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Abstract

In the present paper we introduce and study the twisted γ-filtration on $K_0(G_s)$, where G_s is a split simple linear algebraic group over a field k of characteristic prime to the order of the center of G_s. We apply this filtration to construct nontrivial torsion elements in γ-rings of twisted flag varieties.

1. Introduction

Let X be a smooth projective variety over a field k. Consider the Grothendieck γ-filtration on $K_0(X)$. It is given by subgroups (see [SGA6, § 2.3] and [Kar98, § 2])

$$\gamma^i K_0(X) = \langle c_{n_1}(b_1) \cdot \cdots \cdot c_{n_m}(b_m) \mid n_1 + \cdots + n_m \geqslant i, b_1, \ldots, b_m \in K_0(X) \rangle, \quad i \geqslant 0$$

generated by products of characteristic classes in K_0. Let $\gamma^i(X)$ be the ith subsequent quotient and let $\gamma^*(X) = \bigoplus_{i \geqslant 0} \gamma^i(X)$ be the associated graded ring called the γ-ring of X.

The ring $\gamma^*(X)$ was invented by Grothendieck to approximate the topological filtration on K_0 and, hence, the Chow ring $\mathrm{CH}^*(X)$ of algebraic cycles modulo rational equivalence. Indeed, by the Riemann–Roch theorem (see [SGA6, § 2]) the ith Chern class c_i induces an isomorphism with $\mathbb{Q}$-coefficients, that is, $c_i : \gamma^i(X; \mathbb{Q}) \xrightarrow{\simeq} \mathrm{CH}^i(X; \mathbb{Q})$. Moreover, in some cases the ring $\gamma^*(X)$ can be used to compute $\mathrm{CH}^*(X)$, for example $\gamma^1(X) = \mathrm{CH}^1(X)$, and there is a surjection $\gamma^2(X) \twoheadrightarrow \mathrm{CH}^2(X)$ (see [Ful98, Example 15.3.6]).

In the present paper, we provide a *uniform lower bound* for the torsion part of $\gamma^*(X)$, where $X = {}_\xi\mathfrak{B}_s$ is a twisted form of the variety of Borel subgroups $\mathfrak{B}_s$ of a split simple linear algebraic group G_s by means of a G_s-torsor ξ. Note that the groups $\gamma^2(X)$ and $\mathrm{CH}^2(X)$ had been studied for $G_s = PGL_n$ in [Kar98] and for strongly inner forms in [GZ10]. In particular, it was shown in [GZ10, §§ 3 and 7] that in the strongly inner case the torsion part of $\gamma^2(X)$ determines the Rost invariant.

Our main tool is the twisted γ-filtration on $K_0(G_s)$, where G_s is a split simple linear algebraic group. Roughly speaking, it is defined to be the image (see Definition 4.3) of the γ-filtration on K_0 of the twisted form X under the composition $K_0(X) \to K_0(\mathfrak{B}_s) \to K_0(G_s)$, where the first map is given by the restriction and the second map is induced by taking the quotient. The associated graded ring γ^*_ξ of the twisted γ-filtration has the following properties.

(i) It can be explicitly computed (see Theorem 4.5). Observe that $\gamma^0_\xi = \mathbb{Z}$, $\gamma^1_\xi = 0$ and γ^i_ξ is torsion and finitely generated for $i > 1$.

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(ii) There is a surjective ring homomorphism $\gamma^*(X) \twoheadrightarrow \gamma^*_\xi$. Hence, γ^*_ξ provides a uniform lower bound for the torsion part of the γ-ring of X.

(iii) The assignment $\xi \mapsto \gamma^*_\xi$ respects the base change and, therefore, can be viewed as an invariant of a torsor ξ.

In the last section, we use these properties to construct nontrivial torsion elements in $\gamma^2(X)$ for some twisted flag varieties X (see Examples 5.2 and 5.4). In particular, we establish the connection between the indexes of the Tits algebras of ξ and the order of the special cycle $\theta \in \gamma^2(X)$ constructed in [GZ10].

2. Preliminaries

In the present section, we recall several basic facts concerning linear algebraic groups, characters and the Grothendieck K_0 (see [KMRT98, § 24] and [GZ10, § 1B and § 6]).

Let G_s be a split simple linear algebraic group of rank n over a field k. We assume that the characteristic of k is prime to the order of the center of G_s. We fix a split maximal torus T and a Borel subgroup B such that $T \subset B \subset G_s$.

Let Λ_r and Λ be the root and the weight lattices of the root system of G_s with respect to $T \subset B$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots (a basis of Λ_r) and let $\{\omega_1, \ldots, \omega_n\}$ be the respective set of fundamental weights (a basis of Λ), that is, $\alpha_i^\vee(\omega_j) = \delta_{ij}$. The group of characters T^* of T is an intermediate lattice $\Lambda_r \subset T^* \subset \Lambda$ that determines the isogeny class of G_s. If $T^* = \Lambda$, then the group G_s is simply connected and if $T^* = \Lambda_r$ it is adjoint.

Let $\mathbb{Z}[T^*]$ be the integral group ring of T^*. Its elements are finite linear combinations $\sum_i a_i e^{\lambda_i}$, $\lambda_i \in T^*$. Let $\mathfrak{B}_s$ denote the variety of Borel subgroups G_s/B of G_s. Consider the characteristic map for K_0 (see [Dem74, § 2.8])

$$\mathfrak{c} : \mathbb{Z}[T^*] \to K_0(\mathfrak{B}_s)$$

defined by sending e^λ, $\lambda \in T^*$, to the class of the associated line bundle $[\mathcal{L}(\lambda)]$. Observe that the ring $K_0(\mathfrak{B}_s)$ does not depend on the isogeny class of G_s while the group of characters T^* and, hence, the image of $\mathfrak{c}$ does.

Since $K_0(\mathfrak{B}_s)$ is generated by the classes $[\mathcal{L}(\omega_i)]$, $i = 1, \ldots, n$, the characteristic map $\mathfrak{c}$ is surjective if G_s is simply connected. If G_s is adjoint, then the image of $\mathfrak{c}$ is generated by the classes $[\mathcal{L}(\alpha_i)]$, where

$$\alpha_i = \sum_j c_{ij}\omega_j \quad \text{and therefore} \quad \mathcal{L}(\alpha_i) = \otimes_j \mathcal{L}(\omega_j)^{\otimes c_{ij}},$$

and $c_{ij} = \alpha_i^\vee(\alpha_j)$ are the coefficients of the Cartan matrix of G_s.

The Weyl group W of G_s acts on weights via simple reflections s_{α_i} as

$$s_{\alpha_i}(\lambda) = \lambda - \alpha_i^\vee(\lambda)\alpha_i, \quad \lambda \in \Lambda.$$

For each element $w \in W$, we define (cf. [Ste75, § 2.1]) the weight $\rho_w \in \Lambda$ as

$$\rho_w = \sum_{\{i \in 1,\ldots,n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i).$$

In particular, for a simple reflection $w = s_{\alpha_j}$, we have

$$\rho_w = \sum_{\{i \in 1,\ldots,n \mid s_{\alpha_j}(\alpha_i) < 0\}} s_{\alpha_j}(\omega_i) = s_{\alpha_j}(\omega_j) = \omega_j - \alpha_j.$$

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Observe that the quotient Λ/Λ_r coincides with the group of characters of the center of the simply connected cover of G_s. Since W acts trivially on Λ/Λ_r, we have

$$\bar{\rho}_w = \sum_{\{i\in 1,\ldots,n \mid w^{-1}(\alpha_i)<0\}} \bar{\omega}_i \quad \in \Lambda/T^*,$$

where $\bar{\rho}_w$ denotes the class of $\rho_w \in \Lambda$ modulo T^*. In particular, $\bar{\omega}_i = \bar{\rho}_{s_{\alpha_i}}$.

Let $\mathbb{Z}[\Lambda]^W$ denote the subring of W-invariant elements. Then the integral group ring $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with the basis $\{e^{\rho_w}\}_{w\in W}$ (see [Ste75, Theorem 2.2]). Now let $\epsilon:\mathbb{Z}[\Lambda]\to\mathbb{Z}$, $e^\lambda\mapsto 1$ be the augmentation map. By the Chevalley theorem, the kernel of the surjection $\mathfrak{c}$ is generated by elements $x\in\mathbb{Z}[\Lambda]^W$ such that $\epsilon(x)=0$. Hence, there is an isomorphism

$$\mathbb{Z}[\Lambda]\otimes_{\mathbb{Z}[\Lambda]^W}\mathbb{Z}\simeq\mathbb{Z}[\Lambda]/\ker(\mathfrak{c})\simeq K_0(\mathfrak{B}_s).$$

So, the elements

$$\{g_w=\mathfrak{c}(e^{\rho_w})=[\mathcal{L}(\rho_w)]\}_{w\in W}$$

form a $\mathbb{Z}$-basis of $K_0(\mathfrak{B}_s)$ called the Steinberg basis.

Following [Tit71], we associate with each $\chi\in\Lambda/T^*$ and each cocycle $\xi\in Z^1(k,G_s)$ the central simple algebra $A_{\chi,\xi}$ over k called the Tits algebra. This defines a group homomorphism

$$\beta_\xi:\Lambda/T^*\to Br(k) \quad \text{with } \beta_\xi(\chi)=[A_{\chi,\xi}].$$

Let $\mathfrak{B}={}_\xi\mathfrak{B}_s$ denote the twisted form of the variety of Borel subgroups $\mathfrak{B}_s$ by means of ξ. Consider the restriction map on K_0 over the separable closure k_{sep}:

$$\mathrm{res}:K_0(\mathfrak{B})\to K_0(\mathfrak{B}\times_k k_{\mathrm{sep}})=K_0(\mathfrak{B}_s),$$

where we identify $K_0(\mathfrak{B}\times_k k_{\mathrm{sep}})$ with $K_0(\mathfrak{B}_s)$. By [Pan94, Theorem 4.2], the image of the restriction can be identified with the sublattice

$$\langle \imath_w\cdot g_w\rangle_{w\in W},$$

where $g_w=[\mathcal{L}(\rho_w)]$ is an element of the Steinberg basis and $\imath_w=\mathrm{ind}(\beta_\xi(\bar{\rho}_w))$ is the index of the respective Tits algebra. Observe that if G_s is simply connected, then all indexes $\imath_w$ are trivial and the restriction map becomes an isomorphism.

3. The K_0 of a split simple (adjoint) group

In the present section, we provide an explicit description of the ring $K_0(G_s)$ in terms of generators and relations for every simple split linear algebraic group G_s.

Definition 3.1. Let $\mathfrak{c}:\mathbb{Z}[\Lambda]\to K_0(\mathfrak{B}_s)$ be the characteristic map for the simply connected cover of G_s. We define the ring $\mathfrak{G}_s$ to be the quotient

$$\mathfrak{G}_s:=\mathbb{Z}[\Lambda/T^*]/\overline{(\ker\mathfrak{c})}$$

and the surjective ring homomorphism q to be the composite

$$q:K_0(\mathfrak{B}_s)\xrightarrow[\simeq]{\mathfrak{c}^{-1}}\mathbb{Z}[\Lambda]/(\ker\mathfrak{c})\twoheadrightarrow\mathbb{Z}[\Lambda/T^*]/\overline{(\ker c)}=\mathfrak{G}_s.$$

Observe that if G_s is simply connected, then $\mathfrak{G}_s=\mathbb{Z}$.

Remark 3.2. By [Mer05, Corollary 33] applied to $X = G_s$ and to the simply connected cover $G = \hat{G}_s$ of G_s, there is an isomorphism

$$K_0(G_s) \simeq \mathbb{Z} \otimes_{R(\hat{G}_s)} K_0(\hat{G}_s, G_s),$$

where $R(\hat{G}_s) \simeq \mathbb{Z}[\Lambda]^W$ is the representation ring. By [Mer05, Corollary 5] applied to $G = \hat{G}_s$, $X = \operatorname{Spec} k$ and $G/H = G_s$, there is an isomorphism

$$K_0(\hat{G}_s, G_s) \simeq R(H),$$

where $R(H) \simeq \mathbb{Z}[\Lambda/T^*]$ is the representation ring. Therefore,

$$K_0(G_s) \simeq \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z}[\Lambda/T^*] \simeq \mathfrak{G}_s.$$

Lemma 3.3. *The ideal $\overline{(\ker \mathfrak{c})} \subset \mathbb{Z}[\Lambda/T^*]$ is generated by the elements*

$$d_i(1 - e^{\bar{\omega}_i}), \quad i = 1, \ldots, n,$$

where d_i is the number of elements in the W-orbit of the fundamental weight ω_i.

Proof. By the Chevalley theorem, the subring of invariants $\mathbb{Z}[\Lambda]^W$ can be identified with the polynomial ring $\mathbb{Z}[\rho_1, \ldots, \rho_n]$, where

$$\rho_i = \sum_{\lambda \in W(\omega_i)} e^{\lambda},$$

where $W(\omega_i)$ denotes the W-orbit of the fundamental weight ω_i. Since $d_i = \epsilon(\rho_i)$, we have $\ker \mathfrak{c} = (d_1 - \rho_1, \ldots, d_n - \rho_n)$. To finish the proof, note that $\overline{(d_i - \rho_i)} = d_i(1 - e^{\bar{\omega}_i})$. $\square$

Remark 3.4. Observe that by definition and Lemma 3.3, we have $\mathfrak{G}_s \otimes \mathbb{Q} \simeq \mathbb{Q}$.

In the following examples, we compute the ring $\mathfrak{G}_s \simeq K_0(G_s)$ for every simple split linear algebraic group G_s. We refer to [KMRT98, § 24] for the description of Λ/T^*. Note that in most of the examples provided below, ω_i corresponds to a minuscule representation; in this case d_i is the dimension of the respective fundamental representation that can be found in [Bou05, ch. 8, Table 2].

Λ/T^*	G_s, $m \geqslant 1$	Example
$\mathbb{Z}/m\mathbb{Z}$, $m \geqslant 2$	SL_{n+1}/μ_m	(3.5)
$\mathbb{Z}/2\mathbb{Z}$	O^+_{m+4}, PSp_{2m+2}, HSpin_{4m+4}, E_7^{ad}	(3.6)
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	PGO^+_{4m+4}	(3.7)
$\mathbb{Z}/3\mathbb{Z}$	E_6^{ad}	(3.8)
$\mathbb{Z}/4\mathbb{Z}$	PGO^+_{4m+2}	(3.9)

Example 3.5. Consider the case $G_s = SL_{n+1}/\mu_m$, $m \geqslant 2$. The group G_s has type A_n and $\Lambda/T^* = \langle\sigma\rangle$ is cyclic of order m. The quotient map $\Lambda/\Lambda_r \to \Lambda/T^*$ sends $\bar{\omega}_i \in \Lambda/\Lambda_r$, $i = 1, \ldots, n$, to $(i \bmod m)\sigma \in \Lambda/T^*$. By Definition 3.1 and Lemma 3.3, we have

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(1 - (1 - y)^m, a_1 y, \ldots, a_{m-1} y^{m-1}),$$

where $y = (1 - e^{\sigma})$ and $a_j = \gcd\{\binom{n+1}{i} \mid i \equiv j \mod m,\ i = 1, \ldots, n\}$. In particular, for $G_s = SL_p/\mu_p = PGL_p$, where p is a prime, we obtain

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(\tbinom{p}{1} y, \tbinom{p}{2} y^2, \ldots, \tbinom{p}{p-1} y^{p-1}, y^p).$$

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Example 3.6. Assume that $\Lambda/T^* = \langle\sigma\rangle$ has order two. Then

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(y^2 - 2y, dy),$$

where $y = (1 - e^\sigma)$ and d denotes the greatest common divisor (g.c.d.) of the d_i corresponding to the ω_i with $\bar{\omega}_i = \sigma$. The integer d can be determined as follows.

B_n. We have $\Lambda/\Lambda_r = \{0, \bar{\omega}_n\} \simeq \mathbb{Z}/2\mathbb{Z}$, which corresponds to the adjoint group $G_s = O^+_{2n+1}$. Since $\bar{\omega}_i = 0$ for each $i \neq n$, we have $d = d_n = 2^n$.

C_n. We have $\Lambda/\Lambda_r = \{0, \sigma = \bar{\omega}_1 = \bar{\omega}_3 = \cdots\} \simeq \mathbb{Z}/2\mathbb{Z}$, that is, $G_s = PSp_{2n}$. Since $\bar{\omega}_i = 0$ for even i, we have $d = \text{g.c.d.}(d_1, d_3, \ldots)$.

D_n. If n is odd, then $\Lambda/\Lambda_r = \{0, \bar{\omega}_{n-1}, \bar{\omega}_1, \bar{\omega}_n\} \simeq \mathbb{Z}/4\mathbb{Z}$, where $\bar{\omega}_1 = 2\bar{\omega}_{n-1} = 2\bar{\omega}_n$. Therefore, $\Lambda/T^* \simeq \mathbb{Z}/2\mathbb{Z}$ if it is a quotient of Λ/Λ_r modulo the subgroup $\{0, \bar{\omega}_1\}$. In this case, $\Lambda/T^* = \{0, \sigma = \bar{\omega}_{n-1} = \bar{\omega}_n\}$, which corresponds to the special orthogonal group $G_s = O^+_{2n}$. Since $\bar{\omega}_s = s\bar{\omega}_1$ for $2 \leqslant s \leqslant n-2$ and $\bar{\omega}_1 = 0$ in Λ/T^*, we have $d = \text{g.c.d.}(d_{n-1}, d_n) = 2^{n-1}$.

If n is even, then $\Lambda/\Lambda_r = \{0, \bar{\omega}_{n-1}\} \oplus \{0, \bar{\omega}_n\} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, where $\bar{\omega}_1 = \bar{\omega}_{n-1} + \bar{\omega}_n$. In this case, we have two cases for Λ/T^*.

(i) It is the quotient of Λ/Λ_r modulo the diagonal subgroup $\{0, \bar{\omega}_{n-1} + \bar{\omega}_n\}$. Then $\Lambda/T^* = \{0, \sigma = \bar{\omega}_{n-1} = \bar{\omega}_n\}$, $G_s = O^+_{2n}$ and d is the same as in the odd case, that is, $d = 2^{n-1}$.

(ii) It is the quotient modulo one of the factors, for example $\Lambda/T^* = \{0, \sigma = \bar{\omega}_{n-1}\}$, where $\bar{\omega}_n = 0$. Then $G_s = \mathrm{HSpin}_{2n}$, $\bar{\omega}_1 = \bar{\omega}_3 = \cdots = \bar{\omega}_{n-1}$ and $\bar{\omega}_i = 0$ if i is even. Therefore, $d = \text{g.c.d.}\big(d_1, d_3, \ldots, d_{n-1}\big) = 2^{v_2(n)+1}$, where $v_2(n)$ denotes the 2-adic valuation of n.

E_7. We have $\Lambda/\Lambda_r = \{0, \sigma = \bar{\omega}_7 = \bar{\omega}_5 = \bar{\omega}_2\} \simeq \mathbb{Z}/2\mathbb{Z}$ with $\bar{\omega}_1 = \bar{\omega}_3 = \bar{\omega}_4 = \bar{\omega}_6 = 0$. Therefore, $d = \text{g.c.d.}(d_7, d_5, d_2) = 8$.

Example 3.7. Assume that $\Lambda/T^* = \langle\sigma_1\rangle \oplus \langle\sigma_2\rangle$, where σ_1 and σ_2 are of order two. In this case, $G_s = \mathrm{PGO}^+_{2n}$ is an adjoint group ($T^* = \Lambda_r$) of type D_n with n even. We have $\sigma_1 = \bar{\omega}_{n-1}$ and $\sigma_2 = \bar{\omega}_n$, $\bar{\omega}_s = s\bar{\omega}_1$, $2 \leqslant s \leqslant n-2$, $2\bar{\omega}_1 = 0$ and $\bar{\omega}_1 = \bar{\omega}_{n-1} + \bar{\omega}_n$. Then

$$\mathfrak{G}_s \simeq \mathbb{Z}[y_1, y_2]/(y_1^2 - 2y_1, y_2^2 - 2y_2, a_1 y_1, a_2 y_2, a(y_1 + y_2 - y_1 y_2)),$$

where $y_1 = (1 - e^{\sigma_1})$ and $y_2 = (1 - e^{\sigma_2})$; a_1 (respectively a_2) is the greatest common divisor of the d_i with $\bar{\omega}_i = \bar{\omega}_{n-1}$ (respectively $\bar{\omega}_i = \bar{\omega}_n$), that is, $a_1 = a_2 = 2^{n-1}$; and $a = \gcd(d_1, d_3, \ldots, d_{n-3})$. In particular, for $G_s = \mathrm{PGO}^+_8$, we obtain

$$\mathfrak{G}_s \simeq \mathbb{Z}[y_1, y_2]/(y_1^2 - 2y_1, y_2^2 - 2y_2, 8y_1, 8y_2).$$

Example 3.8. Assume that $\Lambda/T^* = \langle\sigma\rangle$ has order three. Then

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(y^3 - 3y^2 + 3y, a_1 y, a_2 y^2),$$

where $y = (1 - e^\sigma)$ and a_1 (respectively a_2) is the greatest common divisor of the d_i with $\bar{\omega}_i = \sigma$ (respectively $\bar{\omega}_i = 2\sigma$). For the adjoint group of type E_6, we have $\Lambda/\Lambda_r = \{0, \sigma = \bar{\omega}_1 = \bar{\omega}_5, 2\sigma = \bar{\omega}_3 = \bar{\omega}_6\}$ with $\bar{\omega}_2 = \bar{\omega}_4 = 0$. Therefore, $a_1 = a_2 = 27$.

Example 3.9. Assume that $\Lambda/T^* = \langle\sigma\rangle$ has order four. Then

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(y^4 - 4y^3 + 6y^2 - 4y, a_1 y, a_2 y^2, a_3 y^3),$$

where $y = (1 - e^{\sigma})$. For the group PGO^+_{2n} where n is odd, we have $\sigma = \bar{\omega}_{n-1}$, $2\sigma = \bar{\omega}_1$ and $3\sigma = \bar{\omega}_n$. Therefore, $a_1 = a_3 = 2^{n-1}$ and $a_2 = \mathrm{g.c.d.}(d_1, d_3, \ldots, d_{n-2})$.

4. The twisted γ-filtration

In the present section, we introduce and study the twisted γ-filtration.

Let $\gamma = \ker \epsilon$ denote the augmentation ideal in $\mathbb{Z}[\Lambda]$. It is generated by the differences

$$\langle (1 - e^{-\lambda}), \lambda \in \Lambda \rangle.$$

Consider the γ-adic filtration on $\mathbb{Z}[\Lambda]$:

$$\mathbb{Z}[\Lambda] = \gamma^0 \supseteq \gamma \supseteq \gamma^2 \supseteq \cdots .$$

The ith power γ^i is generated by products of at least i differences.

Definition 4.1. We define the filtration on $K_0(\mathfrak{B}_s)$ (respectively on $\mathfrak{G}_s$) to be the image of the γ-adic filtration on $\mathbb{Z}[\Lambda]$ via $\mathfrak{c}$ (respectively via q), that is,

$$\gamma^i K_0(\mathfrak{B}_s) := \mathfrak{c}(\gamma^i) \quad \text{and} \quad \gamma^i \mathfrak{G}_s := q(\gamma^i K_0(\mathfrak{B}_s)), \quad i \geqslant 0.$$

So, we have a commutative diagram of surjective group homomorphisms.

$$\begin{tikzcd} \gamma^i \arrow[r, "\mathfrak{c}"] \arrow[dr] & \gamma^i K_0(\mathfrak{B}_s) \arrow[d, "q"] \\ & \gamma^i \mathfrak{G}_s \end{tikzcd}$$

Lemma 4.2. *The γ-filtration on $K_0(\mathfrak{B}_s)$ coincides with the filtration introduced in Definition 4.1.*

Proof. Since $K_0(\mathfrak{B}_s)$ is generated by the classes of line bundles,

$$\gamma^i K_0(\mathfrak{B}_s) = \langle c_1([\mathcal{L}_1]) \cdot \cdots \cdot c_1([\mathcal{L}_m]) \mid m \geqslant i, \mathcal{L}_j \in K_0(\mathfrak{B}_s) \rangle,$$

where c_1 is the first characteristic class in K_0. Moreover, each line bundle $\mathcal{L}$ is the associated bundle $\mathcal{L} = \mathcal{L}(\lambda)$ for some character $\lambda \in \Lambda$. Therefore, $c_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] = \mathfrak{c}(1 - e^{-\lambda})$ (see [Dem74, § 2.8]). ☐

Definition 4.3. Given a G_s-torsor $\xi \in H^1(k, G_s)$ and the respective twisted form $\mathfrak{B} = {}_\xi\mathfrak{B}_s$, we define the twisted filtration on $\mathfrak{G}_s$ to be the image of the γ-filtration on $K_0(\mathfrak{B})$ via the composite $\mathrm{res} \circ q$, that is,

$$\gamma^i_\xi \mathfrak{G}_s := q(\mathrm{res}(\gamma^i K_0(\mathfrak{B}))), \quad i \geqslant 0.$$

Let $\gamma_\xi^{i/i+1}\mathfrak{G}_s = \gamma_\xi^i \mathfrak{G}_s / \gamma_\xi^{i+1}\mathfrak{G}_s$ denote the ith subsequent quotient. The associated graded ring $\bigoplus_{i \geqslant 0} \gamma_\xi^{i/i+1}\mathfrak{G}_s$ will be called the γ-*invariant* of the torsor ξ and will be denoted simply as γ^*_ξ.

Remark 4.4. Note that the Chern classes commute with restrictions; therefore, the restriction map $\mathrm{res}\colon \gamma^i K_0(\mathfrak{B}) \to \gamma^i K_0(\mathfrak{B}_s)$ is well defined. By definition, there is a surjective ring homomorphism

$$\gamma^*(\mathfrak{B}) \twoheadrightarrow \gamma^*_\xi.$$

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Theorem 4.5. *The twisted filtration $\gamma^i_\xi \mathfrak{G}_s$ can be computed as follows:*

$$\gamma^i_\xi \mathfrak{G}_s = \Big\langle \prod_{j=1}^m \binom{\operatorname{ind}(\beta_\xi(\bar\rho_{w_j}))}{n_j} (1-e^{\bar\rho_{w_j}})^{n_j} \;\Big|\; n_1+\cdots+n_m \geqslant i, w_j \in W \Big\rangle.$$

Proof. Since the characteristic classes commute with restrictions, the image of the restriction $\operatorname{res}\colon \gamma^i K_0(\mathfrak{B}) \to \gamma^i K_0(\mathfrak{B}_s)$ is generated by the products

$$\langle c_{n_1}(\imath_{w_1} g_{w_1}) \cdot \cdots \cdot c_{n_m}(\imath_{w_m} g_{w_m}) \mid n_1+\cdots+n_m \geqslant i, w_1,\ldots,w_m \in W\rangle,$$

where $\{\imath_{w_j}\}$ are the indexes of the respective Tits algebras. Applying the Whitney formula for the characteristic classes [Ful98, § 3.2], we obtain

$$c_j(\imath_w g_w) = \binom{\imath_w}{j} c_1(g_w)^j.$$

Therefore, $q\big(\binom{\imath_w}{j} c_1(g_w)^j\big) = \binom{\imath_w}{j}(1-e^{-\bar\rho_w})^j$, where $\imath_w = \operatorname{ind}(\beta_\xi(\bar\rho_w))$. □

Example 4.6. Since $\gamma^0(X) \simeq \mathbb{Z}$ and $\gamma^1(X) = \operatorname{Pic}(X)$ is torsion free for every smooth projective X, we obtain that $\gamma^0_\xi \simeq \mathbb{Z}$ and $\gamma^1_\xi = 0$ for any ξ.

Example 4.7 (Strongly inner case). If $\beta_\xi = 0$, then $\binom{\imath_{w_j}}{n_j} = 1$ and $\gamma^i_\xi \mathfrak{G}_s = \gamma^i \mathfrak{G}_s$.

Example 4.8 ($\mathbb{Z}/2\mathbb{Z}$-case). As in Example 3.6, assume that $\Lambda/T^* = \langle\sigma\rangle$ has order two and $\beta_\xi \neq 0$. Then there is only one non-split Tits algebra $A = A_{\sigma,\xi}$ and it has exponent 2. Let $\mathsf{i}_A = v_2(\operatorname{ind}(A))$ denote the 2-adic valuation of the index of A. By definition, we have

$$\gamma^i_\xi \mathfrak{G}_s = \Big\langle \binom{2^{\mathsf{i}_A}}{n_1} \cdots \binom{2^{\mathsf{i}_A}}{n_m} 2^{n_1+\cdots+n_m-1} y \;\Big|\; n_1+\cdots+n_m \geqslant i \Big\rangle$$

in $\mathbb{Z}[y]/(y^2-2y, dy)$, where $y = 1-e^\sigma$ and d is given in Example 3.6. Observe that modulo the relation $y^2 = 2y$ these ideals are generated by (for $j \geqslant 1$)

$$\begin{aligned}
&\gamma^{2j-1}_\xi \mathfrak{G}_s = \gamma^{2j}_\xi \mathfrak{G}_s = \langle 2^{2j-1} y\rangle && \text{if } \mathsf{i}_A = 1,\\
&\gamma^{4j-3}_\xi \mathfrak{G}_s = \gamma^{4j-2}_\xi \mathfrak{G}_s = \langle 2^{4j-2} y\rangle,\ \gamma^{4j-1}_\xi \mathfrak{G}_s = \gamma^{4j}_\xi \mathfrak{G}_s = \langle 2^{4j-1} y\rangle && \text{if } \mathsf{i}_A = 2,\\
&\gamma^1_\xi \mathfrak{G}_s = \gamma^2_\xi \mathfrak{G}_s = \langle 2^{\mathsf{i}_A} y\rangle,\ \gamma^3_\xi \mathfrak{G}_s = \gamma^4_\xi \mathfrak{G}_s = \langle 2^{\mathsf{i}_A+1} y\rangle,\ \gamma^5_\xi \mathfrak{G}_s = \langle 2^{\mathsf{i}_A+4} y\rangle, \ldots && \text{if } \mathsf{i}_A > 2.
\end{aligned}$$

Taking these generators modulo the relation $dy = 0$, we obtain the following formulas for the second quotient γ^2_ξ:

$$\text{if } \mathsf{i}_A = 1,\ \text{then } \gamma^2_\xi = \begin{cases} 0 & \text{if } v_2(d) \leqslant 1,\\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2,\\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geqslant 3,\end{cases}$$

$$\text{if } \mathsf{i}_A > 1,\ \text{then } \gamma^2_\xi = \begin{cases} 0 & \text{if } v_2(d) \leqslant \mathsf{i}_A,\\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) > \mathsf{i}_A.\end{cases}$$

Example 4.9 ($\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ case). Following Example 3.7, we assume that $\Lambda/T^* = \langle\sigma_1\rangle \oplus \langle\sigma_2\rangle$, where σ_1, σ_2 have order two. This is the case for the adjoint group PGO^+_{2n} where n is even [KMRT98, § 25]. Assume that $n = 4$, which corresponds to the group of type D_4, that is, PGO^+_8. Let C^+ and C^- denote the Tits algebras corresponding to the generators $\sigma_1 = \bar\omega_3$ and $\sigma_2 = \bar\omega_4$.

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Let A denote the Tits algebra corresponding to the sum $\sigma_1+\sigma_2$. (Note that $C^+\times C^-$ is the even part of the Clifford algebra of the algebra with involution A and $[A]=[C^+\otimes C^-]$ in $Br(k)$.)

By definition, we have in $\mathbb{Z}[y_1,y_2]$ that

$$\gamma^i_\xi\mathfrak{G}_s=\left\langle \binom{\operatorname{ind} C_+}{n_1}y_1^{n_1}\cdot\binom{\operatorname{ind} C_-}{n_2}y_2^{n_2}\cdot\binom{\operatorname{ind} A}{n_3}(y_1+y_2-y_1y_2)^{n_3}\,\middle|\, n_1+n_2+n_3\geqslant i\right\rangle.$$

Modulo the relations $(y_1^2-2y_1,y_2^2-2y_2,8y_1,8y_2)$, we obtain that

$$\gamma^2_\xi\mathfrak{G}_s\simeq\frac{(\operatorname{ind} C_+)\mathbb{Z}}{8\mathbb{Z}}\oplus\frac{(\operatorname{ind} C_-)\mathbb{Z}}{8\mathbb{Z}}\oplus\frac{(\operatorname{ind} A)\mathbb{Z}}{8\mathbb{Z}}.$$

5. Torsion in the γ-filtration

In the present section, we show how the twisted γ-filtration can be used to construct nontrivial torsion elements in the γ-ring of the twisted form $\mathfrak{B}$ of a variety of Borel subgroups. For simplicity, we consider only the case of G_s (see Examples 3.6 and 4.8) with $\Lambda/T^*=\langle\sigma\rangle$ of order two.

Let d denote the greatest common divisor of dimensions of fundamental representations corresponding to σ. Given a G_s-torsor $\xi\in H^1(k,G_s)$, let i_A denote the 2-adic valuation of the index of the Tits algebra $A=A_{\sigma,\xi}$. Let $\mathfrak{B}={}_\xi\mathfrak{B}_s$ denote the twisted form of the variety of Borel subgroups of G_s by means of ξ. Consider the respective twisted filtration $\gamma^i_\xi\mathfrak{G}_s$ on $\mathfrak{G}_s$.

PROPOSITION 5.1. *Assume that $v_2(d)>\mathsf{i}_A\geqslant 3$. Then, for each $\lambda\in\Lambda$ such that $\bar\lambda=\sigma$, there exists a nontrivial torsion element of order two in $\gamma^2(\mathfrak{B})$. Moreover, its image in $\gamma^2_\xi=\mathbb{Z}/2$ (via q) is nontrivial and in $\gamma^2(\mathfrak{B}_s)$ (via* res*) is trivial.*

Proof. The proof of this result was inspired by the proof of [Kar98, Proposition 4.13].

Let $g=[\mathcal{L}(\lambda)]$ denote the class of the associated line bundle. Using the formula for the first Chern class of a tensor product of line bundles for K_0, we obtain

$$c_1(g)^2=2c_1(g)-c_1(g^2).$$

Hence,

$$c_1(g)^4=(2c_1(g)-c_1(g^2))^2=4c_1(g)^2-4c_1(g)c_1(g^2)+c_1(g^2)^2.$$

Therefore,

$$\eta=4c_1(g)^3-c_1(g)^4=4c_1(g)^2-c_1(g^2)^2\in\gamma^3K_0(\mathfrak{B}_s).$$

We claim that the class of $2^{\mathsf{i}_A-3}\eta$ gives the desired torsion element.

Indeed, $c_1(g^2)=c_1([\mathcal{L}(2\lambda)])$. Since $2\lambda\in T^*$, $[\mathcal{L}(2\lambda)]\in\mathfrak{c}(T^*)$ and, therefore, by [GZ12, Corollary 3.1], $c_1(g^2)\in\gamma^1K_0(\mathfrak{B})$. Moreover, we have $2^{\mathsf{i}_A-1}c_1(g)^2=c_2(2^{\mathsf{i}_A}g)$, where $2^{\mathsf{i}_A}g\in K_0(\mathfrak{B})$. Hence, $2^{\mathsf{i}_A-1}c_1(g)^2\in\gamma^2K_0(\mathfrak{B})$. Combining these together, we obtain that $2^{\mathsf{i}_A-3}\eta\in\gamma^2K_0(\mathfrak{B})$.

Now, since $2^{\mathsf{i}_A-3}\eta\in\gamma^2K_0(\mathfrak{B})$, its image in $\gamma^2_\xi\mathfrak{G}_s$ can be computed as

$$q(2^{\mathsf{i}_A-3}\eta)=2^{\mathsf{i}_A-3}q(\eta)=2^{\mathsf{i}_A-1}q(c_1(g)^2)=2^{\mathsf{i}_A-1}(1-e^{-\sigma})^2=2^{\mathsf{i}_A}y.$$

But $q(2^{\mathsf{i}_A-3}\eta)\notin\gamma^3_\xi\mathfrak{G}_s=\langle 2^{\mathsf{i}_A+1}y\rangle$. Therefore, $2^{\mathsf{i}_A-3}\eta\notin\gamma^3K_0(\mathfrak{B})$.

Since $2^{\mathsf{i}_A-2}\eta=2^{\mathsf{i}_A}c_1(g)^3+2^{\mathsf{i}_A-2}c_1(g)^4$ is in $\gamma^3K_0(\mathfrak{B})$, the class of $2^{\mathsf{i}_A-3}\eta$ gives the desired torsion element of order two. □

Example 5.2. Let $G_s=\mathrm{HSpin}_{2n}$ be a half-spin group of rank $n\geqslant 4$. So, G_s is of type D_n, where n is even, $\Lambda/T^*=\langle\sigma=\bar\omega_1\rangle\simeq\mathbb{Z}/2\mathbb{Z}$ and, according to Example 3.6, we have $d=2^{v_2(n)+1}$.

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Let $\xi \in H^1(k, G_s)$ be a nontrivial torsor. Then there is only one Tits algebra $A = A_{\sigma,\xi}$; it has exponent 2 and index $2^{\mathfrak{i}_A}$ such that $\mathfrak{i}_A \leqslant v_2(n) + 1$.

Recall that each such torsor corresponds to an algebra with orthogonal involution (A, δ) with trivial discriminant and trivial component of the Clifford algebra. The respective twisted form $\mathfrak{B} = {}_\xi\mathfrak{B}_s$ then corresponds to the variety of Borel subgroups of the group $\mathrm{PGO}^+(A, \delta)$. Applying Proposition 5.1 to this situation, we obtain that for any such algebra (A, δ) where $8 \mid \mathrm{ind}(A)$ and A is non-division, there exists a nontrivial torsion element of order two in $\gamma^2(\mathfrak{B})$ that vanishes over a splitting field of (A, δ).

Lemma 5.3. *The γ-filtration on $K_0(\mathfrak{B}_s)$ is generated by the first Chern classes $c_1([\mathcal{L}(\omega_i)])$, $i = 1, \dots, n$, that is,*

$$\gamma^i K_0(\mathfrak{B}_s) = \left\langle \prod_{j \in 1,\dots,n} c_1([\mathcal{L}(\omega_j)]) \;\middle|\; \text{the number of elements in the product } \geqslant i \right\rangle.$$

In particular, the second quotient $\gamma^2(\mathfrak{B}_s)$ is additively generated by the products

$$\gamma^2(\mathfrak{B}_s) = \langle c_1([\mathcal{L}(\omega_i)])c_1([\mathcal{L}(\omega_j)]) \mid i, j \in 1, \dots, n\rangle.$$

Proof. Each $b \in K_0(\mathfrak{B}_s)$ can be written as a linear combination $b = \sum_{w \in W} a_w g_w$. Therefore, any Chern class of b can be expressed in terms of $c_1(g_w)$.

Each ρ_w can be written uniquely as a linear combination of fundamental weights $\{\omega_1, \dots, \omega_n\}$. Therefore, by the formula for the Chern class of the tensor product of line bundles [CPZ10, 8.2], each $c_1(g_w)$ can be expressed in terms of $c_1([\mathcal{L}(\omega_i)])$. □

Example 5.4. Let G_s be an adjoint group of type E_7 and let $\xi \in H^1(k, G_s)$ be a nontrivial G_s-torsor. Then there is only one non-split Tits algebra $A = A_{\sigma,\xi}$ of exponent 2 and $\mathfrak{i}_A \leqslant 3$. Let $\mathfrak{B} = {}_\xi\mathfrak{B}_s$ be the respective twisted flag variety.

By Lemma 5.3, any element of $\gamma^2(\mathfrak{B})$ can be written as

$$x = \sum_{ij} a_{ij} c_1([\mathcal{L}(\omega_i)])c_1([\mathcal{L}(\omega_j)]) \in \gamma^2(\mathfrak{B})$$

for certain coefficients $a_{ij} \in \mathbb{Z}$. Since $\sigma = \bar\omega_7 = \bar\omega_5 = \bar\omega_2$ and $\bar\omega_1 = \bar\omega_3 = \bar\omega_4 = \bar\omega_6 = 0$, we obtain that

$$q(x) = C \cdot 2y \in \gamma^2_\xi, \quad \text{where } C = a_{25} + a_{27} + a_{57} + a_{22} + a_{55} + a_{77}.$$

Therefore, $q(x) \neq 0$ in γ^2_ξ if and only if $4 \nmid C$ and $\mathfrak{i}_A \leqslant 2$.

Consider the class $\mathfrak{c}(\theta) \in \gamma^2 K_0(\mathfrak{B}_s)$ of the special cycle θ constructed in [GZ10, Definition 3.3]. Note that the image of θ in $CH^2(\mathfrak{B})$ can be viewed as a generalization of the Rost invariant for split adjoint groups (see [GZ10, § 6]).

If $\mathfrak{i}_A = 1$, then, by [GZ10, Proposition 6.5], we know that $\mathfrak{c}(\theta) \in \gamma^2(\mathfrak{B})$ is a nontrivial torsion element. If $\mathfrak{i}_A = 2$, then, following the proof of [GZ10, Proposition 6.5], we obtain that $2\mathfrak{c}(\theta) \in \gamma^2(\mathfrak{B})$.

We claim that if $\mathfrak{i}_A \leqslant 2$, then $x = 2\mathfrak{c}(\theta)$ is nontrivial. Indeed, in this case $4 \nmid C = a_{22} + a_{55} + a_{77} = 6$; therefore, we have $q(x) \neq 0$, and $x \neq 0$ in $\gamma^2(\mathfrak{B})$. In particular, this shows that for $\mathfrak{i}_A = 1$ the order of the special cycle θ in $\gamma^2(\mathfrak{B})$ is divisible by 4.

Example 5.5. Let $\xi \in H^1(k, \mathrm{PGO}_8^+)$. Applying the same arguments as in Example 5.4 to Example 4.9, we obtain that if $\mathrm{ind}(A), \mathrm{ind}(C_+), \mathrm{ind}(C_-) \leqslant 4$, then $2\mathfrak{c}(\theta) \in \gamma^2(\mathfrak{B})$ is nontrivial.

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