases},$$

since $p^{i+1} - \|c\| > p^i$ and

$$F_i(Z) = Z^{p^i} + \sum_{1 \leq u \leq t} F_{1u}Z^{p^{i-t}}$$

for $F_{1u} \in k[W]$. On the other hand we have

$$(0) = \Phi_c((\sigma_j - 1)f_i) = \Phi_c(F_i((\sigma_j - 1)X_i)^p) + \sum_{1 \leq u \leq m} \Phi_c(F_i((\sigma_j - 1)X_i)\overset{h_{1i}}{\rightarrow})$$

$$= \lambda_{ij}\Phi_c(F_i(Z)^p) + \sum_{1 \leq u \leq d} b_{ju}\Phi_c(F_i(T_u)^p)$$

$$+ \sum_{1 \leq i \leq m} \lambda_{ij}\{\Phi_c(F_i(Z)h_{1i}) + \sum_{1 \leq u \leq d} b_{ju}\Phi_c(F_i(T_u)h_{1u})\}$$

$$= \lambda_{ij}\Phi_c(F_i(Z)^p) + \sum_{1 \leq i \leq m} \lambda_{ij}\Phi_c(F_i(Z)h_{1i}).$$

Therefore this system is reduced to

$$\sum_{1 \leq i \leq m} \lambda_{ij}\{a_i(c) + \sum_{c' \in N^d} \alpha(c')a_i(c')\} = \begin{cases} -\lambda_{ij} & (c = 0) \\ 0 & (c \neq 0) \end{cases}$$

where $\alpha(c') \in k$. The assertion follows from the last equations, because the matrix $[\lambda_{ij}]$ is non-singular.

**Lemma 3.4.** Let $L$ be the subset of

$$\{0\} \times \cdots \times \{0\} \times N^{d-t}$$

$t$ times

consisting of all non-zero elements $c$ such that

$$\|c\| = \omega_0p^i + \sum_{1 \leq i \leq t} \omega_i(p^i - p^{i-1})$$

for $\omega_i \in \mathbb{Z}$ with $\omega_i \leq 0$ ($0 \leq i \leq t - 1$) and $0 < \omega_i < p$. If $c \in L$ then $a_j(c) = 0$ ($1 \leq j \leq m$).

**Proof.** Let $c = (c_1, \ldots, c_d)$ be an element of $L$ such that $a_j(c') = 0$
for all $c' \in L$ with $\|c\| > \|c'\|$. Obviously the equalities

$$\Phi_\zeta(F_\delta((1 - \sigma_j)X)^p) = 0 \quad \text{and} \quad \Phi_\zeta(F_\delta(Z)h_{i_0}) = a_i(c)$$

follow from $p^{i+1} > \|c\|$ and $(c_i, \cdots, c_n) = 0$. Further we can show that

$$\Phi_\zeta(F_\delta(Z)h_{i_0}) - a_i(c) = \beta_i(0)a_i(0) + \sum_{c' \in L} \beta_i(c')a_i(c') \quad (1 < i \leq m)$$

for some $\beta_i(0), \beta_i(c') \in k$, because

$$F_\delta(Z) = Z^{p^i} + \sum_{1 \leq j \leq d} F_{i_j}Z^{p^{i-j}}$$

where $F_{i_j}$ are homogeneous polynomials in $k[W]$. According to (3.3) $a_i(0) = 0 \quad (1 < i \leq m)$ and therefore we must have

$$\Phi_\zeta \left( \left( F_\delta(Z) + \sum_{1 \leq j \leq d} b_{j_u}F_\delta(T_u) \right) h_{i_0} \right) = a_i(c)$$

because $\|c\| \neq p^i$. Now the system

$$\Phi_\zeta(F_\delta((1 - \sigma_j)X)^p) = \sum_{1 \leq j \leq m} \Phi_\zeta(F_\delta((\sigma_j - 1)X)h_{i_0})$$

can be expressed as

$$\sum_{1 \leq j \leq m} \lambda_{ij}a_i(c) = 0 \quad (1 \leq j \leq m),$$

which imply that $a_i(c) = 0 \quad (1 \leq i \leq m)$.

**Lemma 3.5.** If $d > t$, $I_{x_0} \cap 1$ and $I \approx I_{x_0}$, then $a_i(p^ie_i) = 0 \quad (t + 1 < j \leq d)$ for each $i \in I - I_{x_0}$.

**Proof.** Put $\zeta_v = \{vp^i - (v - 1)p^{i-1}\} e_{t+1} \in Z^d \quad (1 \leq v \leq p)$ and let $a_i(\zeta_v) = 0 \quad (1 \leq i \leq m)$. Since $\Phi_\zeta(F_\delta(T_u)h_{i_0}) = 0$ for $u \approx t + 1$, by (2.9) we obtain

$$\Phi_\zeta \left( \sum_{1 \leq j \leq m} F_\delta((\sigma_j - 1)X)h_{i_0} \right) = \sum_{1 \leq j \leq m} \lambda_{ij}\Phi_\zeta(F_\delta(Z)h_{i_0})$$

$$+ \sum_{i \in I} \lambda_{ij}b_{j_u}\Phi_\zeta(F_\delta(T_u)h_{i_0})$$

$$= \sum_{i \in I} \lambda_{ij}\{a_i(\zeta_v) + b_{j_u}a_i((v - 1)(p^i - p^{i-1})e_{t+1})\}$$

$$+ \sum_{i \in I} \lambda_{ij}\{a_i(\zeta_v) - a_i(\zeta_v - 1)\}$$

where $\bar{I} = \{i: \oplus_{u \approx t+1} kT_u \supseteq \mathcal{J}(V^0 \oplus W_t, H_t)\}$. But it follows from (3.4) that

$$a_i((v - 1)(p^i - p^{i-1})e_{t+1}) = 0 \quad (2 \leq v \leq p).$$

Thus for $2 \leq v \leq p$ and $1 \leq j \leq m$ we must have
\[0 = \Phi_{\zeta^i}(F_i((1 - \sigma_j)X_i)^p) = \Phi_{\zeta^i}\left(\sum_{1 \leq i \leq m} F_i((\sigma_j - 1)X_i)h_{1i}\right) = \sum_{i \in I} \lambda_i a_i(\zeta_i) + \sum_{i \in I - I} \lambda_i a_i(\zeta_i) - a_i(\zeta_{-i})\]

which shows \(a_i(p'e_{i+1}) = 0\) for \(i \in I - I\). Further let \(i_0\) be an element of \((I - I_0) \cap I\) if it is non-empty. We may suppose \(\bigoplus_{k \neq t \neq t + 2} kT_u \cong \bigoplus(V^u \oplus W_{t_0}, H_{t_0})\) and set \(\zeta'' = p'e_{t+1} + (v - 1)(p' - p^{-1})e_{t+2}\) \((1 \leq v \leq p)\). Clearly

\[\Phi_{\zeta''}\left(\sum_{1 \leq i \leq m} F_i((\sigma_j - 1)X_i)h_{1i}\right) = \sum_{1 \leq i \leq m} \lambda_i \left(\Phi_{\zeta''}(F_i(Z))h_{1i}\right) + \sum_{u = t + 1, t + 2} b_{iu}\Phi_{\zeta''}(F_i(T_u)h_{1u})\]

for \(2 \leq v \leq p\). On the other hand (2.9) implies

\[\Phi_{\zeta''}(F_i(T_u)h_{1u}) = a_{i_0}(\zeta''_u) - a_{i_0}(\zeta''_{-u})\]

\((2 \leq v \leq p)\) because \(\Phi_{\zeta''}(F_i(T_u)h_{1u})\) \((u = t + 1, t + 2)\) are linear combinations of \(a_i(c)\) such that \(c = (0, \cdots, 0, c_{t+1}, \cdots, c_u)\) and \(||c|| = (v - 1)(p' - p^{-1})\). But we see

\[\Phi_{\zeta''}\left(\sum_{1 \leq i \leq m} F_i((\sigma_j - 1)X_i)h_{1i}\right) = \Phi_{\zeta''}(F_i((1 - \sigma_j)X_i)^p) = 0\]

\((2 \leq v \leq p; 1 \leq j \leq m)\),

and hence this system requires

\[a_{i_0}(p'e_{t+1}) = a_{i_0}(\zeta'_{t}) = \cdots = a_{i_0}(\zeta'_{t+m}) = 0\, .\]

The remainder can be proved in the same way.

Now let \(s_0\) be an integer such that \(I_{s_0} \ni 1\) and put \(\tau_j = \sigma_j\sigma_{s_0}^{s_j} (1 \leq j \leq m)\) where \(j_0 \in J_{s_0}\) and \(n_j \in N\) satisfy \(\lambda_{j_0} = 0\) and \(n_j \lambda_{j_0} = - \lambda_{j_j}\) respectively. According to (3.3)

\[\Phi_{p'e_i}(F_u((\sigma_j - 1)X_u)h_{1u}) = \lambda_{u,j}\Phi_{p'e_i}\left(F_u\left(Z + \sum_{1 \leq u \leq d} b_{ju}T_u\right)h_{1u}\right) = \lambda_{u,j} a_u(p'e_i)\]

for \(2 \leq u \leq m\), and therefore if \(t + 1 \leq i \leq d\) we deduce from (3.5) that

\[0 = \Phi_{p'e_i}(F_i((1 - \sigma_j)X_i)^p) = \sum_{1 \leq u \leq m} \Phi_{p'e_i}(F_u((\sigma_j - 1)X_u)h_{1u}) = \lambda_{i,j} a_i(p'e_i) + b_{ji} a_i(0) + \sum_{u \in I_{s_0} - \{1\}} \lambda_{u,j} a_u(p'e_i) \, .\]

Since \([\lambda_{1}]_{(u,v) \in I_{s_0} \times J_{s_0}}\) is a monomial matrix, these equations imply

\[a_j(p'e_i) = 0\quad (t + 1 \leq i \leq d; 2 \leq j \leq m)\, .\]

So we have

\[a_j(p'e_i) = - b_{ji} a_i(0) = b_{ji}\quad (t + 1 \leq i \leq d)\]
for $1 \leq j \leq m$ with $\lambda_{ij} \neq 0$, and then it follows from the definition of $\tau_j$ that $(\tau_j - 1)X_i \in \oplus_{1 \leq i \leq t} k T_i$ ($1 \leq j \leq m$). By the identities $F_i(T_i) = 0$ ($1 \leq i \leq t$) we can see

$$
\tau_j(f_i) = \tau_j(F_i)^p + \sum_{1 \leq i \leq m} \tau_j(F_i)h_{i,t}
$$

$$
= F_i^p + F_i h_{i,t} + \sum_{2 \leq i \leq m} \tau_j(F_i)h_{i,t}.
$$

Consequently we obtain

$$
(0 =) (\tau_j - 1)f_i = \sum_{2 \leq i \leq m} (c_{ij}F_i(Z) + g_{ij})h_{i,t}
$$

for some homogeneous polynomials $g_{ij}$ in $k[W]$ where

$$
c_{ij} = \frac{(\tau_j - 1)X_i \mod W}{Z \mod W}.
$$

Then, because $F_i(Z) \equiv Z^p \mod \langle W \rangle$, this system requires $h_{i,t} = 0$ ($2 \leq i \leq m$).

For $i \neq j$ we conclude that $h_{i,t} = 0$. Hence $G$ contains subgroups $G_i$ ($i = 1, 2$) which satisfy $k[V]^{G_i} = k[V^\sigma][f_1, X_1, X_2, \ldots, X_m]$ and $k[V]^{G_i} = k[V^\sigma][X_1, f_1, f_2, \ldots, f_m]$. The couple $(V, G)$ has a decomposition $((V^\sigma \oplus k[X_1, G_1], (V^\sigma \oplus \oplus_{2 \leq i \leq m} k[X_i, G_i])$. We have just completed the proof of (3.2).

§ 4. Proof of Theorem 1.3

We begin with

**Proposition 4.1.** Let $(V, G)$ be a quasi-homogeneous couple with $\dim (V, G) \geq 2$. Suppose that $(V, G(W))$ decomposes to one dimensional subcouples for any proper subspace $W$ of $V^\sigma$ with $G(W) \neq \{1\}$. If $k[V]^\sigma$ is a polynomial ring, then $(V, G)$ is decomposable.

**Proof.** Since $(V, G)$ is quasi-homogeneous, there is a subspace $W$ of $V^\sigma$ with codim$_r W = 1$ such that $G(W) = \{1\}$ or $(V, G(W))$ is a homogeneous subcouple which satisfies $\dim (V, G(W)) = \dim (V, G) = m$. Clearly $(V, G)$ is decomposable if $G(W)$ is trivial. Hence we suppose that $(V, G(W))$ decomposes to one dimensional subcouples $(V^\sigma \oplus W_i, H_i)$ ($1 \leq i \leq m$) with $|H_i| = p^i$. Denote by $X_i$ a generator of $W_i$ and let $r$ be the rank of the matrix $[(\sigma_j - 1)X_i \mod W_i]_{\sigma_j \in H_i}$, where $\sigma_j$ runs through all pseudo-reflections in $G - G(W)$. In the case of $r = m$ we have already shown that $(V, G)$ is decomposable. We may assume that $r < m$ and that the submatrix $[(\sigma_j - 1)X_i \mod W_i]_{\sigma_j \in H_i}$ is non-singular.

Let $F_i(X_i)$ be the canonical $(V^\sigma \oplus W_i, H_i)$-invariant on $X_i$. Further
choose \( Z_i \) from \( V \) with \((1 - \sigma_j)V = kZ_j \) and put \( b_{ij} = Z_j^{-1}(\sigma_j - 1)X_i \). Since \( \mathcal{R}(V, G(W)) \) is homogeneous, by (2.8) we see \( \mathcal{R}(V, G) = k[X_1^{\sigma_1}, \ldots, X_r^{\sigma_r}, g_{r+1}, \ldots, g_m] \) where \( \bar{X}_i = X_i \mod V^r \) and \( g_j \) \((r + 1 \leq j \leq m)\) are expressed as

\[
g_j = \bar{X}_j^\rho + \sum_{1 \leq i < r} a_{ij}X_i^\rho
\]

for some \( a_{ij} \in k \). From this the polynomials

\[
F_j(X_i) + \sum_{1 \leq i < r} a_{ij}F_i(X_i) \quad (r + 1 \leq j \leq m)
\]

belong to a regular system of homogeneous parameters of \( k[V]^\rho \). Thus, for \( r + 1 \leq j \leq m \) and \( 1 \leq u \leq r \), we have

\[
-b_{ju}F_j(Z_u) = (1 - \sigma_u)F_j(X_j)
\]

\[
= \sum_{1 \leq i < r} a_{ij}(\sigma_u - 1)F_i(X_i)
\]

\[
= \sum_{1 \leq i < r} b_{iu}a_{ij}F_i(Z_u),
\]

which implies that if \( a_{ij} \neq 0 \)

\[
F_i(Z) = F_i(Z) \quad (1 \leq i \leq r; \quad r + 1 \leq j \leq m)
\]

where \( Z \) denotes a variable. Obviously this requires \( \mathcal{R}(V^\rho \oplus W_i, H_i) = \mathcal{R}(V^\rho \oplus W, H) \). Define \( \theta \in GL(V) \) to satisfy that

\[
\theta(X_j) = X_j + \sum_{1 \leq i < r} a_{ij}X_i \quad (r + 1 \leq j \leq m)
\]

and \( V^{(\rho)} \supseteq \{X_i : 1 \leq i \leq r\} \cup V^\rho \). According to (2.10) \( \theta \) is a \((V^\rho \oplus W, H)\)-admissible transform and \((V, H)\) decomposes to subcouples \((V^\rho \oplus \theta(W)), H_i\) \((1 \leq i \leq m)\) for some subgroups \( H_i \) of \( H \). Then \((V, G)\) decomposes to \((V^\rho \oplus \oplus_{i \leq r \leq m} \theta(W_i), \oplus_{r+1 \leq i \leq m} H_i\) \((V^\rho \oplus \oplus_{r+1 \leq i \leq m} \theta(W_i))\) and \((V^\rho \oplus \oplus_{1 \leq i \leq r} \theta(W_i), L)\) where \( L \) is the stabilizer of \( G \) at \( \oplus_{r+1 \leq i \leq m} \theta(W_i) \).

(4.2) Let \( A_i = K[f_{i1}, f_{i2}, \ldots, f_{in}] \) \((i = 1, 2)\) be graded polynomial algebras with \( \dim A_i = n \) over a field \( K \) where \( f_{ij} \) are homogeneous in \( A_i \). Suppose that \( A_i \) is contained in \( A_2 \) as a graded subalgebra. Then \( A_i = A_2 \) if and only if

\[
\prod_{1 \leq i \leq n} \deg f_{i1} = \prod_{1 \leq i \leq n} \deg f_{ij}
\]

\( q(R) \) denotes the quotient field of an integral domain \( R \).

**Lemma 4.3.** For any couple \((V, G)\) we have the following inequality;
and if the equality holds then \( k[V]^\circ \) is a polynomial ring.

**Proof.** We prove this by induction on \(|G|\). Let \( W \) be a subspace of \( V^\circ \) such that \( \text{codim}_G W = 1 \) and \( W \not\supset \mathcal{A}(V, G) \). Then \( H = G(W) \) is a proper subgroup of \( G \). By the induction hypothesis we have

\[
[q(k[V]/\langle W \rangle), q(k[V]^H/\langle W \rangle^H)] \geq |H|
\]

and if the equality holds \( k[V]^H \) is a polynomial ring. Putting

\[
S = (\overline{k} \otimes_k k[V]/\langle W \rangle^H)^{G/H},
\]

as in the proof of (2.3), we can show that \( S_{\mathfrak{M}_i} \cong S_{\mathfrak{M}_k} \) for any maximal ideals \( \mathfrak{M}_i \) of \( S \) which contain the minimal prime ideal \( \left( \langle \overline{k} \otimes_k V^\circ \rangle^{G/H} \right)^{\langle \overline{k} \otimes_k W \rangle^{G/H}} \). On the other hand it follows easily from (2.3) that \( S \) is normal and hence \( S \) is a polynomial ring over \( \overline{k} \). Since

\[
\overline{k} \otimes_k k[V]/\langle W \rangle^H \cong S
\]

as graded algebras defined over \( \overline{k} \), \( (k[V]^H/\langle W \rangle^H)^{G/H} \) is also a polynomial ring. Clearly \( \mathcal{A}(V, G) \) can be embedded in \( (k[V]^H/\langle W \rangle^H)^{G/H} \) and so we have

\[
[q(k[V]/\langle W \rangle), q(\mathcal{A}(V, G))] \geq |G|.
\]

Now suppose that the equality of (4.3) holds and then we deduce from this

\[
[q(k[V]/\langle W \rangle), q(k[V]^H/\langle W \rangle^H)] = |H|.
\]

Therefore \( k[V]^H \) is a polynomial ring. Moreover by the equality of (4.3) and (2.3) we see that the canonical map

\[
\mathcal{A}(V, G) \longrightarrow (k[V]^H/\langle W \rangle^H)^{G/H} \langle \langle V^\circ \rangle^H / \langle W \rangle^H \rangle^{G/H}
\]

is an isomorphism and that there is an \((n + 1)\)-dimensional graded polynomial subalgebra \( k[f_1, f_2, \ldots, f_{n+1}] \) of \( k[V]^g/\langle W \rangle^g \) with

\[
\prod_{1 \leq i \leq n+1} \deg f_i = |G|.
\]

Here \( n \) denotes the dimension of \((V, G)\) and \( f_i \) \((1 \leq i \leq n + 1)\) are homogeneous elements in \( k[V]/\langle W \rangle \). Then, by (4.2), we must have \( (k[V]^H/\langle W \rangle^H)^{G/H} \) is a polynomial ring which contains \( k[V]^g/\langle W \rangle^g \) as a graded subalgebra.
Further if \( \dim W \geq 2 \) let \( W' \) be a subspace of \( W \) with \( \text{codim}_W W' = 1 \) and put \( H' = G(W')(= H(W')) \). Since \( k[V]^H \) is a polynomial ring, by (2.6) \( k[V]^H \) is also a polynomial ring. Therefore we get the commutative diagram

\[
\begin{array}{ccc}
\frac{k[V]^H}{\langle W' \rangle^H} & \longrightarrow & \frac{k[V]^H}{W'} \\
\downarrow & & \downarrow \\
\frac{(k[V]^H)/\langle W' \rangle^H}{H'}/H & \longrightarrow & \frac{(k[V]^H)/\langle W' \rangle^H}{H'}/H'
\end{array}
\]

of \( kG/H \)-modules with exact rows. From \( (k[V]^H)/\langle W' \rangle^H/k[V]^H = k[V]^H/\langle W' \rangle^H \) the sequence

\[
(k[V]^H)/\langle W' \rangle^H/\langle W' \rangle^G \longrightarrow (k[V]^H)/\langle W' \rangle^G \longrightarrow 0
\]

is exact. Then \( (k[V]^H)/\langle W' \rangle^G \) is a polynomial ring which contains \( k[V]^G/\langle W' \rangle^G \), because \( \langle W' \rangle^G/\langle W' \rangle^G \) is principal. Hence we deduce similarly from the equality of (4.3) and (2.3) that \( k[V]^G/\langle W' \rangle^G = (k[V]^H)/\langle W' \rangle^H \).

If necessary we can continue this procedure. Consequently \( k[V]^G/\langle W' \rangle^G \) is a polynomial ring for a one dimensional subspace \( W \) of \( V^G \). The assertion follows immediately from this.

By the use of (4.1) we establish

**Theorem 4.4.** Let \( (V, G) \) be an indecomposable couple. Then \( k[V]^G \) is a polynomial ring if and only if \( \dim (V, G) = 1 \).

**Proof.** It suffices to prove the "only if" part. Let \( \mathcal{C} \) denote the set of all indecomposable couples \((V_0, G_0)\) with \( \dim (V_0, G_0) \geq 2 \) such that \( k[V]^G_0 \) are polynomial rings. Assume that \( \mathcal{C} \) is non-empty and choose an element \((V, G)\) from \( \mathcal{C} \) which is minimal with respect to the lexicographical preorder of \( \mathcal{C} \) defined by the value \( (\dim (V_0, G_0), \dim V_0) \) for \((V_0, G_0) \in \mathcal{C} \). From (4.1) the couple \((V, G)\) is not quasi-homogeneous. Let \( W \) be a subspace of \( V^G \) with \( \text{codim}_V W = 1 \) and put \( H = G(W) \) and \( u = \dim V^H/V^G \) respectively. Then the \( kH \)-module \( V \) defines a couple \((V, H)\) and by (2.6) \( k[V]^H \) is a polynomial ring. Obviously \( V \) is decomposable as a \( kH \)-module, and hence \((V, H)\) decomposes to one dimensional subcouples \((V^H \oplus W_i, H) \) \((u + 1 \leq i \leq m)\) where \( m = \dim (V, G) \), since \((V, G)\) is minimal in \( \mathcal{C} \). If \((V, H)\) is not homogeneous, we may suppose that

\[
|H_{u+1}| \leq \cdots \leq |H_i| < |H_{u+1}| = \cdots = |H_m|
\]
for some \( v < m \). Otherwise set \( v = u \) (it should be noted that \( u > 0 \) in this case).

Let \( U = V^\alpha \oplus \bigoplus_{u+1 \leq i \leq m} W_i \) (the empty direct sum is regarded as \{0\}) and denote by \( G' \) the stabilizer of \( G \) at \( U \). We can choose homogeneous polynomials \( f_i \in k[V] \) (\( 1 \leq i \leq m \)) such that \( f_i \in k[U] \) (\( 1 \leq i \leq v \)) and \( k[V]^\alpha = k[V]^\alpha[f_1, \cdots, f_m] \), calculating a regular system of parameters of \( \mathcal{B}(V, G) \) through \( k[V]^\alpha/\langle W \rangle^\alpha \) as in the proof of (2.7). Because \( k[V]^\alpha \) is contained in \( k[U][f_{v+1}, \cdots, f_m] \), there is a subgroup \( \bar{G} \) of \( G \) with \( k[V]^\alpha = k[U][f_{v+1}, \cdots, f_m] \). Clearly \( \bar{G} = G' \) and the \( kG' \)-module \( V \) is decomposable. Therefore, from the minimality of \((V, G)\), the couple \((V, G')\) decomposes to one dimensional subcouples \((V^\alpha \oplus W_i, G') \) (\( v + 1 \leq i \leq m \)).

We have

\[
[q(k[U/V^\alpha]) : q(\mathcal{B}(U, G/G'))] = |G/G'|
\]

since \( f_i \in k[U/G'/G] \) (\( 1 \leq i \leq v \)) and \( G/G' \) acts faithfully on \( U \). By (4.3) \( k[U/G'/G] \) is a polynomial ring and so \((U, G/G')\) decomposes to one dimensional subcouples \((U/G'/G, G_i) \) (\( 1 \leq i \leq v \)). It should be noted that \( V^\alpha = U \) and \( U/G'/G = V^\alpha \).

Let \( X_i \) (\( 1 \leq i \leq m \)) denote a generator of \( W_i \) and put \( \bar{G} = G/G' \) and \( p^\sigma = [\bar{G} : \bigoplus_{1 \leq i \leq r} H_i] \) respectively. Because \( k[U]^\alpha = k[V]^\alpha[f_1, \cdots, f_m] \) by (4.2), we deduce from the computation of \( \mathcal{B}(V, G) \) (cf. (2.7)) that there exist pseudo-reflections \( \sigma_i \) (\( 1 \leq i \leq r \)) in \( G - H \) such that the column vectors \( [(\sigma_i - 1)X_i \text{ mod } W_i, 1 \leq j \leq r] \) are linearly independent. Then \( \bar{G}(W) \cap \bigoplus_{1 \leq i \leq r} \langle \sigma_i \text{ mod } G' \rangle = \{1\} \) and hence we see that \( \bar{G}(W) = \bigoplus_{u+1 \leq i \leq m} H_i \). Putting

\[
H_i' = \begin{cases} 
G_i' \cap \bigoplus_{u+1 \leq i \leq v} H_j & (1 \leq i \leq v) \\
G_i' \cap H & (v + 1 \leq i \leq m), 
\end{cases}
\]

we obtain another decomposition

\[
[(V^\alpha \oplus W_i, H_i') : 1 \leq i \leq m \text{ with } H_i' \neq \{1\}]
\]

of \((V, H)\). Since \( \{i : H_i' = \{1\}\} \subseteq \{1, 2, \cdots, v\} \), it may be assumed that \( H_i' = \{1\} \) (\( 1 \leq i \leq u \)).

Let \( F_*(X_i) = X_i \) (\( 1 \leq i \leq u \)) and for \( u + 1 \leq i \leq m \) (resp. \( 1 \leq i \leq m \)) let \( F_*(X_i) \) (resp. \( g_*(X_i) \)) be the canonical \((V^\alpha \oplus W_i, H_i')\)-invariant (resp. \((V^\alpha \oplus W_i, G_i')\)-invariant) on \( X \). Assume that \( G'_i = H'_i \) for some \( u + 1 \leq i \leq v \). Then \((V, G)\) decomposes to \((V^\alpha \oplus W'_{i_0}, H'_{i_0})\) and \((V^\alpha \oplus \bigoplus_{i_0 < i \leq v} W_i, L)\) where \( L \) is the stabilizer of \( G \) at \( W'_{i_0} \) and hence we must have \( |G'_i/H'_i| \)
for all \( u + 1 \leq i \leq v \). Because \( k[V^0] \) is contained in
\[
k[V^0 \oplus \bigoplus_{j \leq u} W^j_i \{ g_i, f_{v+1}, \ldots, f_m \}],
\]
there are pseudo-reflections \( \tau_j (1 \leq j \leq v) \) in \( G - H \) which satisfy the following condition; for \( 1 \leq i \leq u \) \( V^{(v)} \supseteq W^j_i \) if and only if \( i = j \). We may suppose that \( V^{(v)} \supseteq W'_j (1 \leq i \leq u; v + 1 \leq j \leq m) \) and \( \mathcal{A}(V^H \oplus W'_j, H'_j) \supsetneq \mathcal{A}(V^H \oplus W'_j, H'_j) \), applying a \( \{(V^H \oplus W'_j, H'_j) \} \)-
admissible transform on \( V \).

Clearly we may assume that \( \deg f_i = \deg g_i (v + 1 \leq i \leq m) \) and
\[
\deg f_{v+1} = \deg f_{v+2} = \cdots = \deg f_v < \deg f_{v+1} = \cdots = \deg f_m
\]
for some \( y \) with \( v + 1 \leq y \leq m \). Further \( f_i - g_i \) \((v + 1 \leq i \leq y) \) can be regarded as a polynomial \( h_i \) in \( k[U] \), replacing \( f_i \) with linear combinations of them. We deduce from (3.1) that
\[
h_i = \sum_{1 \leq j \leq v} F_j h_{ij} (v + 1 \leq i \leq y)
\]
for some homogeneous polynomials \( h_{ij} \) in \( k[V^0] \), since \( (\tau_j - 1)g_i \in k[V^0] \) \((v + 1 \leq i \leq y; 1 \leq j \leq v) \) and\[
k[U]_{v+1 \leq i \leq u} = \bigoplus_{0 \leq j \leq p \leq m} k[V^0] \{ g_1, \ldots, g_v, F_1, \ldots, F_v \}.
\]

Assume that \( h_{ij} \neq 0 \) and let \( Z_{j_i} \) be an element of \( V \) with \((1 - \tau_j) V = kZ_{j_i} \). Then it follows from \( \tau_j(F_{i}) = f_i \) that
\[
k^* h_{ij} F_i(Z_{j_i}) \ni \frac{(1 - \tau_j) X_{j_i}}{Z_{j_i}} g_i(Z_{j_i}).
\]
So we have \( u + 1 \leq j_i \leq v \) and \( \mathcal{A}(V^H \oplus W'_j, H'_j) \supseteq \mathcal{A}(V^H \oplus W'_j, H'_j) \). Moreover we find a pseudo-reflection \( \sigma \) in \( G'_{j_i} - H'_{j_i} \) because \( F_{j_i} = g_{j_i} \) requires \( \mathcal{A}(V^H \oplus W'_j, H'_j) \supseteq \mathcal{A}(V^H \oplus W'_j, H'_j) \), and choose \( Z \in V \) such that \((1 - \sigma) V = kZ_{j_i} \) and \( Z_{j_i} = Z_{j_i} \mod W \). Let \( \{ T_i \} : 1 \leq i \leq t \) be a \( k \)-basis of \( \mathcal{A}(V^H \oplus W'_j, H'_j) \) and select \( T_i \in V \) \((t + 1 \leq j \leq d) \) to satisfy \( W = \bigoplus_{1 \leq i \leq d} kT_i \) and \( \bigoplus_{1 \leq i \leq d-1} kT_i \supsetneq \mathcal{A}(V^H \oplus W'_j, H'_j) \).

Express \( Z_{j_i} \) as
\[
Z_{j_i} = Z + \sum_{1 \leq i \leq d} a_i T_i
\]
for \( a_i \in k \) \((1 \leq i \leq d) \) and set \( R = k[T_1, \ldots, T_d, Z] \). If \( a_d = 0 \), by (2.9) we have \((1 - \tau_{j_d}) F_{j_d} \in R \) and \( g_{j_d}(Z_{j_d}) \in R \). This implies that \( a_d \neq 0 \). Since \( g_{j_d}(Z_j) = g_{j_d}(Z) = 0 \) \((1 \leq j \leq t) \), we see
Then $g_i(Z_i)$ is a monic polynomial of $T_d$ in $R[T_d]$, but from (2.9) the leading coefficient of $F_i(Z_i)$ as a polynomial of $T_d$ is a non-unit in $R$, which is a contradiction. Therefore we must have $f_i = g_i (v + 1 \leq i \leq y)$.

In the case of $y = m$ it follows that $k[V]^g = k[V^o][g_1, \ldots, g_n]$ and this requires that $(V, G)$ is decomposable. Hence we obtain $y < m$. Because $G'_i = H'_i (v + 1 \leq i \leq y)$, the couple $(V, G)$ decomposes to $(V^o \oplus \bigoplus_{t+1 \leq i \leq y} W'_i, \bigoplus_{t+1 \leq i \leq y} H'_i)$ and $(V^o \oplus \bigoplus_{t+1 \leq i \leq y} W'_i \oplus \bigoplus_{y+1 \leq i \leq m} W'_i, K)$ where $K$ denotes the stabilizer of $G$ at the set $\bigoplus_{t+1 \leq i \leq y} W'_i$. This conflicts with the selection of $(V, G)$. Thus the proof is completed.

Now (1.3) can be reduced to (4.4) by (2.1), (2.2) and (2.4).

References


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