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On smoothness of the Banach space embedding

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For a Banach space X, smoothness at a point of the natural embedding $\hat{X}$ in X^{**}, is characterised by a continuity property of the support mapping from X into X^*. It then becomes clear that there are many non-reflexive Banach spaces with smooth embedding, a matter of interest raised by Ivan Singer [*Bull. Austral. Math. Soc.* 12 (1975), 407-416].

For a normed linear space X with dual space X^*, to each $x \in S(X) \equiv \{x \in X : \|x\| = 1\}$ we consider the set

$$D(x) \equiv \{f \in S(X^*) : f(x) = 1\} ;$$

(by the Hahn-Banach Theorem, $D(x)$ is non-empty). A *support mapping* $x \mapsto f_x$ of X into X^* maps each $x \in S(X)$ to $f_x \in D(x)$ and, for real $\lambda \geq 0$, $f_{\lambda x} = \lambda f_x$. X is *smooth* at $x \in S(X)$ if $D(x)$ contains only one point. The norm of X is *weakly* (*Gâteaux*) *differentiable* at $x \in S(X)$ if, for real λ,

$$\lim_{\lambda \to 0} \frac{\|x+\lambda y\|-\|x\|}{\lambda} \text{ exists for all } y \in S(X) ,$$

and is *strongly* (*Fréchet*) *differentiable* at $x \in S(X)$ if convergence to this limit is uniform for all $y \in S(X)$. X is smooth at $x \in S(X)$ if and only if the norm is weakly differentiable at x [3, p. 109].

We show the equivalence of two sets of conditions used to characterise differentiability of the norm. Our proof uses the extension to the Bishop-

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Phelps Theorem given by Bollobás [1, p. 181].

THE BISHOP-PHELPS-BOLLOBÁS THEOREM. *For a Banach space* X , *given* $0 < \varepsilon < 1$ *and* $x \in S(X)$, $f \in S(X^*)$ *such that* $|f(x)-1| < \varepsilon^2/4$, *there exists a* $y \in S(X)$ *and* $f_y \in D(y)$ *such that* $\|f-f_y\| < \varepsilon$ *and* $\|x-y\| < \varepsilon$.

LEMMA. *For a Banach space* X , *the following conditions are equivalent at* $x \in S(X)$:

1. *for every sequence* $\{f_n\}$ *in* $S(X^*)$ *such that* $f_n(x) \to 1$, *we have that* $\{f_n\}$ *is*

   (i) $\sigma(X^*, X)$- ,

   (ii) $\sigma(X^*, X^{**})$- ,

   (iii) norm-convergent;

2. *every support mapping* $x \mapsto f_x$ *of* X *into* X^* *is continuous on* $S(X)$ *at* x *when* X *has the norm-topology and* X^* *has the*

   (i) $\sigma(X^*, X)$- ,

   (ii) $\sigma(X^*, X^{**})$- ,

   (iii) norm-topology.

Proof. Suppose that 1 holds. It is clear that all such sequences $\{f_n\}$ converge to the same limit f where $\|f\| \le 1$. Then for any sequence $\{x_n\}$ in $S(X)$ where $\{x_n\}$ is norm-convergent to x , we have for $f_{x_n} \in D(x_n)$ that

$$\left|f_{x_n}(x)-1\right| = \left|f_{x_n}(x)-f_{x_n}(x_n)\right| \le \|x-x_n\| ;$$

so $f_{x_n}(x) \to 1$ and, therefore, the sequence $\left\{f_{x_n}\right\}$ is

(i) $\sigma(X^*, X)$- ,

(ii) $\sigma(X^*, X^{**})$- ,

(iii) norm-convergent to f .

Since

$$|f(x)-1| \le \left|f(x)-f_{x_n}(x)\right| + \left|f_{x_n}(x)-f_{x_n}(x_n)\right|$$
$$\le \left|\left(f-f_{x_n}\right)(x)\right| + \|x-x_n\| ,$$

we have that $f \in D(x)$. So there exists a support mapping which is continuous on $S(X)$ at x . However, this implies that X is smooth at x and that every support mapping is continuous at x [3, p. 107 and p. 109].

Suppose that 2 holds. This implies that X is smooth at x . For any sequence $\{f_n\}$ in $S(X^*)$ where $f_n(x) \to 1$, we have, by the Bishop-Phelps-Bollobás Theorem, that there exists a sequence $\{y_n\}$ in $S(X)$ and a sequence $\left\{f_{y_n}\right\}$ where $f_{y_n} \in D(y_n)$, such that $\left\{f_n-f_{y_n}\right\}$ is norm-convergent to 0 and $\{y_n\}$ is norm-convergent to x . But then $\left\{f_{y_n}\right\}$ is

(i) $\sigma(X^*, X)$- ,

(ii) $\sigma(X^*, X^{**})$- ,

(iii) norm-convergent to $f_x \in D(x)$.

Therefore, $\{f_n\}$ is

(i) $\sigma(X^*, X)$- ,

(ii) $\sigma(X^*, X^{**})$- ,

(iii) norm-convergent to f_x .

Šmulian has shown the equivalence of 1 *(i)* to weak differentiability of the norm at x [9, p. 91] and 1 *(iii)* to strong differentiability of the norm at x [10, p. 645]. The equivalence of 2 *(i)* to weak differentiability of the norm at x and 2 *(iii)* to strong differentiability of the norm at x is shown in [3, p. 107]. It is the particular equivalence of 1 *(ii)* and 2 *(ii)* which we require and it will follow from our theorem that these conditions are themselves equivalent to weak differentiability of the norm at $\hat{x} \in X^{**}$.

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Our theorem relies essentially on the following result given by Lindenstrauss and Tzafriri [6, p. 196].

THE PRINCIPLE OF LOCAL REFLEXIVITY. *For a Banach space* X , A *is a finite dimensional subspace of* X^{**} , B *is a finite set in* X^* , *and* $0 < \delta < 1$. *There exists a linear mapping* $T : A \to X$ *such that* $T(\hat{x}) = x$ *for all* $\hat{x} \in A \cap \hat{X}$, $f(T(F)) = F(f)$ *for all* $F \in A$ *and* $f \in B$, *and* $(1-\delta)\|F\| \le \|T(F)\| \le (1+\delta)\|F\|$ *for all* $F \in A$.

THEOREM. *For a Banach space* X , X^{**} *is smooth at* $\hat{x} \in S(\hat{X})$ *if and only if every support mapping* $x \mapsto f_x$ *of* X *into* X^* *is continuous on* $S(X)$ *at* x *when* X *has the norm-topology and* X^* *has the* $\sigma(X^*, X^{**})$*-topology.*

Proof. Suppose that X^{**} is smooth at $\hat{x} \in S(\hat{X})$. Then every support mapping $F \mapsto \mathsf{F}_F$ of X^{**} into X^{***} is continuous on $S(X^{**})$ at $\hat{x}$ when X^{**} has the norm-topology and X^{***} has the $\sigma(X^{***}, X^{**})$-topology, [3, p. 107]. Every support mapping $x \mapsto f_x$ of X into X^* induces a support mapping $\hat{x} \mapsto \hat{f}_x$ of $\hat{X}$ into $\hat{X}^*$ which can be extended to a support mapping $F \mapsto \mathsf{F}_F$ of X^{**} into X^{***} , and so it is continuous on $S(X)$ at x when X has the norm-topology and X^* has the $\sigma(X^*, X^{**})$-topology.

Suppose that X^{**} is not smooth at $\hat{x} \in S(\hat{X})$; that is, there exists an $\hat{f}_x \in S(\hat{X}^*)$ and $\mathsf{F} \in S(X^{***})$ where $\mathsf{F} \ne \hat{f}_x$ such that $f_x(x) = \mathsf{F}(\hat{x}) = 1$. Now there exists an $F \in S(X^{**})$ and an $r > 0$ such that $(\mathsf{F}-\hat{f}_x)(F) \ge r$. Using the Principle of Local Reflexivity as in [11], we have that for every $\delta > 0$ there exists a mapping $T^\delta : \operatorname{span}\{\hat{f}_x, \mathsf{F}\} \to X^*$ such that $T^\delta(\hat{f}_x) = f_x$, $\hat{x}(T^\delta \mathsf{F}) = \mathsf{F}(\hat{x}) = 1$, $F\left(T^\delta \mathsf{F}-f_x\right) = (\mathsf{F}-\hat{f}_x)(F) \ge r$, and $1 - \delta \le \|T^\delta \mathsf{F}\| \le 1 + \delta$. Now consider a sequence $\{\delta_n\}$ where $\delta_n \to 0+$. The sequence $\{f_n\}$ in $S(X^*)$, where $f_n = \left(T^{\delta_n}\mathsf{F}\right)/\left(\|T^{\delta_n}\mathsf{F}\|\right)$, has the property that $f_n(x) \to 1$, but $\liminf F(f_n-f_x) \ge r$, so $\{f_n\}$ is not $\sigma(X^*, X^{**})$-convergent to f_x . It follows from the lemma that there exists a support mapping which is not

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continuous on $S(X)$ at x when X has the norm-topology and X^* has the $\sigma(X^*, X^{**})$-topology.

We deduce, from the continuity characterisation of strong differentiability of the norm [3, p. 107], the following implication of our theorem.

COROLLARY 1. *A Banach space* X *with norm strongly differentiable at* $x \in S(X)$ *has* X^{**} *smooth at* $\hat{x} \in S(\hat{X})$.

For a non-reflexive Banach space X , X^{***} has a non-smooth point in $S(\hat{X}^*)$, [5]. In [8] Singer has shown that for a non-reflexive Banach space X with a certain subspace property, X^{**} has a non-smooth point in $S(\hat{X})$. Our theorem provides us with many examples of non-reflexive Banach spaces with smooth embedding. Using Restrepo's result, [7], that every Banach space X with separable dual X^* can be equivalently renormed to have norm strongly differentiable on $S(X)$, we can specify a significant class of such spaces.

COROLLARY 2. *A Banach space* X *with separable dual* X^* *can be equivalently renormed to have* X^{**} *smooth on* $S(\hat{X})$.

It is of interest to observe that a Banach space X with separable dual X^* , so renormed does not necessarily have X^{**} smooth. If c_0 is so renormed, then $\hat{c}_0$ is smooth; but Day has shown that m cannot be given an equivalent smooth norm [2, p. 522]. However, it is possible to have c_0 so renormed that m is strongly differentiable on $D(l_1)$, [4, p. 395].

Note added in proof (13 June 1975). Dr E.N. Dancer has pointed out that the theorem can also be proved using the fact that $\hat{B}(X^*)$ is $\sigma(X^{***}, X^{**})$-dense in $B(X^{***})$ and without using the principle of local reflexivity.

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