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DENSITIES AND MEASURES OF LINEAR SETS

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1. **Introduction.** If I_n, I_∞ denote the intervals $[0, n)$, $[0, \infty)$ respectively, we propose to examine the properties of an upper and lower density

(1) $$D^*[\mathscr{P}(I_\infty); S] \overset{\text{def}}{=} \limsup \frac{m^*(S \cap I_n)}{|I_n|} \qquad (n \to \infty)$$

(2) $$D_*[\mathscr{P}(I_\infty); S] \overset{\text{def}}{=} \liminf \frac{m_*(S \cap I_n)}{|I_n|} \qquad (n \to \infty)$$

of a set S belonging to the power set $\mathscr{P}(I_\infty)$ of I_∞ where m^*, m_* denote the outer and inner (linear) Lebesgue measures. (The left sides of (1) and (2) will usually be abbreviated to $D^*(S)$ and $D_*(S)$.) With this rather specialized definition of upper and lower density, we shall nevertheless find it possible to reconcile the earlier work of Knopp [3] on densities of arbitrary subsets of a fixed interval $I=[\alpha, \beta)$ with the more recent work of Buck [1], Hintzman [2], and Niven [4] on densities of (infinite) subsets of the set of positive integers Z^+. In the space $\mathscr{P}(I_\infty)$ we can introduce the notion of a *homogeneous set*, which possesses properties corresponding not only to those of Knopp's homogeneous sets on $\mathscr{P}([\alpha, \beta))$ but also to those of Niven's "uniformly distributed" sequences of nonnegative integers; and which, moreover, can be used to obtain their results. First, we state some of the more obvious general properties of D^* and D_*:

Property (i). $D^*(S)$ is a finitely subadditive outer measure on $\mathscr{P}(I_\infty)$.

Property (ii). $D_*(S)=1-D^*(S')$ is the inner measure corresponding to $D^*(S)$.

If $S \in \mathscr{P}([\alpha, \beta))$, then the correspondence $S \mapsto |\beta-\alpha|^{-1}(S-\alpha)$ associates with each $S \in \mathscr{P}([\alpha, \beta))$ a subset of the fixed interval I_1 and so, without essential loss of generality, we shall consider densities of subsets of I_1.

Property (iii). If $S \in \mathscr{P}(I_1)$ and if $\hat{S}=S+Z$, then

$$D^*(\hat{S}) = m^*(S) = d_{I_1}(S),$$

where $d_{I_1}(S)$ is the density of S in I_1, as defined by Knopp [3, p. 412].

Property (iv). If $S \in \mathscr{P}(Z^+)$ and we define $S^\dagger = \bigcup_{k\in S} [k-1, k)$, then

$$D^*(S^\dagger) = \limsup \frac{1}{n} \sum_{k\in S \cap I_n} 1 = \mu^*(S),$$

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where $\mu^*(S)$ is the upper density of S in $\mathscr{P}(Z^+)$, as defined by Hintzman [2, p. 133]. If we introduce the class $\mathscr{M}$ of all measurable sets and the class $\mathscr{D}$ of all sets having a density, i.e.,

$$\text{(3)}\quad \mathscr{M} = \{S \in \mathscr{P}(I_\infty) \mid \forall X \in \mathscr{P}(I_\infty),\ D^*(X) = D^*(X \cap S) + D^*(X \cap S^c)\}$$

and

$$\text{(4)}\quad \mathscr{D} = \{S \in \mathscr{P}(I_\infty) \mid D^*(S) = D_*(S) = D(S), \text{ say}\},$$

then, on selecting $X = I_\infty$ in (3), we see that $S \in \mathscr{M} \Rightarrow S \in \mathscr{D}$, or $\mathscr{M} \subset \mathscr{D}$. In fact, by a routine adaptation of Hintzman's proof of his Theorem 2, we have

Property (v). $\mathscr{M} \neq \mathscr{D}$ and $S \in \mathscr{M} \Leftrightarrow D^*(S) = D_*(S) = 0$ or 1.

Property (vi). If $S \in \mathscr{P}(Z^+)$ and $S^\dagger$ is defined as in (iv), then Property (v) is Hintzman's principal result. [2, Theorems 1 and 2.]

2. Homogeneity.

DEFINITION 1. $S \in \mathscr{P}(I_\infty)$ is said to be upper homogeneous modulo k of upper density $D^*(k, S)$ if, and only if,

$$\text{(5)}\quad D^*(k, S) = \frac{|I_k|}{|I|} D^*(S \cap I(k))$$

is independent of the particular choice of subinterval I of I_k, where $I(k) = I + kZ$. We define lower homogeneity modulo k analogously by replacing "upper" by "lower" everywhere, $D^*(k, S)$ by $D_*(k, S)$ and $D^*(S \cap I(k))$ by $D_*(S \cap I(k))$, in (2.5).

REMARK. If we select $I = I_k$ in (5), then

$$\text{(6)}\quad D^*(k, S) = D^*(S);$$

whence $D^*(k, S)$ and $D_*(k, S)$ are both independent of k and may be replaced by $D^*(S)$ and $D_*(S)$, respectively.

For our discussion of homogeneous sets it is convenient to introduce classes $H^*(k)$, $H_*(k)$, $H'(k)$, $H^*(\infty)$, and $H_*(\infty)$, where $H^*(k)$ and $H_*(k)$ are the classes of all subsets of I_∞ which are respectively upper and lower homogeneous modulo k. Furthermore, $H'(k) = H^*(k), \cap H_*(k)$, and $H^*(\infty) = \bigcap_{1 \leq k < \infty} H^*(k)$, with $H_*(\infty)$ defined analogously. Elements of $H^*(\infty)$ and $H_*(\infty)$ are referred to as being upper homogeneous and lower homogeneous, respectively.

REMARK. Clearly $S = I_\infty$ is an element of $H'(\infty) = \bigcap_{k=1}^\infty H'(k)$, and so none of the classes $H'(k)$, $H^*(k)$, $H_*(k)$, $H^*(\infty)$, and $H_*(\infty)$ is trivially empty. A less obvious example with $S \in H'(\infty)$ is given by $S = \bigcup_{k=1}^\infty [n\theta, n\theta + \alpha)$, where $0 < \alpha < \theta$ and θ is an irrational element of I_1, (see Example 4 below). For the property of homogeneity itself, we introduce a variant on Definition 1 designed so that the *existence* of the limit on the left side of (7) automatically excludes certain uninteresting sets S with $D(S)=0$ [e.g. $S=\bigcup_{n=1}^{\infty}[n^2k, n^2k+1)\in H^*(\infty)$ has upper density $D^*(S)=0$, but cannot be "homogeneous" because all its elements lie in the interval $[0, 1)$ modulo k].

DEFINITION 2. $S\in\mathscr{P}(I_\infty)$ is homogeneous modulo $k \Leftrightarrow$

$$\lim_{n\to\infty}\frac{m(S\cap I(k)\cap I_n)}{m(S\cap I_n)}=\frac{|I|}{|I_k|},\quad \text{for all subintervals } I \text{ of } I_k. \tag{7}$$

Let $H(k)$ denote the class of all elements of $\mathscr{P}(I_\infty)$ which are homogeneous modulo k and put $H(\infty)=\bigcap_{k=1}^{\infty}H(k)$. We shall refer to $H(\infty)$ as the class of all homogeneous subsets of I_∞.

REMARK. Clearly, if $D(S)$ exists (i.e. $D^*(S)=D_*(S)$) and if $D(S)>0$, then (7) may be written as follows:

$$S\in H(k)\Leftrightarrow D(S)=\frac{|I_k|}{|I|}D(S\cap I(k)),$$

is independent of the choice of subinterval I of I_k.

The following examples serve to distinguish the classes $H(k)$, $H'(k)$, and $H(\infty)$.

EXAMPLE 1. Let S be an extremal subset of I_1, (i.e. by classical measure theory), $\exists S\subset I_1$ such that $m^*(S\cap I)=|I|$ and $m_*(S\cap I)=0$, for all $I\subset I_1$. Then $\hat{S}\in H^*(k)$ but $\hat{S}\notin H(k)$.

EXAMPLE 2. Let $S=\bigcup_{n=0}^{\infty}[nk, nk+1)$. Then $D(S\cap I(1)/|I|=k^{-1}$, for all subintervals I of I_1 and so $S\in H(1)$. On the other hand, note that

$$\frac{|I_k|}{|I|}D(S\cap I(k))=\begin{cases}1 & \text{for} \quad I=[0,1)\\ 0 & \text{for} \quad I=[1,2),\end{cases}$$

whence $S\notin H(k)$ and $H(1)\not\subset H(k)$.

EXAMPLE 3. Let $S=\bigcup_{r=0}^{\infty}[rk+a_r, rk+a_r+1)$, where a_r=residue of r modulo k with $0\leqq a_r<k$. Then $S\in H(k)$. However $S\notin H(k^2)$, because

$$D(S\cap I(k^2))\cdot|I_{k^2}|/|I|=0,\qquad k^{-1}\quad\text{for}\quad I=[1,2),[0,1),$$

respectively. Hence $H(k)\notin H(k^2)$.

We propose now to reconcile homogeneity in $\mathscr{P}(I_\infty)$ with that already defined in (a) $\mathscr{P}(I_1)$ and (b) $\mathscr{P}(Z^+)$.

Case (*a*). If $I\subset I_1$ and if $S\in\mathscr{P}(I_1)$, then Knopp [3] defined the density of S in I as $d_I(S)=m^*(S\cap I)/|I|$. If further, $d_I(S)$ is a constant d independent of I for all $I\subset I_1$, then S will be said to be Knopp-homogeneous. He showed [3] that S

is "Knopp-homogeneous" if, and only if, $d=0$ or 1. A complete characterization of Knopp-homogeneity in terms of upper homogeneity modulo 1 is provided by

PROPOSITION 1. *$\hat{S} \in H^*(1) \Leftrightarrow D^*(\hat{S})=0$, or $1 \Leftrightarrow S$ is Knopp-homogeneous.*

Proof of Proposition 1 rests upon the fact that if $D^*(S)\neq 0$ (i.e., $m^*(S)\neq 0$), then

$$\begin{aligned} \hat{S} \in H^*(1) &\Leftrightarrow D^*(\hat{S} \cap I(1))/D^*(\hat{S}) = |I| \qquad \forall I \subset I_1, \\ &\Leftrightarrow m^*(S \cap I)/m^*(S) = |I| \qquad \forall I \subset I_1, \\ &\Leftrightarrow d_I(S) = m^*(S) \text{ is independent of } I \text{ for all } I \subset I_1, \\ &\Leftrightarrow S \text{ Knopp-homogeneous}, \\ &\Leftrightarrow m^*(S) = 1 \text{ and } D^*(\hat{S}) = 1. \end{aligned}$$

Case (*b*). If $S=\{x_n \mid n \in Z^+\}$ is a set of nonnegative integers then we may choose the natural ordering on S and regard it as a strictly increasing sequence of positive integers to be denoted in what follows by $S=\langle x_n\rangle$. Following Niven [4], we let $A(n;j,k)$ denote the number of terms x_i of the sequence S which satisfy the conditions $x_i\leq n$ and $x_i\equiv j \bmod k$; and $A(n)$ the number of elements of S which satisfy $x_i\leq n$. He defined S to be uniformly distributed modulo k ($k \in Z^+$), whenever

$$\lim_{n\to\infty} \frac{A(n;j,k)}{A(n)} = \frac{1}{k} \quad \text{for} \quad j = 1, 2, \ldots, k. \tag{8}$$

A characterization of uniform distribution modulo k in terms of homogeneity modulo k is furnished by the following proposition.

PROPOSITION 2. (i) *$S^\dagger \in H(k) \Leftrightarrow S$ uniformly distributed modulo k.*

(ii) *$S^\dagger \in H(\infty) \Leftrightarrow S$ is uniformly distributed (i.e., uniformly distributed modulo k for all k).*

Proof of Proposition 2 is simply a matter of examining the definitions; thus

$$\begin{aligned} S^\dagger \in H(k) &\Leftrightarrow \lim_{n\to\infty} \frac{m(S^\dagger \cap I(k) \cap I_n)}{m(S^\dagger \cap I_n)} = \frac{|I|}{|I_k|} \quad \text{for all} \quad I \subset I_k, \\ &\Leftrightarrow \lim_{n\to\infty} \frac{m\left(\bigcup_{\alpha\in S} [\alpha-1, \alpha) \cap I(k) \cap I_n\right)}{m\left(\bigcup_{\alpha\in S} [\alpha-1, \alpha) \cap I_n\right)} = \frac{|I|}{|I_k|} \quad \text{for all} \quad I \subset I_k, \\ &\Leftrightarrow \lim_{n\to\infty} \frac{m\left(\bigcup_{\alpha\in S} [\alpha-1, \alpha) \cap [j-1, j) \cap I_n\right)}{m\left(\bigcup_{\alpha\in S} [\alpha-1, \alpha) \cap I_n\right)} = \frac{1}{k} \quad \text{for} \quad j = 1, 2, \ldots, k, \\ &\Leftrightarrow \lim_{n\to\infty} \frac{A(n;j,k)}{A(n)} = \frac{1}{k} \quad \text{for} \quad j = 1, 2, \ldots, k, \end{aligned}$$

as required.

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Finally, we shall relate sequences of real numbers which are uniformly distributed modulo 1 in the classical sense, to corresponding sequences of positive integers which are uniformly distributed in the sense of (ii), Proposition 2.

THEOREM 1. *Let* $\langle x_n \rangle$ *be an infinite sequence of real numbers.*

(i) *If* $\langle x_n/k \rangle$ *is uniformly distributed modulo* 1 (*in the classical sense*) *for each* $k \in Z^+$ *then the sequence of positive integers* $\langle [x_n] \rangle$ *is uniformly distributed.* (*Here* "$[x]$" *is defined to be the largest integer not exceeding* x.)

(ii) *If* $\langle [kx_n] \rangle$ *is uniformly distributed for each* k, *then* $\langle x_n \rangle$ *is uniformly distributed modulo* 1.

Proof of Theorem 1. For "(*i*)" I follow Niven's idea [4, p. 55], noting that, if $\langle x_n/k \rangle$ is uniformly distributed modulo 1, then $\langle x_n \rangle$ is uniformly distributed throughout $[0, k)$ when reduced modulo k. Hence

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{A(n)} \chi_{J(k)}[\langle [x_m] \rangle]}{A(n)} = \frac{1}{k} \quad \text{for all} \quad J = [j, j+1) \tag{9}$$

$$j = 0, 1, 2, \ldots, k-1,$$

and upon applying the definition of $A(n; j, k)$, we find that the left side of (9) reduces to the left side of (8). Hence $\langle [x_n] \rangle$ is uniformly distributed modulo k, in Niven's sense. Since $\langle x_n/k \rangle$ is uniformly distributed modulo 1 for all k, then $\langle [x_n] \rangle$ is uniformly distributed.

For (ii), we observe that if $\langle x_n \rangle$ is not uniformly distributed modulo 1 then there exists an interval $I = [lr^{-1}, (l+1)r^{-1}]$ of I_1 where $l, r \in Z^+$ such that

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} \chi_{I(1)}[\langle x_m \rangle]}{n} = C\,|I| \quad \text{where} \quad 0 \le C < 1. \tag{10}$$

Then, on proceeding as in (i) and introducing Niven's notation, the condition (10) reduces to

$$\lim_{n\to\infty} \frac{A(n; l, r)}{A(n)} = \frac{C}{r} < \frac{1}{r},$$

where $A(n) = \text{card}\{x_m \mid x_m \in \langle [rx_m] \rangle \text{ and } rx_m \le n\}$. Hence $\langle [rx_m] \rangle$ is not uniformly distributed modulo r, contrary to hypothesis.

A consequence of Theorem 1 is an analogue of a result of H. Weyl on the uniform distribution of the fractional parts of $n^k\theta$ (and is, in fact, deduced from it); see [4, p. 55] for the case $k=1$.

EXAMPLE 4. For $0 < \alpha \le \theta$, where θ is a fixed irrational element of I_1.

$$S = \bigcup_{n=1}^{\infty} [n^k\theta, n^k\theta + \alpha) \in H(\infty)$$

and $\langle [n^k\theta] \rangle$ is therefore uniformly distributed (by Proposition 2).

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