Canad. Math. Bull. Vol. 19 (2), 1976

AMENABILITY AND SUBSTANTIAL SEMIGROUPS

BY
JAMES C. S. WONG[(1)]

ABSTRACT. In this paper, we introduce the concept of topological left substantial subsemigroup of a locally compact semigroup S and prove that if T is such a subsemigroup in S and $M(T)$ is the measure algebra of T, then $M(T)^*$ is topological left amenable iff $M(S)^*$ is topological left amenable, an extension of a similar result for discrete semigroups.

1. **Introduction.** Let S be a semigroup and T a subset of S. T is called left thick in S if for every finite subset F or S, there is some $s \in S$ such that $Fs \subset T$. In [5], Mitchell proves that a left thick subsemigroup T (in particular, any left ideal) of S is left amenable if and only if S is left amenable. In this paper, we extend the concept of left thick subsets to locally compact semigroups and prove an analogue (and also an extension) of Mitchell's result for topological left invariant means.

2. **Notations.** For definition of topological left invariant means on locally compact semigroups, we shall follow Wong [6]. Let S be a locally compact semigroup, $M(S)$ its measure algebra with total variation norm and convolution as multiplication and $M_0(S)$ its probability measures. A Borel subset T of S is called *topological left substantial* if the following condition is satisfied: For each compact set $K \subset S$, there is some measure $\mu_K \in M_0(S)$ such that $\mu * \mu_K(T) = 1$ for any $\mu \in M_0(S)$ with $\mu(K) = 1$. In [9], a Borel subset T of S is called *topological left thick* if for each $0 < \varepsilon \leq 1$ and each compact subset K in S, there is some measure $\mu_{(\varepsilon,K)} \in M_0(S)$ such that $\mu * \mu_{(\varepsilon,K)}(T) > 1 - \varepsilon$ for any $\mu \in M_0(S)$ with $\mu(K) = 1$. It is easy to see that any topological left substantial subset of S is necessarily topological left thick, and that for discrete semigroups S, they both coincide with the concept of left thickness introduced in Mitchell [5] (see Wong [9, Theorem 4.2] for a proof). Observe that if T is topological left substantial, the measures $\mu_K \in M_0(S)$ can be chosen such that $\mu_K(T) = 1$. To prove this, let $\phi \neq K \subset S$ be compact. Choose $k \in K$ and let $K_1 = Kk \cup \{k\}$ which is also compact. Consider $\varepsilon_k * \mu_{K_1} \in M_0(S)$, where ε_k is the Dirac measure at k. Since $k \in K$, $\varepsilon_k * \mu_{K_1}(T) = 1$. Moreover, if $\mu \in M_0(S)$ satisfies $\mu(K) = 1$,

Received by the editors March 3, 1975.
AMS Subject Classification (Mathematical Reviews 1973): 22.05, 22.20, 42.50 and 42.56.
Key Words and Phrases: Locally compact semigroups, Measure algebras, Topological left invariant means, Left substantial semigroups, Left ideals.
(1) Research supported by the National Research Council of Canada Grant No. A8227.

then $\mu*(\varepsilon_k*\mu_{K_1})(T)=(\mu*\varepsilon_k)*\mu_{K_1}(T)=1$, since $\mu*\varepsilon_k(K_1)=\int_K \xi_{K_1}(xk)\,d\mu(x)=\mu(K)=1$ (where ξ_{K_1} is the characteristic function of K_1). The measures $\varepsilon_k*\mu_{K_1}$ now have the required property (see also Mitchell [5, §5 Remark (a) p. 256]).

3. Main results.

THEOREM 3.1. *Let T be a topological left substantial locally compact Borel subsemigroup of a locally compact semigroup S. If there is a net $\mu_\beta\in M_0(S)$ such that $\|\mu*\mu_\beta-\mu_\beta\|\to 0$ for every $\mu\in M_0(S)$ with $\mu(T)=1$, then $M(S)^*$ has a topological left invariant mean.*

Proof. Assume there is a net μ_β, $\beta\in E$ such that $\|\mu*\mu_\beta-\mu_\beta\|\to 0$ for each $\mu\in M_0(S)$ with $\mu(T)=1$. Since T is topological left substantial, for each $K\subset S$ compact, there is some $\mu_K\in M_0(S)$ with $\mu_K(T)=1$ such that $\mu*\mu_K(T)=1$ for any $\mu\in M_0(S)$ with $\mu(K)=1$. We now use a standard result on iterated limits (Kelley [4, Theorem 4, p. 69]) to construct a net $\mu_\alpha\in M_0(S)$ such that $\|\mu*\mu_\alpha-\mu_\alpha\|\to 0$ for each $\mu\in M_0(S)$. Let $D=\{K\}$ be the collection of a compact subset in S. Consider the product directed set $D\times\Pi\{E:K\in D\}$. For each $\alpha=(K,f)\in D\times\Pi\{E:K\in D\}$, define $\mu_\alpha=\mu_K*\mu_{f(K)}$. We want to show that μ_α satisfies the required condition by first showing that $\|\mu*\mu_\alpha-\mu_\alpha\|\to 0$ for each $\mu\in M_0(S)$ with compact support. Let K_0 be any (arbitrary but fixed) compact subset of S and $\mu\in M_0(S)$ with $\mu(K_0)=1$. Consider the product directed sets $D_0\times E$ and $D_0\times\Pi\{E:K\in D_0\}$ where $D_0=\{K\in D:K\supset K_0\}$.

Define the maps

$$R:D_0\times\Pi\{E:K\in D_0\}\to D_0\times E$$

and

$$V:D_0\times E\to M(S)\quad\text{(with norm topology)}$$

by

$$R(K,g)=(K,g(K))$$

and

$$V(K,\beta)=\mu*\mu_K*\mu_\beta-\mu_K*\mu_\beta.$$

The iterated limit $\lim_{K\supset K_0}\lim_\beta \mu*\mu_K*\mu_\beta-\mu_K*\mu_\beta$ (in norm topology of $M(S)$) exists and is equal to zero. In fact, for any $K\supset K_0$,

$$\|\mu*\mu_K*\mu_\beta-\mu_K*\mu_\beta\|\le\|\mu*\mu_K*\mu_\beta-\mu_\beta\|+\|\mu_K*\mu_\beta-\mu_\beta\|\xrightarrow{\beta}0.$$

since $\mu(K)=\mu(K_0)=1$ and hence $\mu*\mu_K(T)=\mu_K(T)=1$. (T is topological left substantial.) By Kelley [4, Theorem 4.1, p. 69], the net

$$V\circ R(K,g)=\mu*\mu_K*\mu_{g(K)}-\mu_K*\mu_{g(K)}\to 0$$

with respect to (K,g). It follows that

$$\lim_\alpha\|\mu*\mu_\alpha-\mu_\alpha\|=\lim_{(K,f)}\|\mu*\mu_K*\mu_{f(K)}-\mu_K*\mu_{f(K)}\|$$

$$=\lim_{(K,g)}\|\mu*\mu_K*\mu_{g(K)}-\mu_K*\mu_{g(K)}\|=0$$

for each $\mu\in M_0(S)$ with $\mu(K_0)=1$.

https://doi.org/10.4153/CMB-1976-036-7 Published online by Cambridge University Press

Since the measures in $M_0(S)$ with compact supports are norm dense in $M_0(S)$, the same is true for each $\mu \in M_0(S)$. By [6, Theorem 3.1], $M(S)^*$ has a topological left invariant mean.

THEOREM 3.2. *Let T be a topological left substantial locally compact Borel subsemigroup of a locally compact semigroup S, then $M(T)^*$ has a topological left invariant mean iff $M(S)^*$ has a topological left invariant mean.*

Proof. Assume that $M(S)^*$ has a topological left invariant mean. Since T is topological left substantial, it is topological left thick. By Wong [9, Theorem 4.1, (1) implies (2) and Note (a) after Theorem 4.1], there is a topological left invariant mean M on $M(S)^*$ such that $M(\chi_T)=1$ where χ_T is the characteristic functional of T in S defined by $\chi_T(\mu)=\mu(T)$, $\mu \in M(S)$. Consequently, $M(T)^*$ has a topological left invariant mean by [8, Theorem 4.1].

Conversely, assume that $M(T)^*$ has a topological left invariant mean. Then there is a net $\nu_\beta \in M_0(T)$ such that $\|\nu * \nu_\beta - \nu_\beta\| \to 0$ for each $\nu \in M_0(T)$. Let μ_β be the unique measure in $M_0(T)$ such that $\mu_\beta(T)=1$ and $\mu_{\beta|T}=\nu_\beta$ where $\mu_T=\mu\,|_T \in M_0(T)$ is the restriction of $\mu \in M_0(S)$ to the Borel subsets of T. (In fact, the correspondence $\mu \to \mu_T$ is an isometric isomorphism between the convolution algebra $M(T)$ and the subalgebra of all measures μ in $M(S)$ with $|\mu|\,(T')=0$, see for example [7, Lemmas 3.2 and 3.3] and [8, Lemma 3.1]). Now if $\mu \in M_0(S)$ and $\mu(T)=1$, then $\nu=\mu_T \in M_0(T)$ and

$$\|\mu * \mu_\beta - \mu_\beta\| = \|(\mu * \mu_\beta)_{|T} - \mu_{\beta|T}\| = \|\mu_T * \mu_{\beta|T} - \mu_{\beta|T}\| = \|\nu * \nu_\beta - \nu_\beta\| \to 0.$$

By Theorem 3.1, $M(S)^*$ has a topological left invariant mean. This completes the proof.

COROLLARY 3.3. *Let T be a left ideal of a locally compact semigroup S, then $M(T)^*$ has a topological left invariant mean iff $M(S)^*$ has a topological left invariant mean.*

Proof. It suffices to show that every left ideal is topological left substantial. Let $t \in T$. If $K \subset S$ is compact, then $Kt \subset ST \subset T$. Consider the Dirac measure ε_t at t. For any $\mu \in M_0(S)$ with $\mu(K)=1$, we have $\mu * \varepsilon_t(T) = \int \xi_T(xt)\, d\mu(x) = \int_K \xi_T(xt)\, d\mu(x) = \mu(K) = 1$. Hence T is topological left substantial.

REMARKS.

(a) Theorem 3.2 is a topological analogue and also an extension of a result of Mitchell in [5, Theorem 9, p. 260] for discrete semigroups.

(b) The sufficiency part of Theorem 3.2 remains valid if we assume that T is only topological left thick instead of the (stronger) condition that T be topological left substantial. This is quite evident in the proof of the theorem. However, we were unable to do the same for the necessity part. (See Wong [8] and [9] for other results in this direction.)

References

1. M. Day, *Amenable semigroups*, Illinois J. Math., **1** (1957), 509–544.

2. F. P. Greenleaf, *Invariant Means on Topological Groups*, Van Nostrand Math. Studies, 1969.

3. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, Springer-Verlag, 1963.

4. J. Kelley, *General Topology*, Van Nostrand, New York, 1968.

5. T. Mitchell, *Constant functions and left invariant means on semigroups*, Trans. Amer. Math. Soc., **119** (1965), 244–261.

6. James C. S. Wong, *An ergodic property of locally compact amenable semigroups*, Pacific J. Math. **48** (1973) 615–619.

7. James C. S. Wong, *Absolutely continuous measures on locally compact semigroups*, Canadian Math. Bull. 18(1), 1975.

8. James C. S. Wong, *A characterization of locally compact amenable semigroups*, to appear.

9. James C. S. Wong, *A characterization of topological left thick subsets in locally compact left amenable semigroups*, to appear.

Department of Mathematics,
Statistics & Computing Science,
University of Calgary,
Calgary, Alberta