Representing Multipliers of the Fourier Algebra on Non-Commutative L^p Spaces

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Abstract. We show that the multiplier algebra of the Fourier algebra on a locally compact group G can be isometrically represented on a direct sum on non-commutative L^p spaces associated with the right von Neumann algebra of G. The resulting image is the idealiser of the image of the Fourier algebra. If these spaces are given their canonical operator space structure, then we get a completely isometric representation of the completely bounded multiplier algebra. We make a careful study of the non-commutative L^p spaces we construct and show that they are completely isometric to those considered recently by Forrest, Lee, and Samei. We improve a result of theirs about module homomorphisms. We suggest a definition of a Figa-Talamanca–Herz algebra built out of these non-commutative L^p spaces, say $A_p(\widehat{G})$. It is shown that $A_2(\widehat{G})$ is isometric to $L^1(G)$, generalising the abelian situation.

1 Introduction

The Fourier algebra $A(G)$ is, for a locally compact group G, the space of coefficient functionals $s \mapsto (\lambda(s)\xi|\eta)$ for $s \in G$, where $\xi, \eta \in L^2(G)$. Here λ denotes the left-regular representation of G on $L^2(G)$. For an abelian group, $A(G)$ is nothing but the Fourier transform of $L^1(\widehat{G})$, where $\widehat{G}$ is the Pontryagin dual of G. Eymard defined $A(G)$ for general G in [7]. We can also identify $A(G)$ as the predual of the group von Neumann algebra $VN(G)$; see [34, Chapter VII, Section 3].

In this paper we shall be interested in the multiplier algebra of $A(G)$. This can either be thought of abstractly as the double centraliser algebra (see [17]) of $A(G)$ or, as $A(G)$ is a regular algebra of functions on G, as the space of continuous functions f such that $fa \in A(G)$ for each $a \in A(G)$; see [31], for example. There is now much evidence that $A(G)$ is often best viewed as an *operator space* when given the standard operator space structure as the predual of $VN(G)$. Then it is natural to consider only the *completely bounded* multipliers, leading to $M_{cb}A(G)$ (see [31] or [4]). In [24] a representation of $M_{cb}A(G)$ on $\mathcal{CB}(B(L^2(G)))$ was defined, extending a representation of $M(G)$ defined in [9]. It was shown that these representations are commutants of each other, hence in some sense extending Pontryagin duality. Similar ideas were considered for Kac algebras in [21] and have been extended (along with the commutation ideas) to Locally Compact Quantum Groups in [18].

It is shown in [4] that both $MA(G)$ and $M_{cb}A(G)$ are dual spaces, in such a way that the algebra products are separately weak*-continuous (so these are *dual Banach algebras*); see also [31, Section 6.2]. Now, $\mathcal{CB}(B(L^2(G)))$ is also a dual Banach algebra, and the representation of $M_{cb}A(G)$ constructed in [24] is weak*-weak*-continuous.

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However, it was shown in [3, Corollary 3.8] (and extended in [36] to the completely bounded case) that a dual Banach algebra $\mathcal{A}$ admits an isometric, weak*-weak*-continuous representation on $\mathcal{B}(E)$ for some *reflexive* Banach space E. The space E is built as the large direct sum of real interpolation spaces, and is rather abstract.

In this paper, we shall show that we can represent $MA(G)$ on a direct sum of non-commutative L^p spaces associated with $VN(G)$; we can also represent $M_{cb}A(G)$ on the same space, if it is equipped with the canonical operator space structure. Indeed, our construction is motivated by that of Young in [39]. As Young did not consider multipliers, we sketch his ideas in Section 2 below.

Once we have motivated looking at (non-commutative) L^p spaces, we discuss weights on $VN(G)$ and non-commutative L^p spaces for (possibly) non-semifinite von Neumann algebras in Section 3. This will involve introducing the complex interpolation method. In Section 4 we apply these ideas to the Fourier algebra, leading to a scale of spaces $L^p(\widehat{G})$, for $1 < p < \infty$, which are $A(G)$-modules. We make a careful study of these spaces and prove some approximation results that allow us to work with functions instead of abstract operators in the von Neumann algebra. With this perspective, the $A(G)$-module actions are just point-wise multiplications of functions. We show that our spaces are (completely) isometrically isomorphic to the two families of spaces constructed in [8, Section 6]. We think that our construction is easier and more natural than that of [8], although we have to worry more about the details of the complex interpolation method. The payoff is that, for example, we can easily extend a cohomological result from [8], which we can show to hold for all values of p (and not just $p \geq 2$).

In Section 5 we prove our representation result. Let $p_n \to 1$ in $(1, \infty)$, and let E be the ℓ^2 direct sum of the spaces $L^{p_n}(\widehat{G})$. Then $MA(G)$ is weak*-weak*-continuously isometric to the idealiser of $A(G)$ in $\mathcal{B}(E)$. If we equip E with the canonical operator space structure, then $M_{cb}A(G)$ is weak*-weak*-continuously completely isometric to the idealiser of $A(G)$ in $\mathcal{CB}(E)$. As arguments involving multipliers often use bounded approximate identities, it is worth stressing that our results hold for all locally compact groups G. As hinted at in Section 2, Figa-Talamanca–Herz algebras make a natural appearance, and with our new tools, we define a notion of what $A_p(\widehat{G})$ should be for a non-abelian group G. We show that $A_2(\widehat{G})$ is canonically isometric to $L^1(G)$, but we have been unable to decide if $A_p(\widehat{G})$ is always an algebra.

For Banach algebra notions, we follow [2, 25]; we always write E^* for the dual of a Banach or Operator space E, reserving the notation A' for the commutant. We shall only use standard facts about Operator spaces, for which we refer the reader to [5, 28]. In the few places where we use matrix calculations, we shall simply write $\| \cdot \|$ for the norm on $\mathbb{M}_n(E)$ for any n.

2   Group Convolution Algebras

In this section we quickly review Young's construction in [39, Theorem 4], as applied to multipliers. Let G be a locally compact group, and consider the group convolution algebra $L^1(G)$. The multiplier algebra of $L^1(G)$ can be isometrically isomorphically identified with $M(G)$, the measure algebra of G. This is Wendel's theorem ([37] or

[2, Theorem 3.3.40]).

Let (p_n) be some sequence in $(1, \infty)$ converging to 1. Let E be the direct sum, in an ℓ^2 sense, of the spaces $L^{p_n}(G)$. To be exact, E consists of sequences (ξ_n) where, for each n, $\xi_n \in L^{p_n}(G)$ with

$$\|(\xi_n)\| := \left(\sum_n \|\xi_n\|_{p_n}^2 \right)^{1/2} < \infty.$$

Thus E is reflexive. Then $M(G)$ acts contractively on each $L^{p_n}(G)$ space by convolution, and hence also on E, leading to a contractive homomorphism $\theta \colon M(G) \to \mathcal{B}(E)$.

Theorem 2.1 *With notation as above, θ is isometric and weak*-weak*-continuous.*

We first introduce some further concepts. We write $\widehat{\otimes}$ for the (completed) projective tensor product (see [2, Appendix A3], for example). For any reflexive Banach space F, we thus have that $\mathcal{B}(F) = (F\widehat{\otimes}F^*)^*$. Let $\lambda_p \colon L^1(G) \to \mathcal{B}(L^p(G))$ be the left-regular representation, and let $(\lambda_p)_* \colon L^p(G)\widehat{\otimes}L^{p'}(G) \to L^\infty(G)$ be the adjoint. Here p' is the conjugate index to p, so that $L^p(G)^* = L^{p'}(G)$. For $a \in L^1(G), \xi \in L^p(G)$, and $\eta \in L^{p'}(G)$, we see that

$$\left\langle (\lambda_p)_*(\xi \otimes \eta), a \right\rangle = \left\langle \eta, \lambda_p(a)(\xi) \right\rangle = \int_G \int_G \eta(t)a(s)\xi(s^{-1}t) \, ds \, dt = \langle \omega_{\xi,\eta}, a \rangle.$$

Here $\omega_{\xi,\eta}$ denotes the function $s \mapsto \int_G \xi(s^{-1}t)\eta(t) \, dt$. Thus $\omega_{\xi,\eta}$ is a member of the Figa-Talamanca–Herz algebra $A_p(G)$, identified as a subalgebra of $C_0(G) \subseteq L^\infty(G)$. For further details see [12, 13].

This then suggests an abstract way to define $\tilde{\theta} \colon M(G) \to \mathcal{B}(L^p(G))$, namely

$$\left\langle \eta, \tilde{\theta}(\mu)(\xi) \right\rangle = \langle \mu, \omega_{\xi,\eta} \rangle \qquad \left(\mu \in M(G), \xi \in L^p(G), \eta \in L^{p'}(G) \right).$$

By the above calculation, this extends θ. Furthermore, if $\xi, \eta \in C_{00}(G)$, the space of compactly support continuous functions, then $\xi \in L^p(G)$, $\eta \in L^{p'}(G)$, and for $\mu \in M(G)$ we see that

$$\left\langle \eta, \tilde{\theta}(\mu)(\xi) \right\rangle = \int_G \int_G \xi(s^{-1}t)\eta(t) \, dt \, d\mu(s) = \left\langle \eta, \mu * \xi \right\rangle,$$

where $\mu * \xi$ has the unambiguous meaning of μ convolved with ξ. As such ξ and η are dense, we are justified in saying that $\tilde{\theta}$ is simply the convolution action of $M(G)$ on $L^p(G)$.

Proof of Theorem 2.1 Consider the adjoint map $\theta_* \colon E\widehat{\otimes}E^* \to M(G)^*$ given by

$$\left\langle \theta_*(\xi \otimes \eta), \mu \right\rangle = \left\langle \eta, \theta(\mu)(\xi) \right\rangle = \sum_n \left\langle \eta_n, \tilde{\theta}(\mu)(\xi_n) \right\rangle = \sum_n \langle \mu, \omega_{\xi_n,\eta_n} \rangle,$$

where $\xi = (\xi_n) \in E, \eta = (\eta_n) \in E^*$, and $\mu \in M(G)$. In particular, θ_* maps into $C_0(G)$, the predual of $M(G)$, so that θ is weak*-weak*-continuous.

For $f, g \in C_{00}(G)$, we have that $\omega_{f,g} = g * \check{f}$ as functions, where $\check{f}(s) = f(s^{-1})$ for $s \in G$. Furthermore, we have that

$$\lim_{p \to 1} \|f\|_p = \|f\|_1, \quad \lim_{p' \to \infty} \|g\|_{p'} = \|g\|_\infty.$$

For any $g \in C_{00}(G)$ and $\epsilon > 0$, we can find some $f \in C_{00}(G)$ with $\|f\|_1 = 1$ and $\|g * \check{f} - g\|_\infty < \epsilon$ (for example, see the proof of [2, Lemma 3.3.22]). As $p_n \to 1$, we can find n with $\|g\|_{p_n'} < (1 + \epsilon)\|g\|_\infty$ and $\|f\|_{p_n} < 1 + \epsilon$. It follows that

$$|\langle \mu, g \rangle| \geq |\langle \mu, \omega_{f,g} \rangle| - \epsilon\|\mu\|,$$

and that

$$\|\omega_{f,g}\|_{A_{p_n}(G)} \leq \|f\|_{p_n}\|g\|_{p_n'} < (1 + \epsilon)^2\|g\|_\infty.$$

By taking suitable supremums, it now follows easily that θ is an isometry. ∎

For a Banach algebra $\mathcal{A}$, we say that $\mathcal{A}$ is *faithful* if for $a \in \mathcal{A}$, when $bac = 0$ for all $b, c \in \mathcal{A}$, then $a = 0$. We shall always assume that our algebras are faithful: notice that if $\mathcal{A}$ is unital, or has an approximate identity, then $\mathcal{A}$ is faithful. A pair (L, R) of linear maps $\mathcal{A} \to \mathcal{A}$ is a *multiplier* (or *centraliser*) if

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad aL(b) = R(a)b \qquad (a, b \in \mathcal{A}).$$

The Closed Graph Theorem then shows that $L, R \in \mathcal{B}(\mathcal{A})$. For further details see [2, 17] or [25, Section 1.2]. Indeed, [25, Theorem 1.2.4] shows that if $L, R \colon \mathcal{A} \to \mathcal{A}$ are any maps with $aL(b) = R(a)b$ for $a, b \in \mathcal{A}$, then (L, R) is already a multiplier. Let $M(\mathcal{A})$ be the space of multipliers, normed by embedding into $\mathcal{B}(\mathcal{A}) \times \mathcal{B}(\mathcal{A})$, and made into an algebra for the product $(L, R)(L', R') = (LL', R'R)$. Notice that $\mathcal{A}$ embeds (as $\mathcal{A}$ is faithful) into $M(\mathcal{A})$ by $a \mapsto (L_a, R_a)$, where $L_a(b) = ab, R_a(b) = ba$ for $a, b \in \mathcal{A}$.

Then Wendel's Theorem tells us that for $(L, R) \in M(L^1(G))$ there exists a unique $\mu \in M(G)$ such that $L(a) = \mu a$ and $R(a) = a\mu$ for $a \in L^1(G)$. Indeed, from the proof of [2, Theorem 3.3.40], we have that μ is the weak*-limit of $(L(e_\alpha))$ in $M(G)$, where (e_α) is a bounded approximate identity for $L^1(G)$. It is then easy to show that $L(a) = \mu a$ for $a \in L^1(G)$. Notice then that $R(a)b = aL(b) = a(\mu b) = (a\mu)b$ for $a, b \in L^1(G)$, so as $L^1(G)$ is faithful, $R(a) = a\mu$ as required.

Theorem 2.2 *With notation as above, the image of $\tilde{\theta} \colon M(G) \to \mathcal{B}(E)$ is exactly the idealiser of $\theta(L^1(G))$, namely*

$$\mathfrak{I} = \left\{ T \in \mathcal{B}(E) : T\theta(a), \theta(a)T \in \theta\left(L^1(G)\right) \ \left(a \in L^1(G)\right) \right\}.$$

Proof For $\mu \in M(G)$, we have $\tilde{\theta}(\mu)\theta(a) = \theta(\mu a)$ and $\theta(a)\tilde{\theta}(\mu) = \theta(a\mu)$ for $a \in L^1(G)$, so that $\tilde{\theta}(M(G)) \subseteq \mathfrak{I}$.

Conversely, let $T \in \mathcal{I}$ and define $L, R \colon L^1(G) \to L^1(G)$ by

$$L(a) = \theta^{-1}\big(T\theta(a)\big), \quad R(a) = \theta^{-1}\big(\theta(a)T\big) \qquad (a \in L^1(G)),$$

which makes sense, as θ is injective onto its range. For $a, b \in L^1(G)$ we see that $\theta(a)\theta(L(b)) = \theta(a)T\theta(b) = \theta(R(a))\theta(b)$, so that $aL(b) = R(a)b$. Thus $(L, R) \in M(L^1(G))$. Hence there exists $\mu \in M(G)$ with $L(a) = \mu a$ for $a \in L^1(G)$, so that $\tilde{\theta}(\mu)\theta(a) = T\theta(a)$ for $a \in L^1(G)$.

By the construction of E, we see that $\{\theta(a)\xi : a \in L^1(G), \xi \in E\}$ is linearly dense in E, from which it follows that $T = \tilde{\theta}(\mu)$, completing the proof. ∎

Notice that we implicitly used the Closed Graph Theorem in invoking [25, Theorem 1.2.4]. In the completely bounded setting, this would not be available to us, and indeed, it is unclear to the author if a direct analogue of this result would be true. However, if $\mathcal{A}$ is commutative (or has a bounded approximate identity), then L and R are closely related, allowing a modification of the proof to work; see Theorem 5.4. In relation to this, it is interesting to note that [18] works with *one-sided* multipliers (or centralisers).

It is classical that $L^p(G)$ can be described as a complex interpolation space between $L^1(G)$ and $L^\infty(G)$; see below for definitions or [1, Chapter 4]. We can recover the action of $L^1(G)$ on $L^p(G)$ by interpolation, but some care is needed. Indeed, obviously $L^1(G)$ is an $L^1(G)$-bimodule over itself, and so by duality, $L^\infty(G)$ is an $L^1(G)$-bimodule. However, notice that the resulting left action of $L^1(G)$ on $L^\infty(G)$ is *not* the usual convolution action. With this in mind, the constructions in Section 4 should appear less artificial.

3   Non-Commutative L^p Spaces

In this section we sketch the complex interpolation approach to non-commutative L^p spaces; see [15, 35].

3.1   Weights on Group von Neumann Algebras

For a locally compact group G, let λ and ρ be, respectively, the left- and right-regular representations, defined by

$$\big(\lambda(s)\xi\big)(t) = \xi(s^{-1}t), \quad \big(\rho(s)\xi\big)(t) = \rho(ts)\nabla(s)^{1/2} \qquad \big(\xi \in L^2(G), s, t \in G\big).$$

Here ∇ is the modular function on G. For $f \in L^1(G)$, we shall write $\lambda(f)$ and $\rho(f)$ for the operators induced by integration, for example

$$\big(\rho(f)\xi\big)(s) = \int_G f(t)\xi(st)\nabla(t)^{1/2}\,dt \qquad \big(\xi \in L^2(G)\big).$$

Then the group von Neumann algebra $VN(G)$ is the von Neumann algebra generated by λ, so $VN(G) = \lambda(G)''$. Similarly, the right group von Neumann algebra,

denoted here by $VN_r(G)$, is generated by ρ. We have that $VN(G)' = VN_r(G)$ and $VN_r(G)' = VN(G)$; see [34, Chapter VII, Section 3].

An alternative way to construct $VN(G)$ is to start with $C_{00}(G)$, considered as a left Hilbert algebra. The inner-product is inherited from $L^2(G)$; the product is convolution, and the involution is $f^\sharp(s) = \overline{f(s^{-1})}\nabla(s)^{-1}$ for $f \in C_{00}(G), s \in G$. See [34] or [33] for further details on left Hilbert algebras. One word of caution: for $f \in C_{00}(G)$ (or more generally, for *right bounded* elements of $L^2(G)$) we can define $\pi_r(f) \in VN_r(G)$ (using the notation of [34]). This is *not* equal to $\rho(f)$; we have $\pi_r(f) = \rho(K(f))$ for K defined below.

At this point, we shall stress that henceforth, for functions a, b on G, we denote the convolution product by ab (when this makes sense) and the point-wise product by $a \cdot b$. An exception is that ∇ always acts by point-wise multiplication.

The left Hilbert algebra leads naturally to a weight φ on $VN(G)$. This weight is explored in detail by Haagerup in [11, Section 2]. We let $\mathfrak{n}_\varphi = \{x \in VN(G) : \varphi(x^*x) < \infty\}$ and $\mathfrak{m}_\varphi = \lim \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$, and extend φ to $\mathfrak{m}_\varphi$ in the usual way. Let us just note that

$$\varphi(\lambda(f)) = f(e_G) \qquad (f \in C_{00}(G)^2),$$

where e_G is the unit of G, and $C_{00}(G)^2 = \lim\{fg : f, g \in C_{00}(G)\}$.

Let (π, H, Λ) be the GNS construction for $(VN(G), \varphi)$. We may hence identify H with $L^2(G)$ by $\Lambda(\lambda(f)) = f$ for $f \in C_{00}(G)$ (or more generally for left bounded $f \in L^2(G)$). Henceforth we shall drop π and always regard $VN(G)$ as acting on $L^2(G)$. The *modular conjugation* is the map

$$J\colon L^2(G) \to L^2(G), \quad J\xi(s) = \overline{\xi(s^{-1})}\nabla(s)^{-1/2} \qquad \left(\xi \in L^2(G), s \in G\right).$$

We define a linear version of J to be K, where $K(\xi) = J(\bar{\xi})$ for $\xi \in L^2(G)$. We define the "check map" by $\check{\xi}(s) = \xi(s^{-1})$, so $K\xi = \check{\xi}\nabla^{-1/2}$. We have that $VN_r(G) = VN(G)' = JVN(G)J$, and

$$\lambda(f) = J\rho(\bar{f})J = K\rho(f)K \qquad \left(f \in L^1(G)\right).$$

The *modular operator* is given by point-wise multiplication by ∇, and this leads to the *modular automorphism group* $(\sigma_t)_{t\in\mathbb{R}}$ given by $\sigma_t(\cdot) = \nabla^{it}(\cdot)\nabla^{-it}$.

We shall pick a canonical choice of weight φ' on $VN_r(G)$ by

$$\varphi'(x) = \varphi(Jx^*J) \qquad \left(x \in VN_r(G)^+\right).$$

Then $\mathfrak{m}_{\varphi'} = J\mathfrak{m}_\varphi J$, and the formula above defines φ' on $\mathfrak{m}_{\varphi'}$. Let (π', H', Λ') be the GNS construction for φ'. We can identify H' with H by $\Lambda'(x) = J\Lambda(JxJ)$ for $x \in \mathfrak{n}_{\varphi'} = J\mathfrak{n}_\varphi J$. Hence we identify H' with $L^2(G)$ by

$$\Lambda'(\rho(f)) = J\Lambda(\lambda(\bar{f})) = K(f) \qquad (f \in C_{00}(G)).$$

Again, we suppress π' and regard $VN_r(G)$ as acting on $L^2(G)$. Then the modular conjugation for φ' is simply J. The modular automorphism group for φ' is $(\sigma_t')_{t\in\mathbb{R}}$, and this is given by $\sigma_t'(x) = J\sigma_t(JxJ)J$ for $x \in VN_r(G)$. Some care is required when analytically extending this to complex values; indeed, we have $\sigma_z'(x) = J\sigma_{\bar{z}}(JxJ)J$ for analytic x and $z \in \mathbb{C}$. Consequently,

$$\sigma_z'\big(\rho(f)\big) = \rho\big(\nabla^{-iz}f\big) \qquad \left(f \in C_{00}(G), z \in \mathbb{C}\right).$$

3.2 Non-Commutative L^p Spaces

There is a long history to non-commutative L^p spaces, for which we refer the reader to [29]. For a von Neumann algebra $\mathcal{M}$ with a finite normal trace τ, we can simply let $L^p(\mathcal{M}, \tau)$ be the completion of $\mathcal{M}$ with respect to the norm $\|x\|_p = \tau((x^*x)^{p/2})$, for $1 \le p < \infty$. Similar remarks apply to semi-finite traces, although the framework of "measurable operators" gives a realisation of the completed space. See [34, Chapter IX, Section 2] for further details.

For a general von Neumann algebra that might only admit a weight, Haagerup introduced a crossed-product construction of a non-commutative L^p space in [10]. Building on the work of Connes, Hilsum provided a spatial definition of a non-commutative L^p space in [14], and showed that the resulting space was isometrically isomorphic to Haagerup's. By analogy with the commutative case, we might expect the complex interpolation method to play a role. In [20], Kosaki provided a construction of a non-commutative L^p space associated with a von Neumann algebra with a finite weight (that is, a normal state) using the complex interpolation method. He showed that his space is isometrically isomorphic to Haagerup's. In [35], Terp extended a special case of Kosaki's construction to the semi-finite case, and she showed that her L^p space is isometrically isomorphic to Hilsum's (and hence to Haagerup's).

We shall instead follow Izumi's construction in [15], which simultaneously generalises Kosaki's and Terp's constructions. Of particular interest is that in [16], Izumi makes a detailed study of his spaces, introducing bilinear and sesquilinear products, and showing that his L^2 spaces are canonically isometrically isomorphic to the standard Hilbert space constructed from the underlying weight. As such, Izumi's constructions are self-contained (although we note that, technically, he relies upon Terp's work in a proof in [15]).

First let us define the complex interpolation method. See [1] and [28, Section 2.7] for further details. A *compatible couple* of Banach spaces is a pair (E_0, E_1) continuously embedded into a Hausdorff topological vector space X. We can then make sense of the spaces $E_0 \cap E_1$ and $E_0 + E_1$ and define norms on them by

$$\|x\| = \max(\|x\|_{E_0}, \|x\|_{E_1}) \qquad (x \in E_0 \cap E_1),$$

$$\|x\| = \inf\big\{ \|a\|_{E_0} + \|b\|_{E_1} : x = a + b, a \in E_0, b \in E_1 \big\} \qquad (x \in E_0 + E_1).$$

We need X to be Hausdorff to ensure that we get a norm on $E_0 + E_1$. However, once we can form $E_0 + E_1$, we can always just replace X by $E_0 + E_1$.

Let

$$\mathcal{S} = \{z = x + iy \in \mathbb{C} : 0 \le x \le 1, y \in \mathbb{R}\} \quad \text{and}$$

$$\mathcal{S}_0 = \{z = x + iy \in \mathbb{C} : 0 < x < 1, y \in \mathbb{R}\}.$$

We let $\mathcal{F}$ be the space of functions $f \colon \mathcal{S} \to E_0 + E_1$ such that:

(i) f is continuous and bounded, and analytic on $\mathcal{S}_0$;
(ii) for $j = 0, 1$, we have that $\mathbb{R} \mapsto E_j; t \mapsto f(j + it)$ is continuous, bounded, and tends to 0 as $|t| \to \infty$.

For more details on vector-valued analytic functions, see [34, Appendix], for example. We give $\mathcal{F}$ a norm by setting

$$\|f\| = \max_{j=0,1} \sup_{t \in \mathbb{R}} \|f(j+it)\|_{E_0} \qquad (f \in \mathcal{F}).$$

This is a norm, and then $\mathcal{F}$ becomes a Banach space.

For $0 \leq \theta \leq 1$, we define $(E_0, E_1)_{[\theta]}$ to be the subspace of $E_0 + E_1$ consisting of those x such that $x = f(\theta)$ for some $f \in \mathcal{F}$, together with the quotient norm

$$\|x\|_{[\theta]} = \inf\{\|f\| : f \in \mathcal{F}, f(\theta) = x\}.$$

The following is proved in [1, Theorem 4.1.2].

Theorem 3.1   *With notation as above, we have norm decreasing inclusions $E_0 \cap E_1 \to (E_0, E_1)_{[\theta]} \to E_0 + E_1$. Let (F_0, F_1) be another pair of compatible Banach spaces, and let $T: E_0 + E_1 \to F_0 + F_1$ be a linear map such that for $j = 0, 1$, $T(E_j) \subseteq F_j$, and the restriction $T: E_j \to F_j$ is bounded. Then*

$$T\big((E_0, E_1)_{[\theta]}\big) \subseteq (F_0, F_1)_{[\theta]}, \quad \|T\| \leq \|T: E_0 \to F_0\|^{1-\theta} \|T: E_1 \to F_1\|^{\theta}.$$

Lemma 3.2   *With notation as above, for $j = 0, 1$ let $T_j \in \mathcal{B}(E_j, F_j)$. There exists $T: E_0 + E_1 \to F_0 + F_1$ with $T|_{E_j} = T_j$ for $j = 0, 1$ if and only if T_0 and T_1 map $E_0 \cap E_1$ into $F_0 \cap F_1$ and agree on $E_0 \cap E_1$.*

Proof   If T_0 and T_1 agree on $E_0 \cap E_1$ and map into $F_0 \cap F_1$, then we try to define T by $T(x_0 + x_1) = T_0(x_0) + T_1(x_1)$ for $x_0 \in E_0, x_1 \in E_1$. This is well defined, because if $x_0 + x_1 = x_0' + x_1'$, then $x_0 - x_0' = x_1' - x_1 \in E_0 \cap E_1$ and so $T_0(x_0) - T_0(x_0') = T_1(x_1') - T_1(x_1) \in F_0 \cap F_1$. The converse is clear. ∎

There is also a bilinear version; see [1, Theorem 4.4.1].

Theorem 3.3   *Let (E_0, E_1), (F_0, F_1) and (G_0, G_1) be compatible couples, and let*

$$T: E_0 \cap E_1 \times F_0 \cap F_1 \to G_0 \cap G_1$$

be a bilinear map such that for some constants M_0, M_1, we have

$$\|T(x_j, y_j)\|_{G_j} \leq M_j \|x_j\|_{E_j} \|y_j\|_{F_j} \qquad (j = 0, 1, x_j \in E_j, y_j \in F_j).$$

For $0 < \theta < 1$, there is a bilinear map

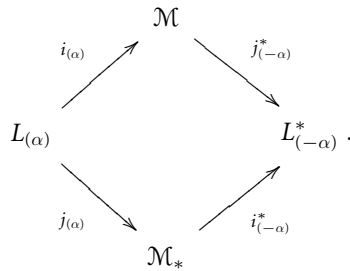$$T_\theta: (E_0, E_1)_{[\theta]} \times (F_0, F_1)_{[\theta]} \to (G_0, G_1)_{[\theta]},$$

which extends T and that is bounded by $M_0^{1-\theta} M_1^{\theta}$.

Now let $\mathcal{M}$ be a von Neumann algebra with normal semi-finite weight φ. Let (H, Λ) be the GNS construction, where we identify $\mathcal{M}$ with a subalgebra of $\mathcal{B}(H)$. Let J be the modular conjugation, and let ∇ be the modular operator. We shall now sketch Izumi's approach to non-commutative L^p spaces. The idea is to turn $(\mathcal{M}, \mathcal{M}_*)$ into a compatible couple; then $L^p(\varphi)$ will be defined as $(\mathcal{M}, \mathcal{M}_*)_{[1/p]}$ for $1 < p < \infty$.

Let (H, Λ) be a GNS construction for φ, so that $\mathfrak{A} = \Lambda(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ is a full left Hilbert algebra in H, which generates $\mathcal{M}$ and induces φ. Let $\mathfrak{A}_0$ be the maximal Tomita algebra associated with $\mathfrak{A}$ (see [34, Chapter VI, Section 2]), and let $\mathfrak{a}_0 = \Lambda^{-1}(\mathfrak{A}_0)$. In particular, each $x \in \mathfrak{a}_0$ is analytic for (σ_t), and $\Lambda(x)$ is in the domain of ∇^α for each $\alpha \in \mathbb{C}$.

For $\alpha \in \mathbb{C}$, we let $L_{(\alpha)}$ be the collection of those $x \in \mathcal{M}$ such that there exists $\varphi_x^{(\alpha)} \in \mathcal{M}_*$ with $\langle y^*z, \varphi_x^{(\alpha)} \rangle = (xJ\nabla^{\overline{\alpha}}\Lambda(y) | J\nabla^{-\alpha}\Lambda(z))$ for $y, z \in \mathfrak{a}_0$. Then $L_{(\alpha)}$ is a subspace of $\mathcal{M}$ that contains $\mathfrak{a}_0^2$, and is hence σ-weakly dense. We norm $L_{(\alpha)}$ by setting $\|x\|_{L_{(\alpha)}} = \max(\|x\|_{\mathcal{M}}, \|\varphi_x^{(\alpha)}\|_{\mathcal{M}_*})$ for $x \in L_{(\alpha)}$. Let $i_{(\alpha)} \colon L_{(\alpha)} \to \mathcal{M}$ be the inclusion map, and let $j_{(\alpha)} \colon L_{(\alpha)} \to \mathcal{M}_*$ be the map $x \mapsto \varphi_x^{(\alpha)}$. These are contractive injections, and $j_{(\alpha)}$ has norm dense range. Izumi proves that we have the following commuting diagram:

$$
\begin{array}{ccc}
 & \mathcal{M} & \\
{\scriptstyle i_{(\alpha)}} \nearrow & & \searrow {\scriptstyle j_{(-\alpha)}^*} \\
L_{(\alpha)} & & L_{(-\alpha)}^* \; . \\
{\scriptstyle j_{(\alpha)}} \searrow & & \nearrow {\scriptstyle i_{(-\alpha)}^*} \\
 & \mathcal{M}_* &
\end{array}
$$

In particular, we have that $\langle y, \varphi_x^{(\alpha)} \rangle = \langle x, \varphi_y^{(-\alpha)} \rangle$ for $x \in L_{(\alpha)}$ and $y \in L_{(-\alpha)}$. By density we have that $i_{(-\alpha)}^*$ and $j_{(-\alpha)}^*$ are injective, and so we can view $(\mathcal{M}, \mathcal{M}_*)$ as a compatible couple. Izumi shows that under this identification, $\mathcal{M} \cap \mathcal{M}_*$ is precisely $L_{(\alpha)}$. We finally set

$$
L_{(\alpha)}^p(\varphi) = (\mathcal{M}, \mathcal{M}_*)_{[1/p]} \qquad (1 < p < \infty).
$$

We shall always view $L_{(\alpha)}^p(\varphi)$ as a subspace of $\mathcal{M} + \mathcal{M}_*$; consequently, by the commuting diagram and Theorem 3.1, we have that $j_{(-\alpha)}^*(x) \in L_{(\alpha)}^p(\varphi)$ for all $x \in L$ and all p.

For most of this paper, we shall actually work just with the case $\alpha = 0$, which is exactly the case that Terp considers in [35]. Set $L = L_{(0)}$, so we actually have the stronger property that $x \in L$ when there exists $\varphi_x \in \mathcal{M}_*$ with

$$
\langle y^*z, \varphi_x \rangle = (Jx^*J\Lambda(z) | \Lambda(y)) = (xJ\Lambda(y) | J\Lambda(z)) \qquad (y, z \in \mathfrak{n}_\varphi).
$$

As shown in [16], there are bilinear maps that satisfy

$$\langle\, \cdot\, ,\, \cdot\, \rangle_{p,(\alpha)}\colon L^p_{(\alpha)}(\varphi) \times L^{p'}_{(-\alpha)}(\varphi) \to \mathbb{C};$$

$$\big\langle\, j^*_{(-\alpha)}(x), j^*_{(\alpha)}(y) \big\rangle = \langle y, \varphi^{(\alpha)}_x \rangle = \langle x, \varphi^{(-\alpha)}_y \rangle,$$

where $1/p + 1/p' = 1$. There are sesquilinear maps that satisfy

$$(\,\cdot\, \mid\, \cdot\,)_{p,(\alpha)}\colon L^p_{(\alpha)}(\varphi) \times L^{p'}_{(\overline{\alpha})}(\varphi) \to \mathbb{C};$$

$$\big(j^*_{(-\alpha)}(x) \mid j^*_{(-\overline{\alpha})}(y) \big)_{p,(\alpha)} = \langle y^*, \varphi^{(\alpha)}_x \rangle = \overline{\langle x^*, \varphi^{(\overline{\alpha})}_x \rangle}.$$

Furthermore, these maps implement dualities between $L^p_{(\alpha)}(\varphi)$ and $L^{p'}_{(-\alpha)}(\varphi)$, and between $L^p_{(\alpha)}(\varphi)$ and $L^{p'}_{(\overline{\alpha})}(\varphi)$, respectively. As such, the dual of $L^p_{(0)}(\varphi)$ can be identified with $L^{p'}_{(0)}(\varphi)$, both linearly and anti-linearly.

We can identify $L^2_{(-1/2)}(\varphi)$ with H_φ, the standard GNS space for φ. Indeed, there is an isometric isomorphism

$$h\colon H_\varphi \to L^2_{(-1/2)}(\varphi); \quad h\big(\Lambda(x)\big) = j^*_{(-1/2)}(x) \qquad (x \in \mathfrak{n}_\varphi).$$

Furthermore, h respects the relevant inner-products, that is

$$(\xi \mid \eta) = \big(h(\xi) \mid h(\eta) \big)_{2,(-1/2)} \qquad (\xi, \eta \in H_\varphi).$$

We can translate this to other values of α by using the fact that there are isometric isomorphisms

$$U_{p,(\beta,\alpha)}\colon L^p_{(\alpha)}(\varphi) \to L^p_{(\beta)}(\varphi); \quad U_{p,(\beta,\alpha)}(j^*_{(-\alpha)}(x)) = j^*_{(-\beta)}\big(\sigma_{i(\beta-\alpha)/p}(x)\big)$$

$$(x \in \mathfrak{a}^2_0, \alpha, \beta \in \mathbb{R}).$$

Then, again for $\alpha, \beta \in \mathbb{R}$, we have that

$$\big(U_{p,(\beta,\alpha)}(\xi) \mid U_{p,(\beta,\alpha)}(\eta) \big)_{p,(\beta)} = (\xi \mid \eta)_{p,(\alpha)} \qquad \big(\xi, \eta \in L^p_{(\alpha)}(\varphi) \big).$$

In particular, there is an isometric isomorphism $k\colon H_\varphi \to L^2_{(0)}(\varphi)$ with

$$k\big(\Lambda(x)\big) = j^*_{(0)}\big(\sigma_{i/4}(x)\big) \quad (x \in \mathfrak{a}^2_0), \qquad (\xi \mid \eta) = \big(k(\xi) \mid k(\eta) \big)_{2,(0)} \quad (\xi, \eta \in H_\varphi).$$

Using convergence theorems for integration, it is easy to show that if (X, μ) is a measure space, and $f \in L^1(\mu) \cap L^\infty(\mu)$, then $f \in L^p(\mu)$ for all $p \in (1, \infty)$, and $\lim_{p \to 1} \|f\|_p = \|f\|_1$. The following is a non-commutative version of this.

Proposition 3.4   *With notation as above, let $x \in L$. Then $\lim_{p \to 1} \|j^*_{(0)}(x)\|_p = \|\varphi_x\|$, where $\|\cdot\|_p$ denotes the norm on $L^p_{(0)}(\varphi)$.*

Proof Firstly, we show that

$$\|j_{(0)}^*(x)\|_p \le \|x\|^{1/p'}\|\varphi_x\|^{1/p} \qquad (x \in L).$$

This is [32, Corollary 2.8], but we give a quick proof. Pick $\epsilon > 0$ and define $F \colon S \to L$ by $F(z) = \exp(\epsilon(z^2 - \theta^2))\|\varphi_x\|^{\theta-z}\|x\|^{z-\theta}x$. Then $F \in \mathcal{F}, F(\theta) = x$, and we can check that

$$\|F\|_{\mathcal{F}} \le \|\varphi_x\|^\theta \|x\|^{1-\theta} \exp\left(\epsilon(1-\theta^2)\right).$$

As $\epsilon > 0$ was arbitrary, we conclude that, as $\theta = 1/p$,

$$\|j_{(0)}^*(x)\|_p \le \|x\|^{1-\theta}\|\varphi_x\|^\theta = \|x\|^{1/p'}\|\varphi_x\|^{1/p}.$$

We now use duality. For $\epsilon > 0$, there exists $p_0 > 1$ such that if $1 < p \le p_0$, then $\|j_{(0)}^*(x)\|_p \le (1+\epsilon)\|\varphi_x\|$. As L is σ-weakly dense in $\mathcal{M}$, by Kaplansky density, we can find $y \in L$ with $\|y\| = 1$ and $\left|\left\langle y, \varphi_x\right\rangle\right| \ge (1-\epsilon)\|\varphi_x\|$. Then there exists $p_1 > 1$ such that if $1 < p \le p_1$, then $\|j_{(0)}^*(y)\|_{p'} \le (1+\epsilon)\|y\| = 1+\epsilon$. Thus, if $1 < p < \min(p_0, p_1)$, then

$$(1+\epsilon)\|\varphi_x\| \ge \|j_{(0)}^*(x)\|_p \ge \left|\left\langle j_{(0)}^*(x), j_{(0)}^*(y)\right\rangle_{p,(0)}\right| \|j_{(0)}^*(y)\|_{p'}^{-1}$$

$$\ge \left|\left\langle y, \varphi_x\right\rangle\right|(1+\epsilon)^{-1} \ge (1-\epsilon)(1+\epsilon)^{-1}\|\varphi_x\|.$$

As $\epsilon > 0$ was arbitrary, this completes the proof. ∎

3.3 Operator Spaces

As noted by Pisier in [26] and [28, Section 2.6], the complex interpolation method interacts very nicely with operator spaces. If E_0 and E_1 are operator spaces that, as Banach spaces, form a compatible couple, then, say, identifying $\mathbb{M}_n(E_0 + F_0)$ with $(E_0 + F_0)^{n^2}$, we turn $(\mathbb{M}_n(E_0), \mathbb{M}_n(E_1))$ into a compatible couple. We then define

$$\mathbb{M}_n\left((E_0, E_1)_{[\theta]}\right) = \left(\mathbb{M}_n(E_0), \mathbb{M}_n(E_1)\right)_{[\theta]}.$$

It is an easy check that these matrix norms satisfy the axioms for an (abstract) operator space. Then the obvious completely bounded version of Theorem 3.1 holds.

Suppose that E and F are Banach spaces that form a sesquilinear dual pair. A typical example would be $E = L^\infty(\mu)$ and $F = L^1(\mu)$ for a probability measure μ, together with the pairing

$$(f \mid g) = \int f\overline{g}\, d\mu \qquad (f \in E, g \in F).$$

Then we can show that $(E, F)_{[1/2]}$ is a Hilbert space if (E, F) is made a compatible couple in the correct way; see [28, Theorem 7.10], for example. In our example, we recover $L^2(\mu)$ for the canonical compatibility. Intrinsic in the proof is that a Hilbert

space H can be canonically identified in an anti-linear way with its own dual, by way of the inner-product.

If E and F are also operator spaces, then we recover a Hilbert space with some operator space structure. There is a unique operator Hilbert space that is anti-linearly completely isometric to its dual: Pisier's *operator Hilbert space*. We write H_{oh} to denote this structure on H. As explained carefully in [28, p. 139], at least when $\mathcal{M}$ is semifinite, we should consider the compatible couple $(\mathcal{M}, \mathcal{M}_*^{\mathrm{op}})$. Here, for an operator space E, E^{op} denotes the space E with the *opposite* structure, namely $\|(x_{ij})\|_{\mathrm{op}} = \|(x_{ji})\|$ for $(x_{ij}) \in \mathbb{M}_n(E)$. If $\mathcal{A}$ is a C*-algebra, then $\mathcal{A}^{\mathrm{op}}$ can be identified with $\mathcal{A}$, but with the product reversed. See also [19, Section 4] for a slightly different perspective.

Indeed, as noted in [19], if $\mathcal{M}$ is in standard position on H with modular conjugation J, then we have a canonical $*$-isomorphism $\phi \colon \mathcal{M}^{\mathrm{op}} \to \mathcal{M}'$, $x \mapsto Jx^*J$ and so $\phi_* \colon \mathcal{M}'_* \to \mathcal{M}_*^{\mathrm{op}}$ is a completely isometric isomorphism of operator spaces. We conclude that the natural operator space structure on $L^p(\mathcal{M})$ will arise from studying the compatible couple $(\mathcal{M}, \mathcal{M}'_*)$. Alternatively, if we privilege $\mathcal{M}_*$, then we should look at $(\mathcal{M}', \mathcal{M}_*)$. When $\mathcal{M}_* = A(G)$, it turns out that this simple observation will guide us as to how to give the resulting non-commutative L^p spaces an $A(G)$-module action.

Let us finish by showing the operator space version of Proposition 3.4.

Proposition 3.5 *With notation as above, let $x \in \mathbb{M}_n(L)$ for some $n \in \mathbb{N}$. Then $\lim_{p \to 1} \|j_{(0)}^*(x)\|_p = \|\varphi_x\|$.*

Proof The norm on $\mathbb{M}_n(L_{(0)}^p(\varphi))$ is given by interpolating $\mathbb{M}_n(\mathcal{M})$ and $\mathbb{M}_n(\mathcal{M}_*^{\mathrm{op}})$, and so we can follow the first part of the proof of Proposition 3.4 to find $p_0 > 1$ such that, if $1 < p < p_0$, then $\|j_{(0)}^*(x)\|_p \leq (1 + \epsilon)\|\varphi_x\|$.

By Smith's Lemma ([5, Proposition 2.2.2]) and as $\mathbb{M}_n(L)$ is σ-weakly dense in $\mathbb{M}_n(\mathcal{M})$, there exists $y \in \mathbb{M}_n(L)$ with $\|y\| = 1$ and $|\langle\!\langle y, \varphi_x \rangle\!\rangle| \geq (1 - \epsilon)\|\varphi_x\|$. We can now proceed as in the end of the proof of Proposition 3.4. ∎

4   Non-Commutative L^p Spaces Associated with the Fourier Algebra

Let G be a locally compact group G. We have that $VN(G)$ is a Hopf-von Neumann algebra, a Kac algebra ([6]), and a locally compact quantum group ([22, 23]). We have a normal $*$-homomorphism

$$\Delta \colon VN(G) \to VN(G)\overline{\otimes}VN(G) = VN(G \times G); \quad \Delta\big(\lambda(s)\big) = \lambda(s) \otimes \lambda(s) \quad (s \in G).$$

It is not obvious that such a map exists, but if we define $W \in \mathcal{B}(L^2(G \times G))$ by $W\xi(s, t) = \xi(ts, t)$ for $s, t \in G$, then W is a unitary, and we can define $\Delta(x) = W^*(1 \otimes x)W$ for $x \in VN(G)$. Then Δ is coassociative, namely $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. Thus Δ induces an associative product on $VN(G)_*$, leading to the Fourier algebra ([7]). For $\xi, \eta \in L^2(G)$ we write $\omega_{\xi,\eta}$ for the normal functional on $VN(G)$ given by $\langle x, \omega_{\xi,\eta} \rangle = (x\xi|\eta)$. As $VN(G)$ is in standard position ([34, Chapter IX, Section 1]), every member of $A(G)$ arises in this way. We define a map, the *Eymard*

embedding, $\Phi\colon VN(G)_* \to C_0(G)$ by

$$\Phi(\omega_{\xi,\eta})(s) = \big\langle \lambda(s), \omega_{\xi,\eta} \big\rangle = \int_G \xi(s^{-1}t)\overline{\eta(t)}\, dt \qquad \big(s \in G, \omega_{\xi,\eta} \in A(G)\big).$$

Then Φ is an algebra homomorphism. This follows [7], but we warn the reader that [34, Chapter VII, Section 3] uses a different map (with s^{-1} replacing s).

Then $VN_r(G) = VN(G)'$ carries a coassociative map Δ' given by $\Delta'(x) = (J \otimes J)\Delta(JxJ)(J \otimes J)$ for $x \in VN_r(G)$. We have that $\Delta'(\rho(s)) = \rho(s) \otimes \rho(s)$ for $s \in G$. Similarly $A_r(G) = VN_r(G)_*$ becomes an algebra. We write $\omega'_{\xi,\eta}$ for the functional on $VN_r(G)$ given by $\big\langle x, \omega'_{\xi,\eta} \big\rangle = (x\xi|\eta)$ for $x \in VN_r(G)$. We similarly define $\Phi'\colon A_r(G) \to C_0(G)$ by

$$\Phi'(\omega'_{\xi,\eta})(s) = \big\langle \rho(s), \omega_{\xi,\eta} \big\rangle = \int_G \xi(ts)\nabla(s)^{1/2}\overline{\eta(t)}\, dt \qquad \big(s \in G, \omega'_{\xi,\eta} \in A_r(G)\big).$$

Guided by the arguments in the previous section, we shall turn $(VN_r(G), A_r(G))$ into a compatible couple in the sense of Terp. As $VN_r(G) = VN(G)'$, we have a canonical $*$-isomorphism

$$\phi\colon VN(G)^{\mathrm{op}} \to VN_r(G); \quad x \mapsto Jx^*J \qquad \big(x \in VN(G)\big).$$

Then we have

$$\phi_*\colon A_r(G) \to A(G)^{\mathrm{op}}; \quad \omega'_{\xi,\eta} \mapsto \omega_{J\eta,J\xi} \qquad \big(\xi,\eta \in L^2(G)\big).$$

This allows us to regard $(VN_r(G), A(G))$ as a compatible couple, and we shall often suppress the implicit ϕ_* involved. We then define

$$L^p(\widehat{G}) = \big(VN_r(G), A(G)\big)_{[1/p]} \qquad (1 < p < \infty).$$

Here we use the "dual group" notation that is common when studying the Fourier algebra. The motivation is that when G is abelian, we have that $VN_r(G) = L^\infty(\widehat{G})$ and $A(G) = L^1(\widehat{G})$ by the Fourier transform, where $\widehat{G}$ is the Pontryagin dual of G, and so $L^p(\widehat{G})$ agrees with the usual meaning. We keep the same notation in the non-abelian case, although now it is purely formal. We give $L^p(\widehat{G})$ the canonical operator space structure

$$\mathbb{M}_n\big(L^p(\widehat{G})\big) = \big(\mathbb{M}_n\big(VN_r(G)\big), \mathbb{M}_n(A(G))\big)_{[1/p]}.$$

We have that

$$\Phi\big(\phi_*(\omega'_{\xi,\eta})\big)(s) = \big(J\lambda(s)^*J\xi|\eta\big) = \big(\rho(s^{-1})\xi \mid \eta\big) = \Phi'(\omega'_{\xi,\eta})(s^{-1})$$
$$\big(s \in G, \xi, \eta \in L^2(G)\big).$$

Hence, under the maps Φ and Φ', ϕ_* induces the "check map". We also have the map K available, which allows us to define a $*$-homomorphism

$$\widehat{\phi} \colon VN(G) \to VN_r(G); \quad x \mapsto KxK \qquad \left(x \in VN(G) \right).$$

The predual of this map is then

$$\widehat{\phi}_* \colon A_r(G) \to A(G); \quad \omega'_{\xi,\eta} \mapsto \omega_{K\xi,K\eta} \qquad \left(\xi, \eta \in L^2(G) \right),$$

so that

$$\Phi \big(\widehat{\phi}_*(\omega'_{\xi,\eta}) \big)(s) = \big(K\lambda(s)K\xi \mid \eta \big) = \big(\rho(s)\xi \mid \eta \big) = \Phi'(\omega'_{\xi,\eta})(s)$$
$$\left(s \in G, \xi, \eta \in L^2(G) \right).$$

Thus, under the maps Φ and Φ', we see that $\widehat{\phi}_*$ is the formal identity.

Lemma 4.1   *For $f, g \in C_{00}(G)$, let $a = f^*g$. Then $\rho(a) \in VN_r(G)$ agrees with $\nabla^{1/2}a \in A(G)$ in $VN_r(G) \cap A(G) = L$.*

Proof   We have that $\rho(f), \rho(g) \in \mathfrak{n}_{\varphi'}$, and so by [35, Proposition 4], we have that $\rho(f^*g) \in L = VN_r(G) \cap A(G)$, with

$$\varphi_{\rho(f^*g)} = \omega'_{J\Lambda'\rho(f), J\Lambda'\rho(g)} = \omega'_{\bar{f}, \bar{g}} = \phi_*^{-1}(\omega_{\Lambda'\rho(g), \Lambda'\rho(f)})$$
$$= \phi_*^{-1}(\omega_{Kg, Kf}) = \phi_*^{-1}\widehat{\phi}_*(\omega'_{g,f}).$$

Now, for $s \in G$,

$$\omega_{Kg, Kf}(s) = \int_G Kg(s^{-1}t)\overline{Kf(t)} \, dt = \int_G g(t^{-1}s)\nabla(t^{-1}s)^{1/2}\overline{f(t^{-1})}\nabla(t^{-1})^{1/2} \, dt$$
$$= \nabla(s)^{1/2} \int_G f^*(t)g(t^{-1}s) \, dt = (\nabla^{1/2}a)(s),$$

which completes the proof.   ∎

We wish to turn $L^p(\widehat{G})$ into a (completely contractive) left $A(G)$-module. For $p = 1$, we obviously have a natural action of $A(G)$ on itself, and so the previous lemma suggests the following action.

Lemma 4.2   *There is a completely contractive action of $A(G)$ on $VN_r(G)$ such that $a \cdot \rho(f) = \rho(a \cdot f)$ for $a \in A(G)$ and $f \in C_{00}(G)$, where $a \cdot f$ denotes the point-wise product.*

Proof   We have that $VN_r(G)$ is a completely contractive $A_r(G)$-module (which is commutative, so we shall not distinguish between left and right actions) such that $a \cdot \rho(f) = \rho(a \cdot f)$ for $a \in A_r(G)$ and $f \in C_{00}(G)$. As above, we have that $\widehat{\phi}_* \colon A_r(G) \to A(G)$ is a completely isometric homomorphism. So our required action is simply $a \cdot x = \widehat{\phi}_*^{-1}(a) \cdot x$ for $a \in A(G), x \in VN_r(G)$.   ∎

The following is a useful approximation result, which allows us to work with concrete functions rather than operators in $VN_r(G)$.

Proposition 4.3 *For $x \in VN_r(G)$, we have that $x \in L$ when there exists $\varphi_x \in A_r(G)$ with $(x(\overline{a})|\overline{b}) = \langle \rho(a^*b), \varphi_x \rangle$ for $a, b \in C_{00}(G)$.*

Proof Let $\mathfrak{A} = \Lambda'(\rho(C_{00}(G))) = C_{00}(G)$, which is a Tomita algebra (but *not* the maximal Tomita algebra). We claim that $\mathfrak{A}$ generates the full left Hilbert algebra $\Lambda'(\mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'})$. This will follow from [33, Lemma 3, Section 10.5] if we can show that $C_{00}(G)$ is a core for the operator S, which is the closure of $\Lambda'(x) \mapsto \Lambda'(x^*)$ for $x \in \mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'}$ (meaning that the closure of the S operator associated with $\mathfrak{A}$ agrees with the canonical one associated with $\mathfrak{n}_{\varphi'}$).

Indeed, for us, S is the map $D(S) \to L^2(G), \xi \mapsto \overline{\check{\xi}}$, where $D(S) = \{\xi \in L^2(G) : \check{\xi} \in L^2(G)\}$. Then $D(S)$ is a Hilbert space for the inner-product $(\xi \mid \eta)_\sharp = (\xi \mid \eta) + (S\eta \mid S\xi)$ for $\xi, \eta \in D(S)$. We claim that $C_{00}(G)$ is dense in $D(S)$, from which it will follow that $C_{00}(G)$ is a core for S. Suppose that $\eta \in D(S)$ is such that $(\xi \mid \eta)_\sharp = 0$ for $\xi \in C_{00}(G)$. Then

$$0 = \int_G \xi(s)\overline{\eta(s)}\, ds + \int_G \xi(s^{-1})\overline{\eta(s^{-1})}\, ds = \int_G \overline{\eta(s)}\xi(s)(1 + \nabla(s)^{-1})\, ds$$

for all $\xi \in C_{00}(G)$. As the set $\{\xi \cdot (1 + \nabla^{-1}) : \xi \in C_{00}(G)\}$ is dense in $L^2(G)$, it follows that $\eta = 0$. So $C_{00}(G)$ is dense in $D(S)$, as required.

As $\mathfrak{A}$ generates $\Lambda'(\mathfrak{n}_{\varphi'} \cap \mathfrak{n}_{\varphi'})$, we can apply the approximation result [34, Theorem 1.26, Chapter VI]. This shows that for $x \in \mathfrak{n}_{\varphi'}$, we can find a sequence (f_n) in $C_{00}(G)$ such that

$$\lim_n \|\Lambda'(x) - \Lambda'\rho(f_n)\| = \lim_n \|\Lambda'(x) - Kf_n\| = 0, \qquad \|\rho(f_n)\| \leq \|x\| \qquad (n \in \mathbb{N}),$$

and that $\rho(f_n) \to x$ strongly.

Finally, suppose that $x \in VN_r(G)$ and $\varphi_x \in A_r(G)$ are such that $(x(\overline{a})|\overline{b}) = \langle \rho(a^*b), \varphi_x \rangle$ for $a, b \in C_{00}(G)$. Choose $\xi, \eta \in L^2(G)$ with $\varphi_x = \omega'_{\xi,\eta}$. Let $y, z \in \mathfrak{n}_{\varphi'}$, so we can find sequences $(a_n), (b_n)$ in $C_{00}(G)$, as above, associated with y and z respectively. Thus

$$\langle y^*z, \varphi_x \rangle = (z\xi \mid y\eta) = \lim_n \big(\rho(b_n)\xi \mid \rho(a_n)\eta\big) = \lim_n \langle \rho(a_n^*b_n), \varphi_x \rangle$$

$$= \lim_n (xJKa_n \mid JKb_n) = \big(xJ\Lambda'(y) \mid J\Lambda'(z)\big).$$

We conclude that $x \in L$ as required. ∎

We can immediately improve Lemma 4.1.

Proposition 4.4 *Let $a \in A(G)$. Then $a \in VN_r(G) \cap A(G)$ if and only if $\check{a}$ is right bounded; that is, there exists $K > 0$ such that $\|f\check{a}\|_2 \leq K\|f\|_2$ for $f \in C_{00}(G)$. In this case, the map $f \mapsto f\check{a}$ extends to an operator $x \in VN_r(G)$, and then $x \in L$ with $a = \phi_*(\varphi_x)$.*

Proof Let $a = \omega_{\xi,\eta}$, so that $\check{a} = \Phi' \phi_*^{-1}(\omega_{\xi,\eta})$. Suppose that $\check{a}$ is right bounded. As convolution on the right commutes with the action of $VN(G)$, we see that $x \in VN_r(G)$. For $f, g \in C_{00}(G)$, we see that

$$\left(x(\overline{f}) \mid \overline{g} \right) = (\overline{f}\check{a} \mid \overline{g}) = \int \overline{f(s)}\check{a}(s^{-1}t)g(t)\, ds\, dt = \int \overline{f(s)}\check{a}(t)g(st)\, dt\, ds$$

$$= \int f^*(s)g(s^{-1}t)\check{a}(t)\, ds\, dt = \left\langle \rho(f^*g), \phi_*^{-1}(\omega_{\xi,\eta}) \right\rangle.$$

So by the previous proposition, $x \in L$ and $a = \phi_*(\varphi_x)$, as claimed.

Conversely, if $a \in VN_r(G) \cap A(G)$, then there exists $x \in L$ with $a = \phi_*(\varphi_x)$. As $f\check{a}$ always exists for $f \in C_{00}(G)$, we can reverse the argument above to conclude that $x(f) = f\check{a}$ for $f \in C_{00}(G)$, so that $\check{a}$ is right bounded. ∎

We can also apply our approximation idea to improve an approximation result of Terp, [35, Theorem 8].

Proposition 4.5 *For $x \in L$, we can find a net (f_i) in $C_{00}(G)^2$ such that $\sup_i \|\rho(f_i)\| < \infty$, $\rho(f_i) \to x$ σ-weakly, and $\varphi_{\rho(f_i)} \to \varphi_x$ in norm.*

Proof By Terp's result [35, Theorem 8] we can find a bounded net (x_i) in $\mathfrak{m}_{\varphi'}$ with $x_i \to x$ σ-weakly and $\varphi_{x_i} \to \varphi_x$ in norm. Indeed, from the proof, we can choose $x_i = y_i^* z_i$ for some $y_i, z_i \in \mathfrak{n}_{\varphi'}$ with (y_i) and (z_i) bounded nets.

For each i, choose a sequence $(a_{i,n})$ in $C_{00}(G)$ with $\rho(a_{i,n}) \to y_i$ strongly, $Ka_{i,n} \to \Lambda'(y_i)$ in norm, and with $\|\rho(a_{i,n})\| \leq \|y_i\|$. Similarly choose $(b_{i,n})$ associated with z_i. It follows (compare with the proof above) that $\rho((a_{i,n})^*b_{i,n}) \to y_i^*z_i = x_i$ σ-weakly, and that $\varphi_{\rho(a_{i,n}^*b_{i,n})} \to \varphi_{x_i}$ in norm. With the diagonal ordering, we see that $(a_{i,n}^*b_{i,n})$ is the required net. ∎

Theorem 4.6 *There is a completely contractive left action of $A(G)$ on $L^p(\widehat{G})$, for $1 < p < \infty$, such that $a \cdot j_{(0)}^* \rho(b) = j_{(0)}^* \rho(a \cdot b)$ for $a \in A(G)$ and $b \in C_{00}(G)^2$.*

Proof Let $a \in A(G)$, and consider the bounded maps

$$T \colon A(G) \to A(G); b \mapsto a \cdot b, \qquad S \colon VN_r(G) \to VN_r(G); x \mapsto \widehat{\phi}_*^{-1}(a) \cdot x.$$

By Lemma 3.2, we wish to show that T and S map L to L and agree on L. If this is so, then we get a map $R \in \mathcal{B}(L^p(\widehat{G}))$ that extends T and S, and is bounded by $\|T\|^{1/p}\|S\|^{1/p'} \leq \|a\|$. Clearly $a \mapsto R$ is a homomorphism, and the resulting action of $A(G)$ on $L^p(\widehat{G})$ is the one stated by Lemma 4.2.

So, for $x \in L$, we need to show that $y = \widehat{\phi}_*^{-1}(a) \cdot x \in L$ and that, furthermore, $a \cdot \phi_*(\varphi_x) = \phi_*(\varphi_y)$. Suppose that $x = \rho(f^*g)$ for $f, g \in C_{00}(G)$, so that from Lemma 4.1, $\Phi\phi_*(\varphi_x) = \nabla^{1/2}f^*g$. By Proposition 4.3, we have that $y \in L$ if

$$\left(y(\overline{c}) \mid \overline{d} \right) = \left\langle \rho(c^*d), \phi_*^{-1}(a \cdot \phi_*(\varphi_x)) \right\rangle \qquad \left(c, d \in C_{00}(G) \right).$$

Now, we have that

$$
\begin{aligned}
\left\langle \rho(c^*d), \phi_*^{-1}(a \cdot \phi_*(\varphi_x)) \right\rangle &= \left\langle \rho(c^*d), \phi_*^{-1}\Phi^{-1}(a \cdot \nabla^{1/2} f^*g) \right\rangle \\
&= \int_G c^*d(s^{-1})a(s^{-1})\nabla(s)^{-1/2}(f^*g)(s^{-1}) \, ds \\
&= \int_G c^*d(s^{-1})a(s)\nabla(s)^{-1/2}(f^*g)(s) \, ds \\
&= \left\langle \widehat{\phi}_*^{-1}(a) \cdot \rho(f^*g), \varphi_{\rho(c^*d)} \right\rangle = \left\langle y, \omega'_{\bar{c},\bar{d}} \right\rangle = \left(y(\bar{c}) \mid \bar{d} \right),
\end{aligned}
$$

using that $\Phi' \phi_*^{-1} \Phi^{-1}$ is the check map. Hence we are done in the case that $x \in \rho(C_{00}(G)^2)$.

For general $x \in L$, choose an approximating net $(f_i) \subseteq C_{00}(G)^2$ as in Proposition 4.5. Then, by the previous paragraph, for $c, d \in C_{00}(G)$,

$$
\begin{aligned}
\left\langle y, \omega'_{\bar{c},\bar{d}} \right\rangle &= \lim_i \left\langle \rho(f_i), \omega'_{\bar{c},\bar{d}} \right\rangle = \lim_i \left\langle \rho(c^*d), \phi_*^{-1}(a \cdot \phi_*(\varphi_{\rho(f_i)})) \right\rangle \\
&= \left\langle \rho(c^*d), \phi_*^{-1}(a \cdot \phi_*(\varphi_x)) \right\rangle,
\end{aligned}
$$

which completes the proof of the claim by another application of Lemma 4.1.

In the completely bounded case, notice that T and S are completely bounded, and hence, so is R. Thus we get a homomorphism $A(G) \to \mathcal{CB}(L^p(\widehat{G}))$. To see that this is completely bounded, it is easier to prove the equivalent statement that $A(G) \times L^p(\widehat{G}) \to L^p(\widehat{G}); (a, \xi) \mapsto R(\xi)$ is jointly completely contractive ([5, Chapter 7]). That is, for $n \in \mathbb{N}$, the map

$$
\mathbb{M}_n(A(G)) \times \mathbb{M}_n(L^p(\widehat{G})) \to \mathbb{M}_{n^2}(L^p(\widehat{G})); (a_{ij}) \times (\xi_{kl}) \mapsto (R_{ij}(\xi_{kl}))_{(i,k),(j,l)}
$$

is contractive. This follows immediately from Theorem 3.3, as the analogous statements hold for T and S. ∎

A slightly curious corollary of this proof is that L is an $A(G)$-submodule of $VN_r(G)$, and hence the image of L in $A(G)$ is a dense ideal. As a final application of our approximation ideas, we have the following.

Proposition 4.7 *For $1 < p < \infty$, we have that $j_{(0)}^* \rho(C_{00}(G)^2)$ is norm dense in $L^p(\widehat{G})$.*

Proof Following the proof of [16, Prop. 6.22], it suffices to show that $\rho(C_{00}(G)^2) \subseteq L$ separates the points of $VN_r(G) + A_r(G) \subseteq L^*$. Indeed, suppose that $x \in VN_r(G)$ and $\omega \in A_r(G)$ are such that

$$
\langle x, \varphi_{\rho(f)} \rangle + \left\langle \rho(f), \omega \right\rangle = 0 \qquad \left(f \in C_{00}(G)^2 \right).
$$

For $y \in L$, use Proposition 4.5 to pick an approximating net (f_i), so that

$$
0 = \lim_i \langle x, \varphi_{\rho(f_i)} \rangle + \left\langle \rho(f_i), \omega \right\rangle = \langle x, \varphi_y \rangle + \langle y, \omega \rangle.
$$

In particular, this holds for any $y \in \mathfrak{m}_\varphi$, so by [35, Proposition 7] (or, essentially by definition) it follows that $x \in L$ with $\varphi_x = -\omega$. Hence $x + \omega = 0$ in $VN_r(G) + A_r(G)$, as required. ∎

Proposition 4.8 *There is an isometric isomorphism $\theta\colon L^2(G) \to L^2(\widehat{G})$ satisfying $\theta(f) = j_{(0)}^* \rho(\nabla^{-3/4} \check{f})$ for $f \in C_{00}(G)^2$. Furthermore, θ intertwines the inner products on $L^2(G)$ and $L^2(\widehat{G})$.*

Proof In Section 3, we discussed the isometric isomorphism $k\colon H_{\varphi'} \to L^2(\widehat{G})$ that is such that $k(\Lambda' \rho(f)) = j_{(0)}^* \sigma_{i/4} \rho(f)$ for $f \in C_{00}(G)^2$. If we identify $H_{\varphi'}$ with $L^2(G)$, then $\Lambda' \rho(f) = Kf$, and so we find a map θ that satisfies $\theta(f) = j_{(0)}^* \sigma_{i/4} \rho(Kf) = j_{(0)}^* \rho(\nabla^{-3/4} \check{f})$ for $f \in C_{00}(G)^2$. ∎

Notice that $L^2(G)$ carries a natural *bilinear* product, $\langle f, g \rangle = \int_G fg$ for $f, g \in L^2(G)$. Similarly $L^2(\widehat{G})$ has the bilinear product $\langle \, \cdot \, , \, \cdot \, \rangle_{2,(0)}$, but θ does not intertwine these products.

4.1   Comparison with Forrest, Lee, and Samei

In [8, Section 6], a different construction of non-commutative L^p spaces associated with $A(G)$ is given. We shall compare their construction to ours.

First, they form the non-commutative L^p space using $VN(G)$ using Izumi's work with $\alpha = -1/2$. Let $\mathcal{O}L_{(-1/2)}^p(VN(G))$ be the operator space version given by interpolating $VN(G)$ and $A(G)^{\mathrm{op}}$. Here we write $(-1/2)$ to indicate the choice of α. Then they define

$$L^p(VN(G)) = \begin{cases} \mathcal{O}L_{(-1/2)}^p(VN(G))^{\mathrm{op}} & : 1 < p \le 2, \\ \mathcal{O}L_{(-1/2)}^p(VN(G)) & : 2 \le p < \infty. \end{cases}$$

Recall that the Hilbert space $\mathcal{O}L_{-1/2}^2(VN(G))$ will carry the operator Hilbert space structure, so that $\mathcal{O}L_{(-1/2)}^2(VN(G)) = \mathcal{O}L_{(-1/2)}^2(VN(G))^{\mathrm{op}}$.

By [8, Theorem 6.3], for $1 < p \le 2$, the $A(G)$ module structure on $L^p(VN(G))$ satisfies

$$a \cdot j_{(1/2)}^* \big(\lambda(\check{f}) \big) = j_{(1/2)}^* \big(\lambda(\check{a} \cdot \check{f}) \big) \qquad \big(a \in A(G), f \in C_{00}(G) \big).$$

Here we use a different, but equivalent, notation to that of [8]. Similarly, by [8, Theorem 6.4], for $2 \le p < \infty$, the module action of $A(G)$ on $L^p(VN(G))$ is

$$a \cdot j_{(1/2)}^*(\lambda(f)) = j_{(1/2)}^*(\lambda(a \cdot f)) \qquad (a \in A(G), f \in C_{00}(G)).$$

Recall the isometric isomorphism $U_{p,(0,-1/2)}\colon L_{(-1/2)}^p(VN(G)) \to L_{(0)}^p(VN(G))$ that satisfies, in particular,

$$U_{p,(0,-1/2)} \big(j_{(1/2)}^* \lambda(f) \big) = j_{(0)}^* \big(\sigma_{i/2p} \lambda(f) \big) = j_{(0)}^* \lambda(\Delta^{-1/2p} f) \qquad \big(f \in C_{00}(G)^2 \big).$$

For $1 < p \leq 2$, we can hence regard $L^p(VN(G))$ as $\mathcal{O}L^p_{(0)}(VN(G))^{\mathrm{op}}$ with the module action

$$a \cdot j^*_{(0)}\lambda(f) = U_{p,(0,-1/2)}\big(a \cdot U^{-1}_{p,(0,-1/2)}j^*_{(0)}\lambda(f)\big) = U_{p,(0,-1/2)}\big(a \cdot j^*_{(0)}\lambda(\Delta^{1/2p}f)\big)$$

$$= U_{p,(0,-1/2)}j^*_{(0)}\lambda(\check{a} \cdot \Delta^{1/2p}f) = j^*_{(0)}\lambda(\check{a} \cdot f)$$

for $a \in A(G)$ and $f \in C_{00}(G)^2$. Similarly, for $2 \leq p < \infty$, we regard $L^p(VN(G))$ as $\mathcal{O}L^p_{(0)}(VN(G))$ with the module action

$$a \cdot j^*_{(0)}\lambda(f) = j^*_{(0)}\lambda(a \cdot f) \qquad \big(a \in A(G), f \in C_{00}(G)^2\big).$$

Proposition 4.9　*For $2 \leq p < \infty$, there exists a completely isometric isomorphism $\widehat{\phi}_p \colon L^p(VN(G)) \to L^p(\widehat{G})$ that is also an $A(G)$-module homomorphism, with*

$$\widehat{\phi}_p\big(j^*_{(0)}\lambda(f)\big) = j^*_{(0)}\rho(f) \qquad (f \in C_{00}(G)^2).$$

Proof　Our $L^p(\widehat{G})$ spaces are formed by interpolating $VN_r(G)$ and $A_r(G)^{\mathrm{op}}$ (identified with $A(G)$). Consider again the maps $\widehat{\phi} \colon VN(G) \to VN_r(G)$ and $\widehat{\phi}^{-1}_* \colon A(G) \to A_r(G)$. We claim that these are compatible, that is, map $L_{(0)}$, for $(VN(G), A(G))$, into $L_{(0)}$, for $(VN_r(G), A_r(G))$. Indeed, let $x \in L_{(0)} \subseteq VN(G)$ with associated $\varphi_x \in A(G)$. Let $a, b \in C_{00}(G)$, so that

$$\big(\widehat{\phi}(x)\overline{a} \mid \overline{b}\big) = \big(xK(\overline{a}) \mid K(\overline{b})\big) = \big(xJ(a) \mid J(b)\big) = \Big(xJ\Lambda\big(\lambda(a)\big) \mid J\Lambda\big(\lambda(b)\big)\Big)$$

$$= \big\langle \lambda(a^*b), \varphi_x \big\rangle = \big\langle K\rho(a^*b)K, \varphi_x \big\rangle = \big\langle \rho(a^*b), \widehat{\phi}^{-1}_*(\varphi_x) \big\rangle,$$

so by Proposition 4.3 we see that $\widehat{\phi}(x) \in L_{(0)}$, for $(VN_r(G), A_r(G))$, with $\varphi_{\widehat{\phi}(x)} = \widehat{\phi}^{-1}_*(\varphi_x)$. Consequently, by Lemma 3.2, we can interpolate these maps, leading to a contraction

$$\widehat{\phi}_p \colon L^p_{(0)}\big(VN(G)\big) \to L^p_{(0)}\big(VN_r(G)\big) = L^p(\widehat{G}).$$

As $\widehat{\phi}^{-1}_*$ is also a complete isometry $A(G)^{\mathrm{op}} \to A_r(G)^{\mathrm{op}}$, we see that $\widehat{\phi}_p$ is a complete contraction. By symmetry, we also have a complete contraction in the other direction, showing that $\widehat{\phi}_p$ is actually a completely isometric isomorphism. In particular,

$$\widehat{\phi}_p j^*_{(0)}\lambda(f) = j^*_{(0)}\big(K\lambda(f)K\big) = j^*_{(0)}\rho(f) \qquad \big(f \in C_{00}(G)^2\big).$$

It is now clear from Proposition 4.7 that this map is an $A(G)$-module homomorphism.　∎

Proposition 4.10　*For $1 < p \leq 2$, there exists a completely isometric isomorphism $\phi_p \colon L^p(VN(G)) \to L^p(\widehat{G})$ that is also an $A(G)$-module homomorphism, with*

$$\phi_p\big(j^*_{(0)}\lambda(f)\big) = j^*_{(0)}\rho(\check{f}\nabla^{-1}) \qquad \big(f \in C_{00}(G)^2\big).$$

Proof For $1 < p \leq 2$, it is clear that

$$L^p\big(VN(G)\big) = \mathcal{O}L^p_{(0)}\big(VN(G)\big)^{\mathrm{op}} = \big(VN(G)^{\mathrm{op}}, A(G)\big)_{[1/p]}.$$

The idea now is to replicate the proof above using the maps

$$\phi \colon VN(G)^{\mathrm{op}} \to VN_r(G) \quad \text{and} \quad \phi_*^{-1} \colon A(G) \to A_r(G)^{\mathrm{op}}.$$

For $x \in L \subseteq VN(G)$, let $\varphi_x = \omega_{\xi,\eta}$ for some $\xi, \eta \in L^2(G)$. Let $y', z' \in \mathfrak{n}_{\varphi'}$ so that $y = Jy'J, z = Jz'J \in \mathfrak{n}_\varphi$, and

$$\big(\phi(x)J\Lambda'(y') \mid J\Lambda'(z')\big) = \big(Jx^*J\Lambda(y) \mid \Lambda(z)\big) = \langle z^*y, \varphi_x\rangle = (z^*y\xi \mid \eta)$$
$$= \big(J(z')^*y'J\xi \mid \eta\big) = \big((y')^*z'J\eta \mid J\xi\big)$$
$$= \big\langle (y')^*z', \phi_*^{-1}(\varphi_x)\big\rangle.$$

Hence $\phi(x) \in L \subseteq VN_r(G)$ with $\varphi_{\phi(x)} = \phi_*^{-1}(\varphi_x)$. Again, we interpolate to find a completely isometric isomorphism $\phi_p \colon L^p(VN(G)) \to L^p(\widehat{G})$. We then see that for $f \in C_{00}(G)^2$,

$$\phi_p j_{(0)}^*\lambda(f) = j_{(0)}^*\phi\big(\lambda(f)\big) = j_{(0)}^*\big(J\lambda(f^*)J\big) = j_{(0)}^*\rho(\check{f}\nabla^{-1}).$$

It is now clear from Proposition 4.7 that this map is an $A(G)$-module homomorphism. ∎

4.2 Application to Homological Questions

The following is an improvement of [8, Proposition 6.8], which only showed the result for $p \geq 2$.

Proposition 4.11   *Let G be a non-discrete group, and let $1 < p < \infty$. Then the only bounded left $A(G)$-module homomorphism $L^p(\widehat{G}) \to A(G)$ is the zero map.*

Proof Let $T \colon L^p(\widehat{G}) \to A(G)$ be a bounded left $A(G)$-module homomorphism, and suppose towards a contradiction that T is not zero. By density, we can find $x \in L$, such that setting $\xi = j_{(0)}^*(x)$, we have that $T(\xi) \neq 0$. Let $a = \phi_*(\varphi_x) \in A(G)$. For $y \in L$, let $\eta = j_{(0)}^*(y)$ and $b = \phi_*(\varphi_y)$. Then, with reference to Theorem 4.6, $z = \widehat{\phi}_*^{-1}(a)\cdot y \in L$ with $\phi_*(\varphi_z) = a\cdot\chi_*(\varphi_y) = ab = ba = b\cdot\phi_*(\varphi_x) = \phi_*(\widehat{\phi}_*^{-1}(b)\cdot x)$. Thus
$$a \cdot T(\eta) = Tj_{(0)}^*(z) = Tj_{(0)}^*\big(\widehat{\phi}_*^{-1}(b)\cdot x\big) = b \cdot T(\xi).$$

Let V be a compact neighbourhood of the identity in G, so that $0 < |V| < \infty$. Let K be a compact neighbourhood of the identity with $KK^{-1} \subseteq V$, let $r \in G$, let $\alpha = |K|^{-1/2}\chi_{r^{-1}K} \in L^2(G)$ and $\beta = |K|^{-1/2}\chi_K \in L^2(G)$. Then $\|\alpha\|_2 = \|\beta\|_2 = 1$, and so $b = \omega_{\alpha,\beta} \in A(G)$ with $\|b\|_{A(G)} \leq 1$. We see that

$$b(s) = \frac{1}{|K|} \int \chi_{r^{-1}K}(s^{-1}t)\chi_K(t)\, dt = \frac{|sr^{-1}K \cap K|}{|K|} \qquad (s \in G).$$

So $b(r) = 1$ and $b(s) \neq 0$ implies that $s \in KK^{-1}r \subseteq Vr$. So b has compact support and is bounded, and hence $b \in L^1(G)$ with $\|b\|_1 \leq |Vr|$. By Proposition 4.4, $b = \phi_*(\varphi_y)$, where $y \in L$ with $y(f) = f\check{b}$ for $f \in C_{00}(G)$. We can check that actually $y = \rho(\nabla^{-1/2}b)$, so that $\|y\| \leq \|\nabla^{-1}b\|_1 \leq |Vr|\|\nabla^{-1}|_{Vr}\|_\infty = K(V)$, say. By the estimate in Proposition 3.4 we see that

$$\|j_{(0)}^*(y)\|_p \leq \|y\|^{1/p'}\|\varphi_y\|^{1/p} \leq K(V)^{1/p'}.$$

With $\eta = j_{(0)}^*(y)$, we see that

$$|T(\xi)(r)| \leq \|b \cdot T(\xi)\|_{A(G)} = \|a \cdot T(\eta)\|_{A(G)} \leq \|a\|_{A(G)}\|T\|K(V)^{1/p'}.$$

In particular, we can make $K(V)$ as small as we like by choosing V small (as G is not discrete). As r was arbitrary, we conclude that $T(\xi) = 0$, giving our contradiction. ∎

We can now follow the proof of [8, Theorem 6.9] to show the following; we refer the reader to [8] for the definition of *operator projective*.

Theorem 4.12 *Let G be a non-discrete group and $1 < p < \infty$. Then $L^p(\widehat{G})$ is not operator projective as a left $A(G)$-module.*

5 Representing the Multiplier Algebra

Let G be a locally compact group, let (p_n) be a sequence in $(1, \infty)$ tending to 1, and let $E = \ell^2 - \bigoplus_n L^{p_n}(\widehat{G})$. In the Banach space case, this is the direct sum in the ℓ^2 sense defined in Section 2. In the operator space case, we regard this as a discrete vector-valued commutative ℓ^2 space, which carries a natural operator space structure; see [38, Section 1] and [27]. Indeed, $E_\infty = \ell^\infty - \oplus L^{p_n}(\widehat{G})$ carries an obvious operator space structure. We give $E_1 = \ell^1 - \oplus L^{p_n}(\widehat{G})$ the operator-space structure arising as a subspace of the dual of $\ell^\infty - \oplus L^{p_n}(\widehat{G})^*$. Then (E_∞, E_1) is a compatible couple, and E is simply $(E_\infty, E_1)_{[1/2]}$. Notice that the underlying Banach space is the same as the usual definition.

Then $A(G)$ acts coordinate wise on E, so that E becomes a (completely) contractive $A(G)$-module. In the operator space case, notice that this is clear for E_1 and E_∞, and hence also for E by bilinear interpolation. In this section, we shall show that $MA(G)$, respectively $M_{cb}A(G)$, have actions on E extending those of $A(G)$, and that the resulting homomorphisms $MA(G) \to \mathcal{B}(E)$ and $M_{cb}A(G) \to \mathcal{CB}(E)$ are weak*-weak*-continuous (complete) isometries.

Proposition 5.1 *For $1 < p < \infty$, there is a natural action of $MA(G)$ on $L^p(\widehat{G})$ extending the action of $A(G)$, such that $a \cdot j_{(0)}^*\rho(f) = j_{(0)}^*\rho(a \cdot f)$ for $a \in MA(G)$ and $f \in C_{00}(G)^2$. Furthermore, this action of $MA(G)$ restricts to give a completely contractive action of $M_{cb}A(G)$ on $L^p(\widehat{G})$.*

Proof We let $MA(G)$ act on $A(G)$ in the canonical way. As in the proof of Lemma 4.2, we note that $MA_r(G)$ acts on $A_r(G)$ and hence on $VN_r(G)$ by duality. This action

satisfies $a \cdot \rho(f) = \rho(a \cdot f)$ for $a \in MA(G)$ and $f \in C_{00}(G)$. We then extend $\widehat{\phi}_*^{-1}$ to an isometric homomorphism $\psi \colon MA(G) \to MA_r(G)$, which completes the argument as in Lemma 4.2. We define ψ by

$$\psi(a)(b) = \widehat{\phi}_*^{-1}\big(a\widehat{\phi}_*(b)\big) \qquad \big(a \in MA(G), b \in A_r(G)\big).$$

As $\widehat{\phi}_*$ is a homomorphism, this does extend $\widehat{\phi}_*^{-1}$ and is itself a homomorphism. Clearly ψ is contractive and has an obvious contractive inverse, so that ψ is isometric as required. Notice that, if we view $a \in MA(G)$ and $\psi(a)$ as functions on G (using Φ and Φ'), then these functions agree.

We now follow Theorem 4.6 and use interpolation to extend this $MA(G)$ action to $L^p(\widehat{G})$. We hence need to show that if $x \in L$, then $y = \psi(a) \cdot x \in L$ with $a \cdot \phi_*(\varphi_x) = \phi_*(\varphi_y)$. As in the proof of Theorem 4.6, by our approximation result, it is enough to show this for $x = \rho(f^*g)$ for $f, g \in C_{00}(G)$. The proof of Theorem 4.6 follows *mutatis mutandis*.

The remark about $M_{cb}A(G)$ will follow if we can show that ψ restricts to a complete contraction $\psi \colon M_{cb}A(G) \to M_{cb}A_r(G)$. However, this follows immediately because $\widehat{\psi}_*$ is a complete isometry. ∎

By [4], $MA(G)$ is a dual Banach algebra with a predual Q, which is the completion of $L^1(G)$ for the norm

$$\|f\|_Q = \sup\left\{\left|\int_G f(s)a(s)\,ds\right| : a \in MA(G), \|a\| \leq 1\right\} \qquad (f \in L^1(G)).$$

Let $\lambda_Q \colon L^1(G) \to Q$ be the inclusion map. Similarly, $M_{cb}A(G)$ has a predual Q_{cb} that is defined in the same way, but taking the supremum over the unit ball of $M_{cb}A(G)$. Define similarly $\lambda_{Q_{cb}} \colon L^1(G) \to Q_{cb}$.

For $1 < p < \infty$, let $\pi^p \colon A(G) \to \mathcal{B}(L^p(\widehat{G}))$ be the contractive homomorphism given by Theorem 4.6, and let $\widehat{\pi}^p \colon MA(G) \to \mathcal{B}(L^p(\widehat{G}))$ be the contractive homomorphism given by Proposition 5.1. Using Izumi's bilinear product, we have that $L^p(\widehat{G})^* = L^{p'}(\widehat{G})$, and so we can consider the map

$$\pi_*^p \colon L^p(\widehat{G})\widehat{\otimes}L^{p'}(\widehat{G}) \to A(G)^* = VN(G); \quad \big\langle \pi_*^p(\xi \otimes \eta), a \big\rangle = \big\langle a \cdot \xi, \eta \big\rangle_{p,(0)},$$

for $a \in MA(G), \xi \in L^p(\widehat{G})$ and $\eta \in L^{p'}(\widehat{G})$). Let $\widehat{\pi}_*^p \colon L^p(\widehat{G})\widehat{\otimes}L^{p'}(\widehat{G}) \to MA(G)^*$ be the analogous map.

Let $\pi^{p,cb} \colon A(G) \to \mathcal{CB}(L^p(\widehat{G}))$ and $\widehat{\pi}^{p,cb} \colon M_{cb}A(G) \to \mathcal{CB}(L^p(\widehat{G}))$ be analogously given by Theorem 4.6 and Proposition 5.1. Similarly, define

$$\pi_*^{p,cb} \colon L^p(\widehat{G})\widehat{\otimes}L^{p'}(\widehat{G}) \to A(G)^* \quad \text{and} \quad \widehat{\pi}_*^{p,cb} \colon L^p(\widehat{G})\widehat{\otimes}L^{p'}(\widehat{G}) \to M_{cb}A(G)^*.$$

Proposition 5.2 *The maps π_*^p and $\pi_*^{p,cb}$ take values in $C_\lambda^*(G)$, the reduced group C^*-algebra. The map $\widehat{\pi}_*^p$ takes values in the predual Q, and $\widehat{\pi}_*^{p,cb}$ takes values in the predual Q_{cb}, so that both $\widehat{\pi}^p$ and $\widehat{\pi}^{p,cb}$ are weak*-weak*-continuous.*

Proof Suppose that $\xi = j_{(0)}^*(f)$ and $\eta = j_{(0)}^*(g)$ for $f, g \in C_{00}(G)^2$. Then, for $a \in MA(G)$, using the calculations of Lemma 4.1,

$$\left\langle \widehat{\pi}_*^p(\xi \otimes \eta), a \right\rangle = \left\langle j_{(0)}^*\rho(a \cdot f), j_{(0)}^*\rho(g) \right\rangle_{p,(0)} = \left\langle \rho(a \cdot f), \varphi_{\rho(g)} \right\rangle$$

$$= \int_G a(s)f(s)\nabla(s)^{-1/2}g(s^{-1})\, ds = \left\langle a, \lambda_Q(f \cdot Kg) \right\rangle.$$

Hence $\widehat{\pi}_*^p(\xi \otimes \eta) = \lambda_Q(f \cdot Kg) \in Q$. By Proposition 4.7, such ξ and η are norm dense, showing that $\widehat{\pi}_*^p$ takes values in Q. It is now standard that $\widehat{\pi}^p$ is weak*-weak*-continuous. The same calculation shows that $\widehat{\pi}_*^{p,cb}(\xi \otimes \eta) = \lambda_{Q_{cb}}(f \cdot Kg) \in Q_{cb}$, so that $\widehat{\pi}_*^{p,cb}$ takes values in Q_{cb}, and hence $\widehat{\pi}^{p,cb}$ is also weak*-weak*-continuous.

We have that $\widehat{\pi}^p$, restricted to $A(G)$, is π^p. Similarly, and for $f \in L^1(G)$, we see that $\lambda_Q(f)$, restricted to $A(G)$, is simply $\lambda(f) \in C_\lambda^*(G) \subseteq VN(G)$. The above calculation hence also shows that π_*^p takes values in $C_\lambda^*(G)$, as claimed. The same argument applies in the completely bounded case. ∎

If $\mathcal{A}$ is a commutative Banach algebra and $(L, R) \in M(\mathcal{A})$, then for $a, b \in \mathcal{A}$, $L(a)b = L(ab) = L(ba) = L(b)a = aL(b) = R(a)b$. If $\mathcal{A}$ is faithful, then $L = R$. We remark that $A(G)$ is faithful by [7, Lemme 3.2]. For any compact $K \subseteq G$ there exists $a \in A(G)$ that is identically 1 on K.

The following is now the $A(G)$ version of the results in Section 2.

Theorem 5.3 *Let G and E be as above. Let $MA(G)$ act on E coordinate wise. Then the resulting homomorphism $\pi\colon MA(G) \to \mathcal{B}(E)$ is an isometry and is weak*-weak*-continuous. Furthermore, the image of π is the idealiser of $\pi(A(G))$ in $\mathcal{B}(E)$.*

Proof Clearly π is contractive. Let $a \in MA(G)$ and $\epsilon > 0$. As $j_{(0)}(L)$ is dense in $A_r(G)$, we can find $x \in L$ with $\|\varphi_x\| = 1$ and

$$\|a \cdot \phi_*(\varphi_x)\| \geq (1 - \epsilon)\|a\|_{MA(G)}.$$

Then, using Proposition 3.4, we see that

$$\|\pi(a)\| \geq \lim_n \|a \cdot j_{(0)}^*(x)\|_{p_n} \|j_{(0)}^*(x)\|_{p_n}^{-1} = \|a \cdot \phi_*(\varphi_x)\|\|\varphi_x\| \geq (1 - \epsilon)\|a\|_{MA(G)}.$$

As $\epsilon > 0$, we conclude that π is an isometry, as required.

Let $\xi = (\xi_n) \in E$ and $\eta = (\eta_n) \in E^*$ be sequences that are eventually zero. For $a \in MA(G)$, we see that

$$\left\langle \pi(a)\xi, \eta \right\rangle = \sum_n \left\langle a, \widehat{\pi}_*^p(\xi_n \otimes \eta_n) \right\rangle,$$

so that $\pi_*(\xi \otimes \eta) \in Q$. As such ξ and η are dense; by continuity we see that $\pi_*\colon E\widehat{\otimes}E^* \to MA(G)^*$ takes values in Q. Again, this implies that π is weak*-weak*-continuous.

Clearly $\pi(MA(G))$ is contained in the idealiser of $\pi(A(G))$. Conversely, given T in the idealiser of $\pi(A(G))$, we can follow the proof of Theorem 2.2 to find $a \in$

$MA(G)$ with $\pi(ab) = T\pi(b)$ and $\pi(ba) = \pi(b)T$ for $b \in A(G)$. For each $L^p(\widehat{G})$, by Proposition 4.7 and again using [7, Lemme 3.2], it follows that $\{\pi(a)\xi : a \in A(G), \xi \in L^p(\widehat{G})\}$ is linearly dense in $L^p(\widehat{G})$. This is enough to show that $T = \pi(a)$ as required to complete the proof. ∎

The completely bounded version of this result requires a subtly different proof.

Theorem 5.4 *Let G and E be as above, where we now regard E as an operator space. Let $M_{cb}A(G)$ act on E coordinate wise. Then the resulting homomorphism $\pi_{cb}\colon M_{cb}A(G) \to \mathcal{CB}(E)$ is a weak*-weak*-continuous complete isometry. Furthermore, the image of π_{cb} is the idealiser of $\pi_{cb}(A(G))$ in $\mathcal{CB}(E)$.*

Proof Again, clearly π_{cb} is completely contractive. As the norm on $\mathbb{M}_n(L^p(\widehat{G}))$ is given by interpolating $\mathbb{M}_n(VN_r(G))$ and $\mathbb{M}_n(A(G))$, we can simply apply the proof of the previous theorem, but working with matrices and using Proposition 3.5 to show that π_{cb} is a complete isometry. Similarly, it follows that π_{cb} is weak*-weak*-continuous.

Clearly $\pi_{cb}(M_{cb}A(G))$ is contained in the idealiser of $\pi_{cb}(A(G))$. Conversely, given T in the idealiser of $\pi_{cb}(A(G))$, we can follow the proof of Theorem 2.2 to find $(L, R) \in M(A(G))$ with $\pi_{cb}(L(a)) = T\pi_{cb}(a)$ and $\pi_{cb}(R(a)) = \pi_{cb}(a)T$ for $a \in A(G)$.

For $n \in \mathbb{N}$, let $i_n\colon L^{p_n}(\widehat{G}) \to E$ be the inclusion map, which is a completely contractive $A(G)$-bimodule homomorphism. Then

$$Ti_n(a\cdot\xi) = T\pi_{cb}(a)i_n(\xi) = \pi_{cb}\big(L(a)\big)\,i_n(\xi) = i_n\big(L(a)\cdot\xi\big) \quad \big(a \in A(G), \xi \in L^{p_n}(\widehat{G})\big).$$

As $A(G) \cdot L^p(\widehat{G})$ is dense in $L^p(\widehat{G})$ for all p, we conclude that there exists $T_n \in \mathcal{CB}(L^{p_n}(\widehat{G}))$ with $Ti_n = i_nT_n$ and $\|T_n\|_{cb} \le \|T\|_{cb}$. It now follows that

$$T_n(a \cdot \xi) = L(a) \cdot \xi, \quad a \cdot T_n(\xi) = R(a) \cdot \xi \qquad \big(a \in A(G), \xi \in L^{p_n}(\widehat{G})\big).$$

Let $A_0 = A(G) \cap VN_r(G)$ regarded as a subspace of $A(G)$ (so that A_0 is $\phi_* j_{(0)}(L)$). Consider the map $i_{(0)}^*\phi_*^{-1}\colon A(G) \to L_{(0)}^*$, which maps A_0 into $L^p(\widehat{G})$ for all p. Let $\iota_n\colon A_0 \to L^{p_n}(\widehat{G})$ be the resulting map. We have that $a \cdot \iota_n(b) = \iota_n(ab)$ for $a \in A(G)$ and $b \in A_0$. Hence we see that

$$a \cdot T_n\iota_n(b) = R(a) \cdot \iota_n(b) = \iota_n\big(R(a)b\big) = \iota_n\big(aL(b)\big) = a \cdot \iota_n\big(L(b)\big)$$
$$\big(a \in A(G), b \in A_0\big).$$

It follows that $T_n\iota_n = \iota_nL$. By much the same argument as at the start of the proof, we see that for $a = (a_{ij}) \in \mathbb{M}_m(A_0)$,

$$\|(L)_m(a)\| = \lim_n \|(\iota_nL(a_{ij}))\| = \lim_n \|(T_n\iota_n(a_{ij}))\|$$
$$\le \|T\|_{cb}\lim_n \|(\iota_n(a_{ij}))\| \le \|T\|_{cb}\|(a_{ij})\|.$$

Thus L is completely bounded, with $\|L\|_{cb} \le \|T\|_{cb}$, and so induces a member of $M_{cb}A(G)$. We can now follow the end of the previous proof to conclude that $T \in \widehat{\pi}_{cb}(M_{cb}A(G))$. ∎

6 Analogues of the Figa-Talamanca–Herz Algebras

In Section 2 we saw that the Figa-Talamanca–Herz algebras $A_p(G)$ naturally appeared. We have now developed enough theory to suggest a definition for analogues of the Figa-Talamanca–Herz algebras, starting with $A(G)$ instead of $L^1(G)$. Indeed, consider the map $\pi^p \colon A(G) \to \mathcal{B}(L^p(\widehat{G}))$ as in the previous section. We define $A_p(\widehat{G})$ to be the image of π_*^p equipped with the quotient norm, so that $A_p(\widehat{G})$ is isometric to $(L^p(\widehat{G})\widehat{\otimes}L^{p'}(\widehat{G}))/\ker \pi_*^p$. By Proposition 5.2 we see that $A_p(\widehat{G})$ is a subspace of $C_\lambda^*(G)$, which we would expect, as this is the "dual" statement to the fact that $A_p(G) \subseteq C_0(G)$.

The following says, informally, that $A_2(\widehat{G}) = L^1(G)$.

Theorem 6.1 *For a locally compact group G, $A_2(\widehat{G})$ is equal to $\lambda(L^1(G))$ as a subset of $C_\lambda^*(G)$, and the norm on $A_2(\widehat{G})$ agrees with that on $L^1(G)$.*

Proof We recall the isometric isomorphism $\theta \colon L^2(G) \to L^2(\widehat{G})$ given by Proposition 4.8, $\theta(f) = j_{(0)}^*\rho(\Delta^{-3/4}\check{f})$ for $f \in C_{00}(G)^2$. Then, from above,

$$\pi_*^2\big(\theta(f) \otimes \theta(g)\big) = \lambda\big(\Delta^{-3/4}\check{f} \cdot K(\Delta^{-3/4}\check{g})\big) = \lambda\big(\Delta^{-1/2}\check{f} \cdot g\big) = \lambda\big(Kf \cdot g\big)$$
$$\big(f, g \in C_{00}(G)^2\big).$$

As $K \colon L^2(G) \to L^2(G)$ is unitary, by continuity, we have that $\pi_*^2(\theta(\xi) \otimes \theta(\eta)) = \lambda(K\xi \cdot \eta)$ for $\xi, \eta \in L^2(G)$. In particular, by Cauchy–Schwarz, we have that $K\xi \cdot \eta \in L^1(G)$ with $\|K\xi \cdot \eta\|_1 \le \|K\xi\|_2\|\eta\|_2$, for $\xi, \eta \in L^2(G)$.

For $\tau \in L^2(G)\widehat{\otimes}L^2(G)$ and $\epsilon > 0$, we can find sequences (ξ_n) and (η_n) in $L^2(G)$ with

$$\tau = \sum_n \xi_n \otimes \tau_n, \quad \|\tau\| \le \sum_n \|\xi_n\|_2\|\eta_n\|_2 < \|\tau\| + \epsilon.$$

Then let $f = \sum_n K\xi_n \cdot \eta_n \in L^1(G)$, the sum converging by Cauchy–Schwarz, with $\|f\|_1 \le \|\tau\| + \epsilon$. We see that

$$\pi_*^2(\theta \otimes \theta)\tau = \lambda\left(\sum_n K\xi_n \cdot \eta_n\right) = \lambda(f).$$

As $(\theta \otimes \theta)$ is an isometric isomorphism, it follows that $A_2(\widehat{G}) \subseteq \lambda(L^1(G))$.

For $f \in L^1(G)$, let $\xi = K(|f|^{1/2}) \in L^2(G)$ and $\eta = f|f|^{-1/2} \in L^2(G)$, so that $\pi_*^2(\theta(\xi) \otimes \theta(\eta)) = f$, and $\|\xi\|_2 = \|\eta\|_2 = \|f\|_1^{1/2}$. We conclude that $A_2(\widehat{G}) = \lambda(L^1(G))$ with the quotient norm on $A_2(\widehat{G})$ agreeing with the L^1 norm on $\lambda(L^1(G))$. ∎

In particular, $A_2(\widehat{G})$ is a subalgebra of $C_\lambda^*(G)$, and, with the quotient norm, $A_2(\widehat{G})$ is a Banach algebra. We have been unable to determine if the same is true for $A_p(\widehat{G})$, for $p \ne 2$. However, we do have the following.

Proposition 6.2 *For $1 < p < \infty$, $A_p(\widehat{G})$ contains a dense subset that is a subalgebra of $C_\lambda^*(G)$.*

Proof Let $a, b, c, d \in C_{00}(G)^2$, let $\xi_1 = j^*_{(0)}(a), \xi_2 = j^*_{(0)}(c) \in L^p(\widehat{G})$, and let $\eta_1 = j^*_{(0)}(b), \eta_2 = j^*_{(0)}(d) \in L^{p'}(\widehat{G})$. Then, as above, $\pi^p_*(\xi_1 \otimes \eta_1) = \lambda(a \cdot Kb)$ and $\pi^p_*(\xi_2 \otimes \eta_2) = \lambda(c \cdot Kd)$. Let $f = (a \cdot Kb)(c \cdot Kd) \in C_{00}(G)^2$.

Pick $g_1 \in C_{00}(G)$ with $\int_G g_1(s) \, ds = 1$. Let $X \subseteq G$ be a compact set containing the support of f, and let $Y \subseteq G$ be a compact set containing the support of g_1. Let $e = |Y|^{-1}\chi_{(XY)^{-1}Y}$ and $f = \chi_Y$, so that $g_0 = e\check{f} \in C_{00}(G)$. Then, for $s \in G$,

$$g_0(s) = \int_G e(t)\check{f}(t^{-1}s) \, dt = \frac{1}{|Y|} \int_{(XY)^{-1}Y} \chi_Y(s^{-1}t) \, dt = \frac{|sY \cap (XY)^{-1}Y|}{|Y|},$$

so if $s \in (XY)^{-1}$, then $sY \subseteq (XY)^{-1}Y$ and so $g_0(s) = |sY||Y|^{-1} = 1$. Now let $g = (\nabla^{-1/2}g_1)(\nabla^{-1/2}g_0) \in C_{00}(G)^2$, so for $s \in X$,

$$(\nabla^{-1/2}g_1)(\nabla^{-1/2}g_0)(s^{-1}) = \int_G \nabla(t)^{-1/2}g_1(t)\nabla(t^{-1}s^{-1})^{-1/2}g_0(t^{-1}s^{-1}) \, dt$$

$$= \nabla(s)^{1/2} \int_Y g_1(t)g_0(t^{-1}s^{-1}) \, dt$$

$$= \nabla(s)^{1/2} \int_Y g_1(t) \, dt = \nabla(s)^{1/2},$$

because if $t \in Y$, then $t^{-1}s^{-1} \in (XY)^{-1}$. Hence, for $s \in X$, we see that $Kg(s) = g(s^{-1})\nabla(s)^{-1/2} = 1$. Thus $f \cdot Kg = f$, showing that

$$\pi^p_*(\xi_1 \otimes \eta_1)\pi^p_*(\xi_2 \otimes \eta_2) = \lambda(f) = \pi^p_*\big(j^*_{(0)}\rho(f) \otimes j^*_{(0)}\rho(g)\big).$$

We conclude that

$$\lin\Big\{\pi^p_*\big(j^*_{(0)}\rho(f) \otimes j^*_{(0)}\rho(g)\big) : f, g \in C_{00}(G)^2\Big\} \subseteq A_p(\widehat{G})$$

is a dense subalgebra. ∎

One could work with $\pi^{p,cb}_*$ instead, which would lead to an operator space version of $A_p(\widehat{G})$, say $OA_p(\widehat{G})$. However, as this would naturally use the operator space projective tensor product, in general we would only have that $A_p(\widehat{G}) \subseteq OA_p(\widehat{G})$. Indeed, in [30], Runde used the natural operator space structure on vector valued *commutative L^p* spaces to define algebras $OA_p(G)$ as an attempt to find an operator space structure on $A_p(G)$. If G is abelian, then by the Fourier transform, $OA_p(\widehat{G})$ has an unambiguous meaning (either ours or Runde's). Let $PM_p(\widehat{G})$ be the weak*-closure of $\pi^p(A(G))$ in $\mathcal{B}(L^p(\widehat{G}))$. After [30, Proposition 2.1], in a remark attributed to Pisier, it is shown that there exist abelian G with $PM_p(\widehat{G}) \not\subseteq \mathcal{CB}(L^p(\widehat{G}))$. It follows that $OA_p(\widehat{G}) \neq A_p(\widehat{G})$. If we wish to view $OA_p(\widehat{G})$ as a generalisation of $A_p(G)$, then this a problem!

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