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THE VECTOR-VALUED TENT SPACES T^1 AND T^∞

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Abstract

Tent spaces of vector-valued functions were recently studied by Hytönen, van Neerven and Portal with an eye on applications to H^∞-functional calculi. This paper extends their results to the endpoint cases $p = 1$ and $p = \infty$ along the lines of earlier work by Harboure, Torrea and Viviani in the scalar-valued case. The main result of the paper is an atomic decomposition in the case $p = 1$, which relies on a new geometric argument for cones. A result on the duality of these spaces is also given.

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1. Introduction

Coifman *et al.* introduced in [4] the concept of tent spaces that provides a neat framework for several ideas and techniques in harmonic analysis. In particular, they defined the spaces T^p, $1 \le p < \infty$, that are relevant for square functions, and consist of functions f on the upper half-space $\mathbb{R}^{n+1}_+$ for which the L^p norm of the conical square function is finite:

$$\int_{\mathbb{R}^n} \bigg(\int_{\Gamma(x)} |f(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \bigg)^{p/2} dx < \infty,$$

where $\Gamma(x)$ denotes the cone $\{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ at $x \in \mathbb{R}^n$. Typical functions in these spaces arise for instance from harmonic extensions u to $\mathbb{R}^{n+1}_+$ of L^p functions on $\mathbb{R}^n$ according to the formula $f(y,t) = t\partial_t u(y,t)$.

Tent spaces were approached by Harboure *et al.* in [5] as L^p spaces of L^2-valued functions, which gave an abstract way to deduce many of their basic properties. Indeed, for $1 < p < \infty$, the mapping $Jf(x) = 1_{\Gamma(x)} f$ is readily seen to embed T^p in $L^p(\mathbb{R}^n; L^2(\mathbb{R}^{n+1}_+))$, when $\mathbb{R}^{n+1}_+$ is equipped with the measure $dy\,dt/t^{n+1}$. Furthermore, they showed that T^p is embedded as a complemented subspace, which not only

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implies its completeness, but also gives a way to prove a few other properties, such as equivalence of norms defined by cones of different aperture and the duality $(T^p)^* \simeq T^{p'}$, where $1/p + 1/p' = 1$.

Treatment of the endpoint cases $p = 1$ and $p = \infty$ requires more careful inspection. First, the space T^∞ was defined in [4] as the space of functions g on $\mathbb{R}^{n+1}_+$ for which

$$\sup_B \frac{1}{|B|} \int_{\widehat{B}} |g(y,t)|^2 \frac{dy\,dt}{t} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ and where $\widehat{B} \subset \mathbb{R}^{n+1}_+$ denotes the 'tent' over B (see Section 4). The tent space duality is now extended to the endpoint case as $(T^1)^* \simeq T^\infty$. Moreover, functions in T^1 admit a decomposition into atoms a each of which is supported in $\widehat{B}$ for some ball $B \subset \mathbb{R}^n$ and satisfies

$$\int_{\widehat{B}} |a(y,t)|^2 \frac{dy\,dt}{t} \le \frac{1}{|B|}.$$

As for the embeddings, it is proven in [5] that T^1 embeds in the $L^2(\mathbb{R}^{n+1}_+)$-valued Hardy space $H^1(\mathbb{R}^n; L^2(\mathbb{R}^{n+1}_+))$, while T^∞ embeds in $\mathrm{BMO}(\mathbb{R}^n; L^2(\mathbb{R}^{n+1}_+))$, the space of $L^2(\mathbb{R}^{n+1}_+)$-valued functions with bounded mean oscillation.

The study of vector-valued analogues of these spaces was initiated by Hytönen, van Neerven and Portal in [7], where they followed the ideas from [5] and proved the analogous embedding results for $T^p(X)$ with $1 < p < \infty$ under the assumption that X is a Banach space with unconditional martingale differences (UMD). It should be noted that, for non-Hilbertian X, the L^2 integrals had to be replaced by stochastic integrals or some equivalent objects, which in turn required further adjustments in proofs, namely the lattice maximal functions that appeared in [5] were replaced by an appeal to Stein's inequality for conditional expectation operators. Later on, Hytönen and Weis provided in [8] a scale of vector-valued versions of the quantity appearing above in the definition of T^∞.

This paper continues the work on the endpoint cases and provides definitions for $T^1(X)$ and $T^\infty(X)$. The main result decomposes a $T^1(X)$ function into atoms using a geometric argument for cones. The original decomposition argument in [4] is inherently scalar-valued and not as such suitable for stochastic integrals. Moreover, the spaces $T^1(X)$ and $T^\infty(X)$ are embedded in certain Hardy and BMO spaces, respectively, much in the spirit of [5]. The theory of vector-valued stochastic integration (see van Neerven and Weis [14]) is used throughout the paper.

2. Preliminaries

2.1. Notation. Random variables are taken to be defined on a fixed probability space whose probability measure and expectation are denoted by $\mathbb{P}$ and $\mathbb{E}$. The integral average (with respect to Lebesgue measure) over a measurable set $A \subset \mathbb{R}^n$ is written as $\fint_A = |A|^{-1} \int_A$, where $|A|$ stands for the Lebesgue measure of A. For a ball B in $\mathbb{R}^n$ we write x_B and r_B for its center and radius, respectively. Throughout the paper X is

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assumed to be a real Banach space and $\langle \xi, \xi^* \rangle$ is used to denote the duality pairing between $\xi \in X$ and $\xi^* \in X^*$. Isomorphism of Banach spaces is expressed using $\simeq$. By $\alpha \lesssim \beta$ it is meant that there exists a constant C such that $\alpha \le C\beta$. Quantities α and β are comparable, $\alpha \eqsim \beta$, if $\alpha \lesssim \beta$ and $\beta \lesssim \alpha$.

2.2. Stochastic integration. We start by discussing the correspondence between Gaussian random measures and stochastic integrals of real-valued functions. Recall that a Gaussian random measure on a σ-finite measure space (M, μ) is a mapping W that takes subsets of M with finite measure to (centered) Gaussian random variables in such a manner that:

(i) the variance $\mathbb{E}W(A)^2 = \mu(A)$;
(ii) for all disjoint A and B the random variables $W(A)$ and $W(B)$ are independent and $W(A \cup B) = W(A) + W(B)$ almost surely.

Since for Gaussian random variables the notions of independence and orthogonality are equivalent, it suffices to consider their pairwise independence in the definition above. Given a Gaussian random measure W, we obtain a linear isometry from $L^2(M)$ to $L^2(\mathbb{P})$, our stochastic integral, by first defining $\int_M 1_A \, dW = W(A)$ and then extending by linearity and density to the whole of $L^2(M)$. On the other hand, if we are in possession of such an isometry, we may define a Gaussian random measure W by sending a subset A of M with finite measure to the stochastic integral of 1_A. For more details, see Janson [9, Ch. 7].

A function $f : M \to X$ is said to be weakly L^2 if $\langle f(\cdot), \xi^* \rangle$ is in $L^2(M)$ for all $\xi^* \in X^*$. Such a function is said to be *stochastically integrable* (with respect to a Gaussian random measure W) if there exists a (unique) random variable $\int_M f \, dW$ in X so that for all $\xi^* \in X^*$

$$\Big\langle \int_M f \, dW, \xi^* \Big\rangle = \int_M \langle f(t), \xi^* \rangle \, dW(t) \quad \text{almost surely.}$$

We also say that a function f is stochastically integrable over a measurable subset A of M if $1_A f$ is stochastically integrable. Note, in particular, that each function $f = \sum_k f_k \otimes \xi_k$ in the algebraic tensor product $L^2(M) \otimes X$ is stochastically integrable and that

$$\int_M f \, dW = \sum_k \Big(\int_M f_k \, dW \Big) \xi_k.$$

A detailed theory of vector-valued stochastic integration can be found in van Neerven and Weis [14], see also Rosiński and Suchanecki [15]. Stochastic integrals have a number of nice properties (see [14]).

(i) Khintchine–Kahane inequality: for every stochastically integrable f we have

$$\Big(\mathbb{E} \Big\| \int_M f \, dW \Big\|^p \Big)^{1/p} \eqsim \Big(\mathbb{E} \Big\| \int_M f \, dW \Big\|^q \Big)^{1/q}$$

whenever $1 \le p, q < \infty$.

(ii) Covariance domination: if a function $g \in L^2(M) \otimes X$ is dominated by a function $f \in L^2(M) \otimes X$ in covariance, that is, if

$$\int_M \langle g(t), \xi^* \rangle^2 \, d\mu(t) \leq \int_M \langle f(t), \xi^* \rangle^2 \, d\mu(t)$$

for all $\xi^* \in X^*$, then

$$\mathbb{E} \left\| \int_M g \, dW \right\|^2 \leq \mathbb{E} \left\| \int_M f \, dW \right\|^2.$$

(iii) Dominated convergence: if a sequence (f_k) of stochastically integrable functions is dominated in covariance by a single stochastically integrable function and

$$\int_M \langle f_k(t), \xi^* \rangle^2 \, d\mu(t) \to 0$$

for all $\xi^* \in X^*$, then

$$\mathbb{E} \left\| \int_M f_k \, dW \right\|^2 \to 0.$$

In particular, if a sequence (A_k) of measurable sets satisfies $1_{A_k} \to 0$ pointwise almost everywhere, then for every f in $L^2(M) \otimes X$,

$$\mathbb{E} \left\| \int_{A_k} f \, dW \right\|^2 \to 0.$$

The expression

$$\left(\mathbb{E} \left\| \int_M f \, dW \right\|^2 \right)^{1/2}$$

defines a norm on the space of (equivalence classes of) strongly measurable stochastically integrable functions $f : M \to X$. However, the norm is not generally complete, unless X is a Hilbert space. For convenience, we operate mainly with functions in $L^2(M) \otimes X$ and denote their completion under the norm above by $\gamma(M; X)$.

This space can be identified with the space of γ-radonifying operators from $L^2(M)$ to X (see [14] and the survey [13]). We note the following facts.

(i) Given an $m \in L^\infty(M)$, the multiplication operator $f \mapsto mf$ on $L^2(M) \otimes X$ has norm $\|m\|_{L^\infty(M)}$.

(ii) For K-convex X (see [13, Section 10]) the duality $\gamma(M; X)^* = \gamma(M; X^*)$ holds and realizes for $f \in L^2(M) \otimes X$ and $g \in L^2(M) \otimes X^*$ via

$$\langle f, g \rangle = \int_M \langle f(t), g(t) \rangle \, d\mu(t).$$

A family $\mathcal{T}$ of operators in $\mathcal{L}(X)$ is said to be *γ-bounded* if for every finite collection of operators $T_k \in \mathcal{T}$ and vectors $\xi_k \in X$,

$$\mathbb{E} \left\| \sum_k \gamma_k T_k \xi_k \right\|^2 \lesssim \mathbb{E} \left\| \sum_k \gamma_k \xi_k \right\|^2,$$

where (γ_k) is an independent sequence of standard Gaussians.

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Observe, that families of operators obtained by composing operators from (a finite number of) other γ-bounded families are also γ-bounded. It follows from covariance domination and Fubini's theorem, that the family of operators $f \mapsto mf$ is γ-bounded on $L^p(\mathbb{R}^n; X)$ whenever the multipliers m are chosen from a bounded set in $L^\infty(\mathbb{R}^n)$.

The following continuous-time result for γ-bounded families is common knowledge (to be found in Kalton and Weis [10]).

Lemma 2.1. *Assume that X does not contain a closed subspace isomorphic to c_0. If the range of an X-strongly measurable function $A : M \to \mathcal{L}(X)$ is γ-bounded, then for every strongly measurable stochastically integrable function $f : M \to X$ the strongly measurable function $t \mapsto A(t)f(t) : M \to X$ is also stochastically integrable and satisfies*

$$\mathbb{E}\left\|\int_M A(t)f(t)\,dW(t)\right\|^2 \lesssim \mathbb{E}\left\|\int_M f(t)\,dW(t)\right\|^2.$$

Recall that X-strong measurability of a function $A : M \to \mathcal{L}(X)$ requires $A(\cdot)\xi : M \to X$ to be strongly measurable for every $\xi \in X$. For simple functions $A : M \to \mathcal{L}(X)$ the lemma above is immediate from the definition of γ-boundedness and requires no assumption regarding containment of c_0, as the function $t \mapsto A(t)f(t) : M \to X$ is also in $L^2(M) \otimes X$. Assuming A to be simple is anyhow too restrictive for applications and to consider nonsimple functions A we need to handle more general stochastically integrable functions than just those in $L^2(M) \otimes X$.

Our choice of (M, μ) will be the upper half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty)$ equipped with the measure $dy\,dt/t^{n+1}$. We will simplify our notation and write $\gamma(X) = \gamma(\mathbb{R}^{n+1}_+; X)$; in what follows, stochastic integration is performed on $\mathbb{R}^{n+1}_+$.

2.3. The UMD property and averaging operators. It is often necessary to assume that our Banach space X is UMD. This has the crucial implication, known as *Stein's inequality* (see Bourgain [2] and Clément *et al.* [3]), that every increasing family of conditional expectation operators is γ-bounded on $L^p(X)$ whenever $1 < p < \infty$. While this is proven in the given references only in the case of probability spaces, it can be generalized to the σ-finite case such as ours with no difficulty. Namely, let us consider filtrations on $\mathbb{R}^n$ generated by systems of dyadic cubes, that is, by collections $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$, where each $\mathcal{D}_k$ is a disjoint cover of $\mathbb{R}^n$ consisting of cubes Q of the form $x_Q + [0, 2^{-k})^n$ and every $Q \in \mathcal{D}_k$ is a union of 2^n cubes in $\mathcal{D}_{k+1}$. The conditional expectation operators or averaging operators are then given for each integer k by

$$f \mapsto \sum_{Q\in\mathcal{D}_k} 1_Q \fint_Q f, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; X).$$

Composing such an operator with multiplication by an indicator 1_Q of a dyadic cube Q, we arrive through Stein's inequality to the conclusion that the family $\{A_Q\}_{Q\in\mathcal{D}}$ of localized averaging operators

$$A_Q f = 1_Q \fint_Q f,$$

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is γ-bounded on $L^p(\mathbb{R}^n; X)$ whenever $1 < p < \infty$. The following result of Mei [11] allows us to replace dyadic cubes by balls.

LEMMA 2.2. *There exist $n + 1$ systems of dyadic cubes such that every ball B is contained in a dyadic cube Q_B from one of the systems and $|B| \lesssim |Q_B|$.*

Stein's inequality together with the lemma above guarantees that the family $\{A_B : B \text{ ball in } \mathbb{R}^n\}$ is γ-bounded on $L^p(\mathbb{R}^n; X)$ whenever $1 < p < \infty$. Indeed, for each ball B we can write

$$A_B = 1_B \frac{|Q_B|}{|B|} A_{Q_B} 1_B.$$

This was proven already in [7].

It will be useful to consider smoothed or otherwise different versions of indicators $1_B(x) = 1_{[0,1)}(|x - x_B|/r_B)$. Given a measurable $\psi : [0, \infty) \to \mathbb{R}$ with $1_{[0,1)} \le |\psi| \le 1_{[0,\alpha)}$ for some $\alpha > 1$, we define the averaging operators

$$A^{\psi}_{y,t} f(x) = \psi\Big(\frac{|x - y|}{t}\Big) \frac{1}{c_\psi t^n} \int_{\mathbb{R}^n} \psi\Big(\frac{|z - y|}{t}\Big) f(z)\, dz, \quad x \in \mathbb{R}^n,$$

where

$$c_\psi = \int_{\mathbb{R}^n} \psi(|x|)^2\, dx.$$

Again, under the assumption that X is UMD and $1 < p < \infty$, the γ-boundedness of the family $\{A^{\psi}_{y,t} : (y, t) \in \mathbb{R}^{n+1}_+\}$ of operators on $L^p(\mathbb{R}^n; X)$ follows at once when we write

$$A^{\psi}_{y,t} = \psi\Big(\frac{|\cdot - y|}{t}\Big) \frac{|Q_{B(y,\alpha t)}|}{c_\psi t^n} A_{Q_{B(y,\alpha t)}} \psi\Big(\frac{|\cdot - y|}{t}\Big).$$

Observe, that the function $(y, t) \mapsto A^{\psi}_{y,t}$ from $\mathbb{R}^{n+1}_+$ to $\mathcal{L}(L^p(\mathbb{R}^n; X))$ is $L^p(\mathbb{R}^n; X)$-strongly measurable. Recall also that every UMD space is K-convex and cannot contain a closed subspace isomorphic to c_0.

3. Overview of tent spaces

3.1. Tent spaces $T^p(X)$. Let us equip the upper half-space $\mathbb{R}^{n+1}_+$ with the measure $dy\, dt/t^{n+1}$ and a Gaussian random measure W. Recall the definition of the cone $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}$ at $x \in \mathbb{R}^n$.

Let $1 \le p < \infty$. We wish to define a norm on the space of functions $f : \mathbb{R}^{n+1}_+ \to X$ for which $1_{\Gamma(x)} f \in L^2(\mathbb{R}^{n+1}_+) \otimes X$ for almost every $x \in \mathbb{R}^n$ by

$$\|f\|_{T^p(X)} = \Big(\int_{\mathbb{R}^n} \Big(\mathbb{E}\Big\|\int_{\Gamma(x)} f\, dW\Big\|^2\Big)^{p/2} dx\Big)^{1/p}$$

and use the Khintchine–Kahane inequality to write

$$\|f\|_{T^p(X)} \eqsim \Big(\mathbb{E}\Big\|\int_{\Gamma(\cdot)} f\, dW\Big\|^p_{L^p(\mathbb{R}^n; X)}\Big)^{1/p},$$

but issues concerning measurability need closer inspection.

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Lemma 3.1. *Suppose that $f : \mathbb{R}^{n+1}_+ \to X$ is such that $1_{\Gamma(x)} f \in L^2(\mathbb{R}^{n+1}_+) \otimes X$ for almost every $x \in \mathbb{R}^n$. Then:*

(1) *the function $x \mapsto 1_{\Gamma(x)} f$ is strongly measurable from $\mathbb{R}^n$ to $\gamma(X)$;*

(2) *the function $x \mapsto \int_{\Gamma(x)} f\, dW$ is strongly measurable from $\mathbb{R}^n$ to $L^2(\mathbb{P}; X)$ and may be considered, when $\|f\|_{T^p(X)} < \infty$, as a random $L^p(\mathbb{R}^n; X)$ function;*

(3) *the function $x \mapsto (\mathbb{E}\| \int_{\Gamma(x)} f\, dW\|^2)^{1/2}$ agrees almost everywhere with a lower semicontinuous function so that the set*

$$\left\{x \in \mathbb{R}^n : \left(\mathbb{E}\left\|\int_{\Gamma(x)} f\, dW\right\|^2\right)^{1/2} > \lambda\right\}$$

is open whenever $\lambda > 0$.

Proof. Denote by A_k the set $\{(y, t) \in \mathbb{R}^{n+1}_+ : t > 1/k\}$ and write $f_k = 1_{A_k} f$. It is clear that for each positive integer k, the functions $x \mapsto 1_{\Gamma(x)} f_k$ and $x \mapsto \int_{\Gamma(x)} f_k\, dW$ are strongly measurable and continuous since

$$\mathbb{E}\left\|\int_{\Gamma(x)\Delta\Gamma(x')} f_k\, dW\right\|^2 \to 0, \quad \text{as} \quad x \to x'.$$

Furthermore, $1_{\Gamma(x)} f_k \to 1_{\Gamma(x)} f$ in $\gamma(X)$ for almost every $x \in \mathbb{R}^n$ since

$$\mathbb{E}\left\|\int_{\Gamma(x)} (f - f_k)\, dW\right\|^2 = \mathbb{E}\left\|\int_{\Gamma(x)\setminus A_k} f\, dW\right\|^2 \to 0.$$

Consequently, $x \mapsto 1_{\Gamma(x)} f$ and $x \mapsto \int_{\Gamma(x)} f\, dW$ are strongly measurable. Moreover, the pointwise limit of an increasing sequence of real-valued continuous functions is lower semicontinuous, which proves the third claim. □

Definition 3.2. Let $1 \leq p < \infty$. The tent space $T^p(X)$ is defined as the completion under $\|\cdot\|_{T^p(X)}$ of the space of (equivalence classes of) functions $\mathbb{R}^{n+1}_+ \to X$ (in what follows, '$T^p(X)$ functions') such that $1_{\Gamma(x)} f \in L^2(\mathbb{R}^{n+1}_+) \otimes X$ for almost every x in $\mathbb{R}^n$ and $\|f\|_{T^p(X)} < \infty$.

As was mentioned in the previous section, it is useful to consider the more general situation where the indicator of a ball is replaced by a measurable function $\phi : [0, \infty) \to \mathbb{R}$ with $1_{[0,1)} \leq |\phi| \leq 1_{[0,\alpha)}$ for some $\alpha > 1$. Let us assume, in addition, that ϕ is continuous at zero. For functions $f : \mathbb{R}^{n+1}_+ \to X$ such that $(y, t) \mapsto \phi(|x - y|/t) f(y, t) \in L^2(\mathbb{R}^{n+1}_+) \otimes X$ for almost every $x \in \mathbb{R}^n$, the strong measurability of

$$x \mapsto \left((y, t) \mapsto \phi\left(\frac{|x - y|}{t}\right) f(y, t)\right) \quad \text{and} \quad x \mapsto \int_{\Gamma(x)} \phi\left(\frac{|x - y|}{t}\right) f(y, t)\, dW(y, t)$$

are treated as in the case of $\phi(|x - y|/t) = 1_{[0,1)}(|x - y|/t) = 1_{\Gamma(x)}(y, t)$.

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3.2. Embedding $T^p(X)$ into $L^p(\mathbb{R}^n;\gamma(X))$. A collection of results from the paper [7] by Hytönen, van Neerven and Portal is presented next. Following the idea of Harboure, Torrea and Viviani [5], the tent spaces are embedded into L^p spaces of $\gamma(X)$-valued functions by

$$Jf(x) = 1_{\Gamma(x)} f, \quad x \in \mathbb{R}^n.$$

Furthermore, for simple $L^2(\mathbb{R}^{n+1}_+) \otimes X$-valued functions F on $\mathbb{R}^n$ we define an operator N by

$$(NF)(x; y, t) = 1_{B(y,t)}(x) \fint_{B(y,t)} F(z; y, t)\, dz, \quad x \in \mathbb{R}^n, (y, t) \in \mathbb{R}^{n+1}_+.$$

Assuming that X is UMD, we can now view $T^p(X)$ as a complemented subspace of $L^p(\mathbb{R}^n;\gamma(X))$:

THEOREM 3.3. *Suppose that X is UMD and let $1 < p < \infty$. Then N extends to a bounded projection on $L^p(\mathbb{R}^n;\gamma(X))$ and J extends to an isometry from $T^p(X)$ onto the image of $L^p(\mathbb{R}^n;\gamma(X))$ under N.*

The following result shows the comparability of different tent space norms.

THEOREM 3.4. *Suppose that X is UMD, let $1 < p < \infty$ and let $1_{[0,1)} \le |\phi| \le 1_{[0,\alpha)}$. For every function f in $T^p(X)$ the function $(y, t) \mapsto \phi(|x - y|/t) f(y, t)$ is stochastically integrable for almost every $x \in \mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} \mathbb{E} \Big\| \int_{\mathbb{R}^{n+1}_+} \phi\Big(\frac{|x-y|}{t}\Big) f(y, t)\, dW(y, t) \Big\|^p dx \eqsim \int_{\mathbb{R}^n} \mathbb{E} \Big\| \int_{\Gamma(x)} f\, dW \Big\|^p dx.$$

The proof relies on the boundedness of modified projection operators

$$(N_\phi F)(x; y, t) = \phi\Big(\frac{|x-y|}{t}\Big) \fint_{B(y,t)} F(z; y, t)\, dz, \quad x \in \mathbb{R}^n,\ (y, t) \in \mathbb{R}^{n+1}_+$$

and the observation that the embedding

$$J_\phi f(x; y, t) = \phi\Big(\frac{|x-y|}{t}\Big) f(y, t), \quad x \in \mathbb{R}^n,\ (y, t) \in \mathbb{R}^{n+1}_+.$$

can be written as $J_\phi f = N_\phi J f$. In particular, this shows that the norms given by cones of different apertures are comparable. Indeed, choosing $\phi = 1_{[0,\alpha)}$ gives the norm where $\Gamma(x)$ is replaced by the cone $\Gamma_\alpha(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < \alpha t\}$ with aperture $\alpha > 1$.

Identification of tent spaces $T^p(X)$ with complemented subspaces of $L^p(\mathbb{R}^n;\gamma(X))$ gives a powerful way to deduce their duality.

THEOREM 3.5. *Suppose that X is UMD and let $1 < p < \infty$. Then the dual of $T^p(X)$ is $T^{p'}(X^*)$, where $1/p + 1/p' = 1$, and the duality is realized for functions $f \in T^p(X)$ and $g \in T^{p'}(X^*)$ via*

$$\langle f, g \rangle = c_n \int_{\mathbb{R}^{n+1}_+} \langle f(y, t), g(y, t) \rangle \frac{dy\, dt}{t},$$

where c_n is the volume of the unit ball in $\mathbb{R}^n$.

The following theorem combines results from [7, Theorem 4.8] and [8, Corollary 4.3 and Theorem 1.3]. The tent space $T^\infty(X)$ is defined in the next section.

THEOREM 3.6. *Suppose that X is UMD and let Ψ be a Schwartz function with vanishing integral. Then the operator*

$$T_\Psi f(y,t) = \Psi_t * f(y)$$

is bounded from $L^p(\mathbb{R}^n;X)$ to $T^p(X)$ whenever $1 < p < \infty$, from $H^1(\mathbb{R}^n;X)$ to $T^1(X)$ and from $\mathrm{BMO}(\mathbb{R}^n;X)$ *to $T^\infty(X)$.*

4. Tent spaces $T^1(X)$ and $T^\infty(X)$

Having completed our overview of tent spaces $T^p(X)$ with $1 < p < \infty$ we turn to the endpoint cases $p = 1$ and $p = \infty$, of which the latter remains to be defined. As for the case $p = 1$, our aim is to show that $T^1(X)$ is isomorphic to a complemented subspace of the Hardy space $H^1(\mathbb{R}^n;\gamma(X))$ of $\gamma(X)$-valued functions on $\mathbb{R}^n$. In the case $p = \infty$, we introduce the space $T^\infty(X)$, which is shown to embed in $\mathrm{BMO}(\mathbb{R}^n;\gamma(X))$, that is, the space of $\gamma(X)$-valued functions whose mean oscillation is bounded. The idea of these embeddings was originally put forward by Harboure *et al.* in the scalar-valued case (see [5]).

Recall that the tent over an open set $E \subset \mathbb{R}^n$ is defined by $\widehat{E} = \{(y,t) \in \mathbb{R}^{n+1}_+ : B(y,t) \subset E\}$ or equivalently by

$$\widehat{E} = \mathbb{R}^{n+1}_+ \Big\backslash \bigcup_{x\notin E} \Gamma(x).$$

Observe that while cones are open, tents are closed. Truncated cones are also needed: for $x \in \mathbb{R}^n$ and $r > 0$ we define $\Gamma(x;r) = \{(y,t) \in \Gamma(x) : t < r\}$.

In [8] Hytönen and Weis adjusted the quantities that define scalar-valued atoms and T^∞ functions in terms of tents to more suitable ones that rely on averages of square functions. More precisely for scalar-valued g on $\mathbb{R}^{n+1}_+$ we have

$$\begin{aligned}\int_B \int_{\Gamma(x;r_B)} |g(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\,dx &= \int_B \int_{\mathbb{R}^n\times(0,r_B)} 1_{B(y,t)}(x)|g(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\,dx \\ &= \int_0^{r_B} \int_{2B} |g(y,t)|^2 |B\cap B(y,t)| \frac{dy\,dt}{t^{n+1}},\end{aligned}$$

from which one reads

$$\int_{\widehat{B}} |g(y,t)|^2 \frac{dy\,dt}{t} \lesssim \int_B \int_{\Gamma(x;r_B)} |g(y,t)|^2 \frac{dy\,dt}{t^{n+1}}\,dx \lesssim \int_{\widehat{3B}} |g(y,t)|^2 \frac{dy\,dt}{t}.$$

This motivates the definition of a $T^1(X)$ *atom* as a function $a : \mathbb{R}^{n+1}_+ \to X$ such that for some ball B we have $\operatorname{supp} a \subset \widehat{B}$, $1_{\Gamma(x)}a \in L^2(\mathbb{R}^{n+1}_+) \otimes X$ for almost every $x \in B$ and

$$\int_B \mathbb{E}\Big\| \int_{\Gamma(x)} a\,dW \Big\|^2 dx \le \frac{1}{|B|}.$$

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Then $1_{\Gamma(x)} a$ differs from zero only when $x \in B$ and so

$$\|a\|_{T^1(X)} = \int_{\mathbb{R}^n} \Big(\mathbb{E} \Big\| \int_{\Gamma(x)} a \, dW \Big\|^2 \Big)^{1/2} dx \le |B|^{1/2} \Big(\int_B \mathbb{E} \Big\| \int_{\Gamma(x)} a \, dW \Big\|^2 dx \Big)^{1/2} \le 1.$$

Furthermore, for (equivalence classes of) functions $g : \mathbb{R}^{n+1}_+ \to X$ such that $1_{\Gamma(x;r)} g \in L^2(\mathbb{R}^{n+1}_+) \otimes X$ for every $r > 0$ and almost every $x \in \mathbb{R}^n$ we define

$$\|g\|_{T^\infty(X)} = \sup_B \Big(\fint_B \mathbb{E} \Big\| \int_{\Gamma(x;r_B)} g \, dW \Big\|^2 dx \Big)^{1/2} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Definition 4.1. The tent space $T^\infty(X)$ is defined as the completion under $\| \cdot \|_{T^\infty(X)}$ of the space of (equivalence classes of) functions $g : \mathbb{R}^{n+1}_+ \to X$ such that $1_{\Gamma(x;r)} g \in L^2(\mathbb{R}^{n+1}_+) \otimes X$ for every $r > 0$ and almost every $x \in \mathbb{R}^n$ and for which $\|g\|_{T^\infty(X)} < \infty$.

4.1. The atomic decomposition. In an atomic decomposition, we aim to express a $T^1(X)$ function as an infinite sum of (multiples of) atoms. The original proof for scalar-valued tent spaces by Coifman, Meyer and Stein [4, Theorem 1(c)] rests on a lemma that allows one to exchange integration in the upper half-space with 'double integration', which is something unthinkable when 'double integration' consists of both standard and stochastic integration. The following argument provides a more geometrical reasoning. We start with a covering lemma.

Lemma 4.2. *Suppose that an open set $E \subset \mathbb{R}^n$ has finite measure. Then there exist disjoint balls $B^j \subset E$ such that*

$$\widehat{E} \subset \bigcup_{j \ge 1} \widehat{5B^j}.$$

Proof. We start by writing $d_1 = \sup_{B \subset E} r_B$ and choosing a ball $B^1 \subset E$ with radius $r_1 > d_1/2$. Then we proceed inductively: suppose that balls $B^1, \dots, B^k$ have been chosen and write

$$d_{k+1} = \sup\{ r_B : B \subset E, B \cap B^j = \emptyset, j = 1, \dots, k \}.$$

If possible, we choose $B^{k+1} \subset E$ with radius $r_{k+1} > d_{k+1}/2$ so that B^{k+1} is disjoint from all $B^1, \dots, B^k$. Let then $(y, t) \in \widehat{E}$. In order to show that $B(y, t) \subset 5B^j$ for some j we note that $B(y, t)$ has to intersect some B^j: indeed, if there are only finitely many balls B^j, then $y \in \overline{B^j}$ for some j. On the other hand, if there are infinitely many balls B^j and they are all disjoint from $B(y, t)$, then $r_j > d_j/2 > t/2$ and E has infinite measure, which is a contradiction. Thus, there exists a j for which $B(y, t) \cap B^j \ne \emptyset$ and so $B(y, t) \subset 5B^j$ because $t \le d_j \le 2r_j$ by construction. □

Given a $0 < \lambda < 1$, we define the extension of a measurable set $E \subset \mathbb{R}^n$ by

$$E^*_\lambda = \{ x \in \mathbb{R}^n : M 1_E(x) > \lambda \}.$$

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Here M is the Hardy–Littlewood maximal operator assigning the maximal function

$$Mf(x) = \sup_{B \ni x} \fint_B |f(y)|\,dy, \qquad x \in \mathbb{R}^n,$$

to every locally integrable real-valued f. Note that the lower semicontinuity of Mf guarantees that E^*_λ is open while the weak $(1,1)$ inequality for the maximal operator assures us that $|E^*_\lambda| \leq \lambda^{-1}|E|$.

We continue by constructing sectors opening in finite number of directions of our choice. To do this, we fix vectors $v_1, \ldots, v_N$ in the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$ such that

$$\max_{1 \leq m \leq N} v \cdot v_m \geq \frac{\sqrt{3}}{2}$$

for every $v \in \mathbb{S}^{n-1}$. In other words, every $v \in \mathbb{S}^{n-1}$ makes an angle of no more than $30°$ with one of v_m. We write

$$S_m = \left\{ v \in \mathbb{S}^{n-1} : v \cdot v_m \geq \frac{\sqrt{3}}{2} \right\}$$

and observe that the angle between two $v, v' \in S_m$ is at most $60°$, i.e. $v \cdot v' \geq \frac{1}{2}$. Consequently, $|v - v'| \leq 1$.

For every $x \in \mathbb{R}^n$ and $t > 0$, write

$$R_m(x,t) = \left\{ y \in B(x,t) : \frac{y - x}{|y - x|} \in S_m \text{ or } y = x \right\}$$

for the sector opening from x in the direction of v_m. For any two $y, y' \in R_m(x,t)$, the angle between $y - x$ and $y' - x$ is at most $60°$ (when y and y' are different from x), implying that $|y - y'| \leq t$. Hence the proportion of $R_m(x,t)$ in $B(y,t)$ for any $y \in R_m(x,t)$ is a dimensional constant, in symbols,

$$\frac{|R_m(x,t)|}{|B(y,t)|} = c(n), \qquad y \in R_m(x,t).$$

For every $0 < \lambda < c(n)$ it thus holds that $M1_{R_m(x,t)} > \lambda$ in $B(y,t)$ whenever $y \in R_m(x,t)$. Writing $E^* = E^*_{c(n)/2}$ we have now proven the following result.

Lemma 4.3. *If $E \subset \mathbb{R}^n$ is measurable and $y \in R_m(x,t) \subset E$, then $B(y,t) \subset E^*$.*

Note that the next lemma follows easily when $n = 1$ and holds even without the extension. Indeed, if E is an open interval in $\mathbb{R}$ and $x \in E$, then one can choose x_1 and x_2 to be the endpoints of E and obtain $\Gamma(x) \setminus \widehat{E} \subset \Gamma(x_1) \cup \Gamma(x_2)$. On the other hand, for $n \geq 2$ the extension is necessary, which can be seen already by taking E to be an open ball.

Lemma 4.4. *Suppose that an open set $E \subset \mathbb{R}^n$ has finite measure. Then for every $x \in E$ there exist $x_1, \ldots, x_N \in \partial E$, with N depending only on the dimension n, such that*

$$\Gamma(x) \setminus \widehat{E^*} \subset \bigcup_{m=1}^{N} \Gamma(x_m).$$

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Proof. For every $1 \le m \le N$ we may pick $x_m \in \partial E$ in such a manner that

$$\frac{x_m - x}{|x_m - x|} \in S_m$$

and $|x_m - x|$, which we denote by t_m, is minimal (while positive, since E is open). In other words, $R_m(x, t_m) \subset E$. We need to show that for every $(y, t) \in \Gamma(x) \setminus \widehat{E^*}$ the point y is less than t away from one of the x_m. Thus, let $(y, t) \in \Gamma(x) \setminus \widehat{E^*}$, which translates to $|x - y| < t$ and $B(y, t) \not\subset E^*$.

Consider first the case of y not belonging to any $R_m(x, t_m)$. Then for some m,

$$\frac{y - x}{|y - x|} \in S_m \quad \text{and} \quad |y - x| \ge t_m.$$

Now the point

$$z = t_m \frac{y - x}{|y - x|} + x$$

sits in the line segment connecting x and y and satisfies $|z - x| = t_m$. Hence the calculation

$$\begin{aligned} |y - x_m| &\le |y - z| + |z - x_m| \\ &= |y - z| + t_m \left| \frac{z - x}{t_m} - \frac{x_m - x}{t_m} \right| \\ &= |y - z| + |z - x| \left| \frac{z - x}{|z - x|} - \frac{x_m - x}{|x_m - x|} \right| \\ &\le |y - z| + |z - x| \\ &= |y - x| < t, \end{aligned}$$

where we used the fact that $|v - v'| \le 1$ for any two $v, v' \in S_m$, shows that $(y, t) \in \Gamma(x_m)$.

On the other hand, if $y \in R_m(x, t_m)$ for some m, then $|y - x_m| \le t_m$, since the diameter of $R_m(x, t_m)$ does not exceed t_m. Also $B(y, t_m) \subset E^*$ by Lemma 4.3 so that $t_m < t$ since $B(y, t) \not\subset E^*$, which shows that $(y, t) \in \Gamma(x_m)$. □

We are now ready to state and prove the atomic decomposition for $T^1(X)$ functions.

Theorem 4.5. *For every function f in $T^1(X)$ there exist countably many atoms a_k and real numbers λ_k such that*

$$f = \sum_k \lambda_k a_k \quad \text{and} \quad \sum_k |\lambda_k| \lesssim \|f\|_{T^1(X)}.$$

Proof. Let f be a function in $T^1(X)$ and write

$$E_k = \left\{ x \in \mathbb{R}^n : \left(\mathbb{E} \left\| \int_{\Gamma(x)} f \, dW \right\|^2 \right)^{1/2} > 2^k \right\}$$

for each integer k. By Lemma 3.1, each E_k is open. For each k, apply Lemma 4.2 to the open set E_k^* in order to get disjoint balls $B_k^j \subset E_k^*$ for which

$$\widehat{E_k^*} \subset \bigcup_{j \ge 1} \widehat{5B_k^j}.$$

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Further, for each of these covers, take a (rough) partition of unity, that is, a collection of functions χ_k^j for which

$$0 \le \chi_k^j \le 1, \quad \sum_{j=1}^{\infty} \chi_k^j = 1 \text{ on } \widehat{E_k^*} \quad \text{and} \quad \operatorname{supp} \chi_k^j \subset \widehat{5B_k^j}.$$

For instance, one can define χ_k^1 as the indicator of $\widehat{5B_k^1}$ and χ_k^j for $j \ge 2$ as the indicator of

$$\widehat{5B_k^j} \Big\backslash \bigcup_{i=1}^{j-1} \widehat{5B_k^i}.$$

Write $A_k = \widehat{E_k^*} \setminus \widehat{E_{k+1}^*}$. We are now in the position to decompose f as

$$f = \sum_{k \in \mathbb{Z}} 1_{A_k} f = \sum_{k \in \mathbb{Z}} \sum_{j \ge 1} \chi_k^j 1_{A_k} f = \sum_{k \in \mathbb{Z}} \sum_{j \ge 1} \lambda_k^j a_k^j,$$

where

$$\lambda_k^j = |5B_k^j|^{1/2} \Big(\int_{5B_k^j} \mathbb{E} \Big\| \int_{\Gamma(x) \cap A_k} f \, dW \Big\|^2 \, dx \Big)^{1/2}.$$

Observe that $a_k^j = \chi_k^j 1_{A_k} f / \lambda_k^j$ is an atom supported in $\widehat{5B_k^j}$.

It remains to estimate the sum of λ_k^j. For $x \notin E_{k+1}$,

$$\mathbb{E} \Big\| \int_{\Gamma(x) \cap A_k} f \, dW \Big\|^2 \, dx \le 4^{k+1}$$

by the definition of E_{k+1}. The cones at points $x \in E_{k+1}$ are the problematic ones and so in order to estimate λ_k^j, we need to exploit the fact that $1_{A_k} f$ vanishes on $\widehat{E_{k+1}^*}$. Let $x \in E_{k+1}$ and use Lemma 4.4 to pick $x_1, \ldots, x_N \in \partial E_{k+1}$, where $N \le c'(n)$, such that

$$\Gamma(x) \setminus \widehat{E_{k+1}^*} \subset \bigcup_{m=1}^{N} \Gamma(x_m).$$

Now $x_1, \ldots, x_N \notin E_{k+1}$ which allows us to estimate

$$\mathbb{E} \Big\| \int_{\Gamma(x) \cap A_k} f \, dW \Big\|^2 \le \Big(\sum_{m=1}^{N} \Big(\mathbb{E} \Big\| \int_{\Gamma(x_m)} f \, dW \Big\|^2 \Big)^{1/2} \Big)^2 \le N^2 4^{k+1}.$$

Hence, integrating over $5B_k^j$ we obtain

$$\int_{5B_k^j} \mathbb{E} \Big\| \int_{\Gamma(x) \cap A_k} f \, dW \Big\|^2 \, dx \le |5B_k^j| c'(n)^2 4^{k+1}.$$

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Consequently,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{j\geq 1}\lambda_k^j &\leq c'(n)\sum_{k\in\mathbb{Z}}2^{k+1}\sum_{j\geq 1}|5B_k^j|\\ &\leq c'(n)5^n\sum_{k\in\mathbb{Z}}2^{k+1}|E_k^*|\\ &\leq c'(n)\lambda(n)^{-1}5^n\sum_{k\in\mathbb{Z}}2^{k+1}|E_k|\\ &\leq c'(n)\lambda(n)^{-1}5^n\|f\|_{T^1(X)}.\end{aligned}$$

□

It is perhaps surprising that the UMD assumption is not needed for the atomic decomposition.

4.2. Embedding $T^1(X)$ into $H^1(\mathbb{R}^n;\gamma(X))$ and $T^\infty(X)$ into BMO$(\mathbb{R}^n;\gamma(X))$. Armed with the atomic decomposition we proceed to the embeddings. Suppose that $\psi:[0,\infty)\to\mathbb{R}$ is smooth, that $1_{[0,1)}\leq|\psi|\leq 1_{[0,\alpha)}$ for some $\alpha>2$ and that $\int_{\mathbb{R}^n}\psi(|x|)\,dx=0$. For functions $f:\mathbb{R}^{n+1}_+\to X$ we define

$$J_\psi f(x;y,t)=\psi\Big(\frac{|x-y|}{t}\Big)f(y,t),\quad x\in\mathbb{R}^n,\ (y,t)\in\mathbb{R}^{n+1}_+,$$

and note immediately that $\int_{\mathbb{R}^n}J_\psi f(x)\,dx=0$.

Recall also that functions in the Hardy space $H^1(\mathbb{R}^n;\gamma(X))$ are composed of atoms $A:\mathbb{R}^n\to\gamma(X)$ each of which is supported on a ball $B\subset\mathbb{R}^n$, has zero integral and satisfies

$$\int_B\mathbb{E}\Big\|\int_{\mathbb{R}^{n+1}_+}A(x;y,t)\,dW(y,t)\Big\|^2\,dx\leq\frac{1}{|B|}.$$

For further references, see Blasco [1] and Hytönen [6].

Theorem 4.6. *Suppose that X is UMD. Then J_ψ embeds $T^1(X)$ into $H^1(\mathbb{R}^n;\gamma(X))$ and $T^\infty(X)$ into* BMO$(\mathbb{R}^n;\gamma(X))$.

Proof. We argue that J_ψ takes $T^1(X)$ atoms to (multiples of) $H^1(\mathbb{R}^n;\gamma(X))$ atoms. If a $T^1(X)$ atom a is supported in $\widehat{B}$ for some ball $B\subset\mathbb{R}^n$, then $J_\psi a$ is supported in αB and $\int J_\psi a=0$. Moreover, since X is UMD, we may use the equivalence of $T^2(X)$ norms (Theorem 3.4) and write

$$\int_{\alpha B}\mathbb{E}\Big\|\int_{\mathbb{R}^{n+1}_+}\psi\Big(\frac{|x-y|}{t}\Big)a(y,t)\,dW(y,t)\Big\|^2\,dx\lesssim\int_B\mathbb{E}\Big\|\int_{\Gamma(x)}a\,dW\Big\|^2\,dx\leq\frac{1}{|B|}.$$

The boundedness of J_ψ from $T^1(X)$ to $H^1(\mathbb{R}^n;\gamma(X))$ follows. In addition, since $1_{[0,1)}\leq|\psi|$, it follows that $\|f\|_{T^1(X)}\leq\|J_\psi f\|_{L^1(\mathbb{R}^n;\gamma(X))}\leq\|J_\psi f\|_{H^1(\mathbb{R}^n;\gamma(X))}$ and so J_ψ is also bounded from below.

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To see that J_ψ maps $T^\infty(X)$ boundedly into $\mathrm{BMO}(\mathbb{R}^n;\gamma(X))$, we need to show that

$$\left(\fint_B \mathbb{E}\left\|\int_{\mathbb{R}^{n+1}_+}\left(J_\psi g(x;y,t) - \fint_B J_\psi g(z;y,t)\,dz\right)dW(y,t)\right\|^2 dx\right)^{1/2} \lesssim \|g\|_{T^\infty(X)}$$

for all balls $B \subset \mathbb{R}^n$. We partition the upper half-space into $\mathbb{R}^n \times (0, r_B)$ and the sets $A_k = \mathbb{R}^n \times [2^{k-1}r_B, 2^k r_B)$ for positive integers k and study each piece separately.

On $\mathbb{R}^n \times (0, r_B)$,

$$\begin{aligned}\left(\fint_B \mathbb{E}\left\|\int_{\mathbb{R}^n\times(0,r_B)} \psi\left(\frac{|z-y|}{t}\right)g(y,t)\,dW(y,t)\right\|^2 dz\right)^{1/2} &\le \left(\fint_B \mathbb{E}\left\|\int_{\Gamma_\alpha(x;r_B)} g\,dW\right\|^2 dx\right)^{1/2}\\ &\lesssim \|g\|_{T^\infty}\end{aligned}$$

since $|\psi| \le 1_{[0,\alpha)}$ and the $T^2(X)$ norms are comparable (Theorem 3.4). Furthermore, as one can justify by approximating ψ with simple functions,

$$\begin{aligned}&\left(\mathbb{E}\left\|\int_{\mathbb{R}^n\times(0,r_B)} g(y,t)\fint_B \psi\left(\frac{|z-y|}{t}\right)dz\,dW(y,t)\right\|^2\right)^{1/2}\\ &\qquad\le \left(\fint_B \mathbb{E}\left\|\int_{\mathbb{R}^n\times(0,r_B)} \psi\left(\frac{|z-y|}{t}\right)g(y,t)\,dW(y,t)\right\|^2 dz\right)^{1/2},\end{aligned}$$

which can be estimated from above by $\|g\|_{T^\infty}$, as above.

For each k and $x \in B$, we claim that

$$\left|\fint_B\left(\psi\left(\frac{|x-y|}{t}\right) - \psi\left(\frac{|z-y|}{t}\right)\right)dz\right| \lesssim 2^{-k}1_{\Gamma_{\alpha+2}(x)}(y,t),$$

whenever $(y,t) \in A_k$. Indeed, if $(y,t) \in A_k \cap \Gamma_{\alpha+2}(x)$, we may use the fact that

$$\left|\psi\left(\frac{|x-y|}{t}\right) - \psi\left(\frac{|z-y|}{t}\right)\right| \lesssim \sup|\psi'|\frac{|x-z|}{t} \lesssim \frac{r_B}{2^k r_B} = 2^{-k}$$

for all $z \in B$, while for $(y,t) \in A_k \setminus \Gamma_{\alpha+2}(x)$ we have $|y-x| \ge (\alpha+2)t \ge \alpha t + 2r_B$ so that $|y-z| \ge |y-x| - |x-z| \ge \alpha t$ for each $z \in B$, which results in

$$\int_B\left(\psi\left(\frac{|x-y|}{t}\right) - \psi\left(\frac{|z-y|}{t}\right)\right)dz = 0.$$

This gives

$$\begin{aligned}&\left(\fint_B \mathbb{E}\left\|\int_{A_k}\frac{g(y,t)}{|B|}\int_B\left(\psi\left(\frac{|x-y|}{t}\right) - \psi\left(\frac{|z-y|}{t}\right)\right)dz\,dW(y,t)\right\|^2 dx\right)^{1/2}\\ &\qquad\le 2^{-k}\left(\fint_B \mathbb{E}\left\|\int_{A_k\cap\Gamma_{\alpha+2}(x)} g\,dW\right\|^2 dx\right)^{1/2}.\end{aligned}$$

But every $A_k \cap \Gamma_{\alpha+2}(x)$ with $x \in B$ is contained in any $\Gamma_{\alpha+6}(z)$ with $z \in 2^k B$. Indeed, for all $(y,t) \in A_k \cap \Gamma_{\alpha+2}(x)$,

$$|y-z| \le |y-x| + |x-z| \le (\alpha+2)t + (2^k+1)r_B \le (\alpha+6)t.$$

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Hence,

$$\fint_B \mathbb{E}\Big\| \int_{A_k \cap \Gamma_{\alpha+2}(x)} g\, dW \Big\|^2 dx \le \fint_{2^k B} \mathbb{E}\Big\| \int_{\Gamma_{\alpha+6}(z)} g\, dW \Big\|^2 dz.$$

Summing up, we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \Big(\fint_B \mathbb{E}\Big\| \int_{A_k} g(y,t) \fint_B \Big(\psi\Big(\frac{|x-y|}{t}\Big) - \psi\Big(\frac{|z-y|}{t}\Big)\Big)\, dz\, dW(y,t) \Big\|^2 dx \Big)^{1/2} \\
\le \sum_{k=1}^{\infty} 2^{-k} \Big(\fint_{2^k B} \mathbb{E}\Big\| \int_{\Gamma_{\alpha+6}(z)} g\, dW \Big\|^2 dz \Big)^{1/2} \\
\lesssim \|g\|_{T^\infty(X)}.
\end{aligned}$$

To see that $\|g\|_{T^\infty(X)} \lesssim \|J_\psi g\|_{\mathrm{BMO}(\mathbb{R}^n;\gamma(X))}$ it suffices to fix a ball $B \subset \mathbb{R}^n$ and show, that for every $x \in B$,

$$1_{\Gamma(x;r_B)}(y,t) \le \Big| \psi\Big(\frac{|x-y|}{t}\Big) - \fint_{(\alpha+2)B} \psi\Big(\frac{|z-y|}{t}\Big)\, dz \Big|,$$

since this gives us

$$\begin{aligned}
\fint_B \mathbb{E}\Big\| \int_{\Gamma(x;r_B)} g\, dW \Big\|^2 dx &\le \fint_B \mathbb{E}\Big\| \int_{\mathbb{R}^{n+1}_+} g(y,t)\Big(\psi\Big(\frac{|x-y|}{t}\Big) - \fint_{(\alpha+2)B} \psi\Big(\frac{|z-y|}{t}\Big)\, dz \Big) \Big\|^2 dx \\
&\le (\alpha+2)^n \|J_\psi g\|_{\mathrm{BMO}(\mathbb{R}^n;\gamma(X))}.
\end{aligned}$$

Now that $1_{[0,1)} \le |\psi|$ and $\int_{\mathbb{R}^n} \psi(|x|)\, dx = 0$, it is enough to prove for a fixed $x \in B$, that

$$\operatorname{supp} \psi\Big(\frac{|\cdot - y|}{t}\Big) \subset (\alpha+2)B$$

for every $(y,t) \in \Gamma(x; r_B)$, i.e. that $B(y, \alpha t) \subset (\alpha+2)B$ whenever $|x-y| < t < r_B$. This is indeed true, as every $z \in B(y, \alpha t)$ satisfies

$$|z-x| \le |z-y| + |y-x| < (\alpha+1) r_B.$$

We have established that, also in this case, J_ψ is bounded from below. □

It follows that different $T^1(X)$ norms are equivalent in the sense that whenever $1_{[0,1)} \le |\phi| \le 1_{[0,\alpha)}$ for some $\alpha > 1$, we can take smooth $\psi : [0,\infty) \to \mathbb{R}$ with $|\phi| \le |\psi| \le 1_{[0,2\alpha)}$ to obtain

$$\|f\|_{T^1(X)} \le \|J_\phi f\|_{L^1(\mathbb{R}^n;\gamma(X))} \le \|J_\psi f\|_{L^1(\mathbb{R}^n;\gamma(X))} \le \|J_\psi f\|_{H^1(\mathbb{R}^n;\gamma(X))} \lesssim \|f\|_{T^1(X)}.$$

To identify $T^1(X)$ as a complemented subspace of $H^1(\mathbb{R}^n;\gamma(X))$ we define a projection first on the level of test functions. Let us write

$$T(X) = \{ f : \mathbb{R}^{n+1}_+ \to X : \ 1_{\Gamma(x)} f \in L^2(\mathbb{R}^{n+1}_+) \otimes X \text{ for almost every } x \in \mathbb{R}^n \}$$

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and

$$S(\gamma(X)) = \operatorname{span}\{F : \mathbb{R}^n \times \mathbb{R}^{n+1}_+ \to X : \ F(x; y, t) = \Psi(x; y, t) f(y, t) \text{ for some } \Psi \in L^\infty(\mathbb{R}^n \times \mathbb{R}^{n+1}_+) \text{ and } f \in T(X)\}.$$

Observe that J_ψ maps $T(X)$ into $S(\gamma(X))$ and that $S(\gamma(X))$ intersects $L^p(\mathbb{R}^n; \gamma(X))$ densely for all $1 < p < \infty$ and likewise for $H^1(\mathbb{R}^n; \gamma(X))$.

For F in $S(\gamma(X))$ we define

$$(N_\psi F)(x; y, t) = \psi\Big(\frac{|x-y|}{t}\Big)\frac{1}{c_\psi t^n}\int_{\mathbb{R}^n} \psi\Big(\frac{|z-y|}{t}\Big) F(z; y, t)\, dz,$$

where $c_\psi = \int_{\mathbb{R}^n} \psi(|x|)^2\, dx$. Now N_ψ is a projection and satisfies $N_\psi J_\psi = J_\psi$. Also, for every $F \in S(\gamma(X))$ we find an $f \in T(X)$ so that $N_\psi F = J_\psi f$, namely

$$f(y, t) = \frac{1}{c_\psi t^n}\int_{\mathbb{R}^n} \psi\Big(\frac{|z-y|}{t}\Big) F(z; y, t)\, dz, \quad (y, t) \in \mathbb{R}^{n+1}_+.$$

THEOREM 4.7. *Suppose that X is UMD. Then N_ψ extends to a bounded projection on $H^1(\mathbb{R}^n; \gamma(X))$ and J_ψ extends to an isomorphism from $T^1(X)$ onto the image of $H^1(\mathbb{R}^n; \gamma(X))$ under N_ψ.*

PROOF. Let $1 < p < \infty$. For simple $L^2(\mathbb{R}^{n+1}_+) \otimes X$ -valued functions F defined on $\mathbb{R}^n$ the mapping $(y, t) \mapsto F(\cdot; y, t) : \mathbb{R}^{n+1}_+ \to L^p(\mathbb{R}^n; X)$ is in $L^2(\mathbb{R}^{n+1}_+) \otimes L^p(\mathbb{R}^n; X)$ and we may express N_ψ using the averaging operators as

$$(N_\psi F)(\cdot; y, t) = A^\psi_{y,t}(F(\cdot; y, t)).$$

Since X is UMD, Stein's inequality guarantees γ-boundedness for the range of the strongly $L^p(\mathbb{R}^n; X)$-measurable function $(y, t) \mapsto A^\psi_{y,t}$, and so by Lemma 2.1,

$$\mathbb{E}\Big\|\int_{\mathbb{R}^{n+1}_+} A^\psi_{y,t}(F(\cdot; y, t))\, dW(y, t)\Big\|^p_{L^p(\mathbb{R}^n;X)} \lesssim \mathbb{E}\Big\|\int_{\mathbb{R}^{n+1}_+} F(\cdot; y, t)\, dW(y, t)\Big\|^p_{L^p(\mathbb{R}^n;X)}.$$

In other words, $\|N_\psi F\|^p_{L^p(\mathbb{R}^n;\gamma(X))} \lesssim \|F\|^p_{L^p(\mathbb{R}^n;\gamma(X))}$.

We wish to define a suitable $\mathcal{L}(\gamma(X))$-valued kernel K that allows us to express N_ψ as a Calderón–Zygmund operator

$$N_\psi F(x) = \int_{\mathbb{R}^n} K(x, z) F(z)\, dz, \quad F \in L^p(\mathbb{R}^n; \gamma(X)).$$

For distinct $x, z \in \mathbb{R}^n$ and we define $K(x, z)$ as multiplication by

$$(y, t) \mapsto \psi\Big(\frac{|x-y|}{t}\Big)\frac{1}{c_\psi t^n}\psi\Big(\frac{|z-y|}{t}\Big),$$

and so

$$\|K(x, z)\|_{\mathcal{L}(\gamma(X))} = \sup_{(y,t)\in\mathbb{R}^{n+1}}\Big|\psi\Big(\frac{|x-y|}{t}\Big)\frac{1}{c_\psi t^n}\psi\Big(\frac{|z-y|}{t}\Big)\Big|.$$

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For $|x-z| > \alpha t$,

$$\psi\Big(\frac{|x-y|}{t}\Big)\frac{1}{c_\psi t^n}\psi\Big(\frac{|z-y|}{t}\Big) = 0$$

while $|x-z| \le \alpha t$ guarantees that

$$\Big|\psi\Big(\frac{|x-y|}{t}\Big)\frac{1}{c_\psi t^n}\psi\Big(\frac{|z-y|}{t}\Big)\Big| \le \frac{1}{c_\psi t^n} \le \frac{\alpha^n}{c_\psi |x-z|^n}.$$

Hence,

$$\|K(x,z)\|_{\mathcal{L}(\gamma(X))} \lesssim \frac{1}{|x-z|^n}.$$

Similarly,

$$\|\nabla_x K(x,z)\|_{\mathcal{L}(\gamma(X))} = \sup_{(y,t)\in\mathbb{R}^{n+1}_+}\Big|\psi'\Big(\frac{|x-y|}{t}\Big)\frac{1}{c_\psi t^{n+1}}\psi\Big(\frac{|z-y|}{t}\Big)\Big| \lesssim \frac{1}{|x-z|^{n+1}}.$$

Thus K is indeed a Calderón–Zygmund kernel.

Now $\int_{\mathbb{R}^n}\psi(|x|)\,dx = 0$ implies that $\int_{\mathbb{R}^n} N_\psi F(x)\,dx = 0$ for $F \in H^1(\mathbb{R}^n;\gamma(X))$, which guarantees that N_ψ maps $H^1(\mathbb{R}^n;\gamma(X))$ into itself (see Meyer and Coifman [12, Ch. 7, Section 4]). □

We proceed to the question of duality of $T^1(X)$ and $T^\infty(X^*)$. Assuming that X is UMD, it is both reflexive and K-convex so that the duality

$$H^1(\mathbb{R}^n;\gamma(X))^* \simeq \mathrm{BMO}(\mathbb{R}^n;\gamma(X)^*) \simeq \mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$$

holds (recall the discussion in Section 2) and we may define the adjoint of N_ψ by $\langle F, N_\psi^* G\rangle = \langle N_\psi F, G\rangle$, where $F \in H^1(\mathbb{R}^n;\gamma(X))$ and $G \in \mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$. Moreover, as $T^1(X)$ is isomorphic to the image of $H^1(\mathbb{R}^n;\gamma(X))$ under N_ψ, its dual $T^1(X)^*$ is isomorphic to the image of $\mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$ under the adjoint N_ψ^* and the question arises whether the latter is isomorphic to $T^\infty(X^*)$. For J_ψ to give this isomorphism (and to be onto) one could try and follow the proof strategy of the case $1 < p < \infty$ and give an explicit definition of N_ψ^* on a dense subspace of $\mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$. Even though the properties of the kernel K of N_ψ guarantee that N_ψ^* formally agrees with N_ψ on $L^p(\mathbb{R}^n;\gamma(X^*))$, it is problematic to find suitable dense subspaces of $\mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$.

In order to address these issues in more detail, we specify another pair of test function classes, namely

$$\begin{aligned}\widetilde{T}(X) = \{g : \mathbb{R}^{n+1}_+ \to X :\ & 1_{\Gamma(x;r)} g \in L^2(\mathbb{R}^{n+1}_+) \otimes X \text{ for every } r > 0 \\ & \text{and for almost every } x \in \mathbb{R}^n\}\end{aligned}$$

and

$$\begin{aligned}\widetilde{S}(\gamma(X)) = \operatorname{span}\{G : \mathbb{R}^n \times \mathbb{R}^{n+1}_+ \to X :\ & G(x;y,t) = \Psi(x;y,t)g(y,t) \\ & \text{for some } \Psi \in L^\infty(\mathbb{R}^n \times \mathbb{R}^{n+1}_+) \text{ and } g \in \widetilde{T}(X)\} / \{\text{constant functions}\}.\end{aligned}$$

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Since $\int_{\mathbb{R}^n} \psi(|x|)\,dx = 0$, the projection N_ψ is well-defined on $\widetilde{S}(\gamma(X))$. Moreover, given any $G \in \widetilde{S}(\gamma(X))$ we can write

$$g(y,t) = \frac{1}{c_\psi t^n} \int_{\mathbb{R}^n} \psi\Big(\frac{|z-y|}{t}\Big) G(z;y,t)\,dz$$

to define a function $g \in \widetilde{T}(X)$ for which $N_\psi G = J_\psi g$. But $\widetilde{S}(\gamma(X))$ has only weak*-dense intersection with $\mathrm{BMO}(\mathbb{R}^n;\gamma(X))$ (recall that $X \simeq X^{**}$). Nevertheless, J_ψ is an isomorphism from $T^\infty(X)$ onto the closure of the image of $\widetilde{S}(\gamma(X)) \cap \mathrm{BMO}(\mathbb{R}^n;\gamma(X))$ under N_ψ. It is not clear whether test functions are dense in the closure of their image under the projection.

The following relaxed duality result is still valid.

Theorem 4.8. *Suppose that X is UMD. Then $T^\infty(X^*)$ isomorphic to a norming subspace of $T^1(X)^*$ and its action is realized for functions $f \in T^1(X)$ and $g \in T^\infty(X^*)$ via*

$$\langle f, g\rangle = c \int_{\mathbb{R}^{n+1}_+} \langle f(y,t), g(y,t)\rangle \frac{dy\,dt}{t},$$

where c depends on the dimension n.

Proof. Fix a smooth $\psi : [0,\infty) \to \mathbb{R}$ such that $1_{[0,1)} \le |\psi| \le 1_{[0,\alpha)}$ for some $\alpha > 2$ and $\int_{\mathbb{R}^n} \psi(|x|)\,dx = 0$. By Theorem 4.7, $T^1(X)$ is isomorphic to the image of $H^1(\mathbb{R}^n;\gamma(X))$ under N_ψ, from which it follows that the dual $T^1(X)^*$ is isomorphic to the image of $\mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$ under the adjoint projection N_ψ^*, which formally agrees with N_ψ. The space $T^\infty(X^*)$, on the other hand, is isomorphic to the closure of the image of $\widetilde{S}(\gamma(X^*)) \cap \mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$ under N_ψ in $\mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$ and hence is a closed subspace of $T^1(X)^*$. We can pair a function $f \in T^1(X)$ with a function $g \in T^\infty(X^*)$ using the pairing of $J_\psi f$ and $J_\psi g$ and the atomic decomposition of f to obtain

$$\begin{aligned}\langle f, g\rangle = \sum_k \langle J_\psi a_k, J_\psi g\rangle &= \sum_k \lambda_k \int_{\mathbb{R}^n}\int_{\mathbb{R}^{n+1}_+} \psi\Big(\frac{|x-y|}{t}\Big)^2 \langle a_k(y,t), g(y,t)\rangle \frac{dy\,dt}{t^{n+1}}\\ &= c_n c_\psi \sum_k \lambda_k \int_{\mathbb{R}^{n+1}_+} \langle a_k(y,t), g(y,t)\rangle \frac{dy\,dt}{t}\\ &= c_n c_\psi \int_{\mathbb{R}^{n+1}_+} \langle f(y,t), g(y,t)\rangle \frac{dy\,dt}{t},\end{aligned}$$

where c_n denotes the volume of the unit ball in $\mathbb{R}^n$. The space $L^\infty(\mathbb{R}^n) \otimes L^2(\mathbb{R}^{n+1}_+) \otimes X^*$ is weak*-dense in $\mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$ and hence a norming subspace for $H^1(\mathbb{R}^n;\gamma(X))$. As it is contained in $\widetilde{S}(\gamma(X^*)) \cap \mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$, we obtain

$$\begin{aligned}\|f\|_{T^1(X)} \eqsim \|J_\psi f\|_{H^1(\mathbb{R}^n;\gamma(X))} &= \sup_G |\langle J_\psi f, G\rangle| = \sup_G |\langle N_\psi J_\psi f, G\rangle|\\ &= \sup_G |\langle J_\psi f, N_\psi^* G\rangle| \eqsim \sup_g |\langle J_\psi f, J_\psi g\rangle| = \sup_g |\langle f, g\rangle|,\end{aligned}$$

where the suprema are taken over $G \in \widetilde{S}(\gamma(X^*)) \cap \mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))$ with $\|G\|_{\mathrm{BMO}(\mathbb{R}^n;\gamma(X^*))} \le 1$ and $g \in T^\infty(X^*)$ with $\|g\|_{T^\infty(X^*)} \le 1$. □

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References

[1] O. Blasco, 'Hardy spaces of vector-valued functions: duality', *Trans. Amer. Math. Soc.* **308** (1988), 495–507.

[2] J. Bourgain, 'Vector-valued singular integrals and the H^1-BMO duality', in: *Probability Theory and Harmonic Analysis (Cleveland, Ohio, 1983)*, Monographs and Textbooks in Pure and Applied Mathematics, 98 (Dekker, New York, 1986), 1–19.

[3] P. Clément, B. de Pagter, F. A. Sukochev and H. Witvliet, 'Schauder decomposition and multiplier theorems', *Studia Math.* **138** (2000), 135–163.

[4] R. R. Coifman, Y. Meyer and E. M. Stein, 'Some new function spaces and their applications to harmonic analysis', *J. Funct. Anal.* **62**(2) (1985), 304–335.

[5] E. Harboure, J. L. Torrea and B. E. Viviani, 'A vector-valued approach to tent spaces', *J. Anal. Math.* **56** (1991), 125–140.

[6] T. Hytönen, 'Vector-valued wavelets and the Hardy space $H^1(\mathbb{R}^n, X)$', *Studia Math.* **172** (2006), 125–147.

[7] T. Hytönen, J. M. A. M. van Neerven and P. Portal, 'Conical square function estimates in UMD Banach spaces and applications to H^∞-functional calculi', *J. Anal. Math.* **106** (2008), 317–351.

[8] T. Hytönen and L. Weis, 'The Banach space-valued BMO, Carleson's condition, and paraproducts', *J. Fourier Anal. Appl.* **16** 495–513.

[9] S. Janson, *Gaussian Hilbert Spaces*, Cambridge Tracts in Mathematics, 129 (Cambridge University Press, Cambridge, 1997).

[10] N. Kalton and L. Weis, The H^∞-functional calculus and square function estimates, in preparation.

[11] T. Mei, 'BMO is the intersection of two translates of dyadic BMO', *C. R. Math. Acad. Sci. Paris.* **336** (2003), 1003–1006.

[12] Y. Meyer and R. R. Coifman, 'Wavelets', *Calderón–Zygmund and multilinear operators, Translated from the 1990 and 1991 French originals by David Salinger*, Cambridge Studies in Advanced Mathematics, 48 (Cambridge University Press, Cambridge, 1997).

[13] J. M. A. M. van Neerven, 'γ-radonifying operators—a survey', in: *The AMSI–ANU Workshop on Spectral Theory and Harmonic Analysis*, Proceedings of the Centre for Mathematics and its Applications, 44 (Australian National University, Canberra, 2010), 1–61.

[14] J. M. A. M. van Neerven and L. Weis, 'Stochastic integration of functions with values in a Banach space', *Studia Math.* **166** (2005), 131–170.

[15] J. Rosiński and Z. Suchanecki, 'On the space of vector-valued functions integrable with respect to the white noise', *Colloq. Math.* **43** (1980), 183–201; 1981.

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