Ergod. Th. & Dynam. Sys. (1984), **4**, 89–104
Printed in Great Britain

Kakutani equivalence of ergodic $\mathbb{Z}^n$ actions

ANDRES DEL JUNCO AND DANIEL J. RUDOLPH
Ohio State University, Newark, OH 48055, *USA*; *University of Maryland, College Park, MD* 20742, *USA*

(*Received* 13 *April* 1983)

Abstract. We define two families of relations between ergodic $\mathbb{Z}^n$ actions, both indexed equivariantly by non-singular $n \times n$ matrices. The first is to be Katok cross-sections of the same flow, indexed in a natural way by the matrices. The second is determined by the existence of an orbit preserving injection with an extra asymptotic linearity condition. We demonstrate that these two families are identical. In one dimension this is the classical theory of Kakutani equivalence.

The purpose of this paper is to lift the notion of Kakutani equivalence of finite measure preserving ergodic $\mathbb{Z}$-actions, and its correspondence to cross-sections of a one-dimensional flow [5] to a notion of equivalence between $\mathbb{Z}^n$-actions with a correspondence to Katok cross-sections [3] of an n-dimensional flow.

The classical definition of Kakutani equivalence says $T_1 \leadsto T_2$ if there exist subsets A_1 and A_2 so that the induced maps T_{1,A_1} and T_{2,A_2} are isomorphic. A second identical relation is $T_1 \leadsto T_2$ if T_1 and T_2 arise as measurable cross-sections of the same ergodic flow.

For T_1 and T_2 ergodic actions of $\mathbb{Z}^n$ we can define $T_1 \leadsto T_2$ if T_1 and T_2 arise as Katok cross-sections of the same $\mathbb{R}^n$ flow. We will give a natural intrinsic definition of $\sim$ and show that it is identical to $\leadsto$.

In fact, we will do more. In 1-dimension, both $\sim$ and $\leadsto$ can be indexed by a positive real, $T_1 \overset{r}{\rightsquigarrow} T_2$ if $\mu_2(A_2)/\mu_1(A_1) = r$, and $T_1 \overset{r}{\leadsto} T_2$ if $\int_{\Omega_1} f_1 \, d\mu_1 / \int_{\Omega_2} f_2 \, d\mu_2 = r$, where f_i is the ceiling function over Ω_i. In fact, $\overset{r}{\sim}$ and $\overset{r}{\leadsto}$ are identical, and clearly

$$T_1 \overset{r_1}{\rightsquigarrow} T_2 \overset{r_2}{\rightsquigarrow} T_3 \quad \text{implies} \quad T_1 \overset{r_1 \circ r_2}{\rightsquigarrow} T_3,$$

i.e. we have a family of relations indexed equivariantly by $\mathbb{R}^+$. We will obtain a similar family of relations among $\mathbb{Z}^n$-actions, only now equivariant with non-singular $n \times n$ matrices, i.e. we will have relations $T_1 \overset{M}{\rightsquigarrow} T_2$ and $T_1 \overset{M}{\leadsto} T_2$, where

$$T_1 \overset{M_1}{\rightsquigarrow} T_2 \overset{M_2}{\rightsquigarrow} T_3 \quad \text{implies} \quad T_1 \overset{M_2 \circ M_1}{\rightsquigarrow} T_3.$$

We would like to thank A. Katok for the insight that no further restriction need be placed on M than non-singularity.

We first discuss Katok cross-sections and $T_1 \overset{M}{\leadsto} T_2$. If T is an ergodic $\mathbb{Z}^n$ action, whose elements we write as T^v, $v \in \mathbb{Z}^n$, then define $\bar{T}^v$, the ergodic $\mathbb{R}^n$ action obtained by suspending T. Thus $\bar{\Omega}=\Omega\times[0,1]^n$ and

$$\bar{T}^v(\omega, v_0)=(t^m(\omega), v_1)$$

where $v_0+v=m+v_1$, $m\in\mathbb{Z}^n$, $v_1\in[0,1)^n$.

If T is a $\mathbb{Z}^n$ action, and ω_1, ω_2 are on the same T orbit, then we define $v(\omega_1,\omega_2)$ as that $v\in\mathbb{Z}^n$ with $T^v(\omega_1)=\omega_2$. If we have two actions T_1 and T_2 then v_1 and v_2 will be the corresponding coordinatizations of the orbits. For $\mathbb{R}^n$ actions $\bar{T}_1$ or $\bar{T}_2$ similarly we get $\bar{v}_1$ and $\bar{v}_2\in\mathbb{R}^n$. Although it is not standard, we refer to v and $\bar{v}$ as 'cocycles.' If we have two $\mathbb{Z}^n$ or $\mathbb{R}^n$ cocycles on the same orbits, we say they differ by a time change.

For fixed ω_0, $v(\omega_0,\omega)=f_{v,\omega_0}(\omega)$ maps the orbit of ω_0 invertibly to $\mathbb{Z}^n$, and $\bar{v}(\omega_0,\omega)=f_{\bar{v},\omega_0}(\omega)$ to $\mathbb{R}^n$.

If $\bar{v}_1$ and $\bar{v}_2$ differ by a time change and for a.e. ω_0,

$$f_{v_2,\omega_0}\circ f^{-1}_{v_1,\omega_0}$$

is a homeomorphism we say the time change is continuous, if C^1 then a C^1-time change, etc.

We say v_1 and v_2 differ by a coboundary if

$$v_1(\omega_1,\omega_2)=v_2(\omega_1,\omega_2)+v(\omega_1)-v(\omega_2)$$

where $v:\Omega\to\mathbb{Z}$ and $T_1^{v(\omega)}(\omega)=\varphi(\omega)$ is an invertible map. In this case, setting $\omega_2=T_2^v(\omega_1)$,

$$\begin{aligned}\varphi(T_2^v(\omega_1)) &= T_1^{v(\omega_2)}(T_2^{v_2(\omega_1,\omega_2)}(\omega_1))\\ &= T_1^{v(\omega_2)}(T_1^{v_1(\omega_1,\omega_2)}(\omega_1))\\ &= T_1^{v_2(\omega_1,\omega_2)}(T_1^{v(\omega_1)}(\omega_1))\\ &= T_1^v(\varphi(\omega_1)),\end{aligned}$$

and T_1 and T_2 are isomorphic by a map φ which fixes orbits, i.e. a coboundary. We can now define $\bar{T}_1\overset{M}{\rightsquigarrow}\bar{T}_2$.

Definition 1. We say $\bar{T}_1\overset{M}{\rightsquigarrow}\bar{T}_2$ if there are versions $\bar{T}_1'$ and $\bar{T}_2'$ acting on the same space $\bar{\Omega}$ with the same orbits, so that for any $\varepsilon>0$, there is a cocycle $\bar{v}_{2,\varepsilon}$, differing by a co-boundary from $\bar{v}_2$, with $\bar{v}_{2,\varepsilon}$ a C^∞ time change of $\bar{v}_1$ and

$$\left|\frac{\partial(\bar{v}_{2,\varepsilon}(\omega, T^v(\omega))}{\partial v}(0)-M\right|<\varepsilon$$

uniformly on $\bar{\Omega}$. These are the 'barely linear' time changes of [**6**].

Two facts will be helpful to us as we continue. The first we simply quote from [**6**]. It will be very helpful for the reader to be familiar with the results of [**6**], as much of our argument is explicitly parallel to that.

PROPOSITION 1. *Suppose that for a basis $e_1\cdots e_n$ for $\mathbb{R}^n$, there are constants a and b and sets $A_1\cdots A_n$ of positive measure so that if $\omega_1,\omega_2\in A_i$, $\bar{v}_1(\omega_1,\omega_2)=ne_i$, then $|\bar{v}_2(\omega_1,\omega_2)|<an+b$. Then $\bar{v}_1$ and $\bar{v}_2$ differ by a barely linear time change, i.e. for some M, $\bar{T}_1\overset{M}{\rightsquigarrow}\bar{T}_2$.*

(Theorem 3.4 of [**6**]).

PROPOSITION 2. *If $\bar{v}_2$ and $\bar{v}'_2$ differ by a coboundary, and $\bar{v}_2$ differs from $\bar{v}_1$ by a barely linear time change with matrix M, and $\bar{v}'_2$ differs from $\bar{v}_1$ by a barely linear time change with matrix M', then $M = M'$.*

Proof. Construct $\bar{v}_{2,\varepsilon}$ and $\bar{v}'_{2,\varepsilon}$, which now must differ by a C^∞ coboundary, i.e.

$$\bar{v}_{2,\varepsilon}(\omega_1, \omega_2) = \bar{v}_{2,\varepsilon}(\omega_1, \omega_2) + v_\varepsilon(\omega_1) - v_\varepsilon(\omega_2).$$

Now

$$\varepsilon < \left|\frac{\partial \bar{v}_{2,\varepsilon}(\omega, T_1^v(\omega))}{\partial v}(0) - M\right|$$

$$= \left|\frac{\partial(\bar{v}'_{2,\varepsilon}(\omega, T_1^v(\omega)) + v_\varepsilon(\omega) - v_\varepsilon(T_1^v(\omega)))}{\partial v}(0) - M\right|$$

$$= \left|\frac{\partial(\bar{v}'_{2,\varepsilon}(\omega, T_1^v(\omega))}{\partial v} - \frac{\partial(v_\varepsilon(T_1^v(\omega))}{\partial v} - M\right|$$

so

$$\left|M' - M - \frac{\partial(v_\varepsilon(T_1^v(\omega))}{\partial v}(0)\right| < 2\varepsilon.$$

But now for $|u| = 1$,

$$|v_\varepsilon(T_1^{Lu}(\omega)) - v_\varepsilon(\omega)| = \left|\int_{\substack{\lambda=0\\ \omega' = T^{\lambda u}(\omega)}}^{K} \left[\frac{\partial(v_\varepsilon(T_1^v T^{\lambda u}_{(\omega)}))}{\partial v}(0)\right](u)\, d\lambda\right|$$

$$\geq |L[M' - M](u)| - 2\varepsilon L.$$

But by Poincaré recurrence, for any $u, |v_\varepsilon(T_1^{Lu}(\omega)) - v_\varepsilon(\omega)|$ must, for arbitrarily large L, be less than one. Hence $|M - M'| \leq 2\varepsilon$. □

This is in fact a very simple result, but is useful to keep in mind. It is possible for $\bar{T}_1 \overset{M}{\rightsquigarrow} \bar{T}_2$ and $\bar{T}_1 \overset{\bar{M}}{\rightsquigarrow} \bar{T}_2$ where $M \neq \bar{M}$, but the orbit map achieving $\bar{T}_1 \overset{\bar{M}^{-1}\circ M}{\rightsquigarrow} \bar{T}_1$ cannot fix orbits.

Definition 2. We say $T_1 \overset{M}{\rightsquigarrow} T_2$ for two ergodic $\mathbb{Z}^n$ actions if $\bar{T}_1 \overset{M}{\rightsquigarrow} \bar{T}_2$.

A number of comments are in order. Katok [3] has shown that for any ergodic $\mathbb{R}^n$ action $\bar{S}$, there is an ergodic $\mathbb{Z}^n$ action T with $\bar{T} \overset{\text{id}}{\rightsquigarrow} \bar{S}$. Replacing $\bar{S}$ with $\bar{S}_M$, where

$$\bar{S}^v_M(\omega) = \bar{S}^{M(v)}(\omega),$$

it is clear $\bar{S} \overset{M}{\rightsquigarrow} \bar{S}_M$, so if $\bar{T} \overset{\text{id}}{\rightsquigarrow} \bar{S}_M$, then $\bar{T} \overset{M}{\rightsquigarrow} \bar{S}$, and we conclude that for any ergodic $\mathbb{Z}^n$ action T_1, and non-singular M there are $\mathbb{Z}^n$ actions T_2 with $T_1 \overset{M}{\rightsquigarrow} T_2$. We also know that $h(\bar{T}) = h(T)$ (the entropy of the unit suspension is the entropy of the base map), and if $\bar{T} \overset{M}{\rightsquigarrow} \bar{S}$ then $h(\bar{T}) = h(\bar{S})/|\det(M)|$, [4] so if $T_1 \overset{M}{\rightsquigarrow} T_2$ then

$$h(T_1) = \frac{h(T_2)}{|\det(M)|}.$$

We now wish to define an intrinsic relation $\overset{M}{\rightarrow}$, without lifting T_1 and T_2.

We will give two conditions. The first is a version of barely linear for $\mathbb{Z}^n$-actions. The second is, on the face of it, much weaker and more geometric a condition. As such we will indicate it $\underset{w}{\overset{M}{\rightsquigarrow}}$. Of course, in the end, $\overset{M}{\leadsto}$, $\overset{M}{\rightsquigarrow}$ and $\underset{w}{\overset{M}{\rightsquigarrow}}$ will all turn out to be the same relation.

Definition 3. We say $T_1 \overset{M}{\rightsquigarrow} T_2$, where $|\det(M)| \geq 1$, if there is a 1–1, measurable, non-singular map $\varphi : \Omega_1 \to \Omega_2$, mapping orbits into distinct orbits (we call such a φ an orbit injection), so that for any $\varepsilon > 0$, there is a set $A(\varepsilon)$, $\mu_1(A(\varepsilon)) > 1 - \varepsilon$ and a constant $N(\varepsilon)$ so that if $\omega_1, \omega_2 \in A(\varepsilon)$, on the same orbit, and $|v_1(\omega_1, \omega_2)| \geq N(\varepsilon)$, then

$$|v_2(\varphi(\omega_1), \varphi(\omega_2)) - M(v_1(\omega_1, \omega_2))| \leq \varepsilon \max\{|v_1(\omega_1, \omega_2)|, |M(v_1(\omega_1, \omega_2))|\}.$$

This definition can be modified in a number of ways, all leading to the same condition. One alternative is:

Definition 3′. We say $T_1 \overset{M}{\rightsquigarrow} T_2$, where $|\det(M)| \geq 1$, if there is an orbit injection $\varphi : \Omega_1 \to \Omega_2$ so that for any $\varepsilon > 0$, there is a set $\bar{A}(\varepsilon) \subset \Omega$, $\mu_1(\bar{A}(\varepsilon)) > 1 - \varepsilon$, and a $b(\varepsilon)$ so that if $\omega_1, \omega_2 \in \bar{A}(\varepsilon)$ and on the same orbit, then

$$\begin{aligned}&|v_2(\varphi(\omega_1), \varphi(\omega_2)) - M(v_1(\omega_1, \omega_2))|\\ &\qquad \leq \varepsilon \max\{|v_1(\omega_1, \omega_2)|, |M(v_1(\omega_1, \omega_2))|\} + b(\varepsilon).\end{aligned}$$

That satisfying definition 3′ implies satisfying definition 3 is easy. The other implication follows by selecting $\bar{A}(\varepsilon)$ to be a subset of $A(\varepsilon/2)$ where $\{v_2(\omega_1, T_1^v(\omega_1)); |v| \leq N(\varepsilon)\}$ is bounded by some, perhaps very large, bound B so that $\mu_1(\bar{A}(\varepsilon)) > 1 - \varepsilon$. We will take such a step many times as we continue. Now $b(\varepsilon) = B + N(\varepsilon)\|M\|$.

In either of these versions of the definition we could replace the $\varepsilon \max\{|v_1(\omega_1, \omega_2)|, |M(v_1(\omega_1, \omega_2)|\}$ by any one of $\varepsilon|v_1(\omega_1, \omega_2)|$, $\varepsilon|M(v_1(\omega_1, \omega_2))|$, $\varepsilon|v_2(\varphi(\omega_1), \varphi(\omega_2))|$, $\varepsilon|M^{-1}(v_2(\varphi(\omega_1), \varphi(\omega_2)))|$, or $\varepsilon \min\{|v_1(\omega_1, \omega_2)|, |M(v_1(\omega_1, \omega_2))|\}$ and get the same conditions, as any one of these as a bound forces the ratio of any two of these possible bounds to be uniformly bounded. In our arguments we will stick to definition 3 as $N(\varepsilon)$ is usually easier to construct than $b(\varepsilon)$, and the max is the weakest of the possible bounds.

PROPOSITION 3. *If* $T_1 \overset{M_1}{\rightsquigarrow} T_2 \overset{M_2}{\rightsquigarrow} T_3$ *then* $T_1 \overset{M_2 \circ M_1}{\rightsquigarrow} T_3$.

Proof. Using definition 3′, index the sets and parameters of the two relations with subscripts 1 and 2. Pick a $\delta < \varepsilon/2$ so small that

$$A(\varepsilon) = A_1(\delta) \cap \varphi_1^{-1}(A_2(\delta))$$

satisfies $\mu_1(A(\varepsilon)) > 1 - \varepsilon$ and $\|M_2\| < \varepsilon/2\delta$. Now if $\omega_1, \omega_2 \in A(\varepsilon)$,

$$\begin{aligned}&|v_3(\varphi_2 \circ \varphi_1(\omega_1), \varphi_2 \circ \varphi_1(\omega_2)) - M_2 \circ M_1(v_1(\omega_1, \omega_2))|\\ &\quad \leq |v_3(\varphi_2 \circ \varphi_1(\omega_1), \varphi_2 \circ \varphi_1(\omega_2)) - M_2(v_2(\varphi_1(\omega_1), \varphi_1(\omega_2)))|\\ &\qquad + |M_2(v_2(\varphi_1(\omega_1), \varphi_1(\omega_2)) - M_1(v_1(\omega_1, \omega_2)))|\\ &\quad \leq \delta|v_3(\varphi_2 \circ \varphi_1(\omega_1), \varphi_2 \circ \varphi_1(\omega_2))| + b_2(\varepsilon)\\ &\qquad + (\varepsilon/2\delta)\|M_2\|(|v_1(\omega_1, \omega_2)|) + \|M_2\| b_1(\varepsilon)\end{aligned}$$

https://doi.org/10.1017/S0143385700002297 Published online by Cambridge University Press

$$\leq \varepsilon \max \{|v_3(\varphi_2 \circ \varphi_1(\omega_1), \varphi_2 \circ \varphi_1(\omega_2))|, |v_1(\omega_1, \omega_2)|\} + b(\varepsilon),$$

where $b(\varepsilon) = b_2(\varepsilon) + \|M_2\| b_1(\varepsilon)$. This is enough. □

Notice that as φ is non-singular, the ergodic theorem applied to T_2 says a.e. T_2 orbit spends a fraction about $\mu_2(\varphi(A(\varepsilon)))$ of a large cube centred at the origin in $\varphi(A(\varepsilon))$. But as φ is an orbit injection, by definition 3, this fraction is approximately $\mu(A(\varepsilon))|\det(M^{-1})|$. Letting $\varepsilon \to 0$, we conclude $\mu_2(\varphi(\Omega_1)) = 1/|\det(M)|$. Thus it is clearly necessary for $|\det(M)| \geq 1$.

We could remove this condition by allowing φ not to be 1–1. This leads to a reasonable development of the theory, but we feel it loses some of the flavour of the classical one-dimensional case, is no easier to work with, and follows quite easily from our development here.

To give our second version, $\overset{M}{\underset{w}{\rightsquigarrow}}$, we need some notation. For a rectangle $R(r_1, \ldots, r_n) = \bigotimes_{i=1}^n \{0, 1, \ldots, v_i - 1\}$, let $t(R) = \min_{i,j}(r_i/r_j)$ and $\operatorname{dia}(R) = \min_i (r_i)$.

Definition 3*w*. We say $T_1 \overset{M}{\underset{w}{\rightsquigarrow}} T_2$, where $|\det(M)| \geq 1$, if there is an orbit injection φ, $\mu(\varphi(\Omega_1)) = 1/\det(M)|$, and an $\alpha > 0$ so that for a.e. ω,

$$\lim_{\substack{\operatorname{dia}(R) \to \infty, \\ A(R) \geq \alpha}} \left(\frac{\operatorname{card}(\{v_2(\varphi(\omega), \varphi(T_1^v(\omega))); v \in R\} - \text{c.h.}(M(R)))}{\operatorname{card}(R)} \right) = 0.$$

By c.h. (S) we mean the lattice points of $\mathbb{Z}^n$ in the convex hull in $\mathbb{R}^n$ of S.

As we shall see later, this condition could be stated more weakly and still imply $T_1 \overset{M}{\rightsquigarrow} T_2$. As stated, though, as a limit condition on the images of rectangles satisfying a Vitali-type condition ($t(R) \geq \alpha$), it is the most natural of such possibilities. That definition 3 implies definition 3*w* follows from the fact that the pointwise ergodic theorem holds uniformly on rectangles $t(R) \geq \alpha$, as $\operatorname{dia}(R) \to \infty$. Just apply it to $\chi_{A(\varepsilon)}$. The extra condition $\mu((\Omega_1)) = 1/|\det(M)|$ will force uniqueness of M in the reverse argument.

We now give a series of ever weaker conditions on an orbit injection map φ, all of which imply $T_1 \overset{M}{\rightsquigarrow} T_2$. These conditions parallel the development in **[6]** for barely linear equivalence.

Condition 1. There is a set of linearly independent vectors $e_1, \ldots, e_n \in \mathbb{Z}^n$, and a non-singular matrix M so that for any $\varepsilon > 0$ there is an $N(\varepsilon)$ and a set $A(\varepsilon)$ with $\mu_1(A(\varepsilon)) > 1 - \varepsilon$ so that if $\omega_1, \omega_2 \in A(\varepsilon)$, and $v_1(\omega_1, \omega_2) = Ne_i$, where $N \geq N(\varepsilon)$ then

$$|v_2(\varphi(\omega_1), \varphi(\omega_2)) - M(v_1(\omega_1, \omega_2))| \leq \varepsilon |v_1(\omega_1, \omega_2)|.$$

Condition 2. There is a constant K and a linearly independent set $e_1, \ldots, e_n \in \mathbb{Z}^n$ so that for any $\varepsilon > 0$ there is a set $A(\varepsilon)$, $\mu(A(\varepsilon)) > 1 - \varepsilon$, and an $N(\varepsilon)$ so that if ω_1, $\omega_2 \in A(\varepsilon)$ and $v_1(\omega_1, \omega_2) = Ne_i$, $N > N(\varepsilon)$, then

$$|v_2(\varphi(\omega_1), \varphi(\omega_2))| \leq K |v_1(\omega_1, \omega_2)|.$$

Condition 3. There is a constant K and a linearly independent set $e_1, \ldots, e_n \in \mathbb{Z}^n$ so that for any $\varepsilon > 0$ there are sets $A_i(\varepsilon)$, $\mu_1(A_i(\varepsilon)) > 1 - \varepsilon$, and an $N(\varepsilon)$ so that if

$\omega_1, \omega_2 \in A_i(\varepsilon)$ and $v_1(\omega_1, \omega_2) = Ne_i$ where $N \geq N(\varepsilon)$, then

$$|v_2(\varphi(\omega_1), \varphi(\omega_2))| < K|v_1(\omega_1, \omega_2)|.$$

Condition 4. There is a constant K and a linearly independent set $e_1, \ldots, e_n \in \mathbb{Z}^n$ and sets $A_1, \ldots, A_n$ each of positive measure so that if $\omega_1, \omega_2 \in A_i$ and $v_1(\omega_1, \omega_2) = Ne_i$, $N > K$, then

$$|v_2(\varphi(\omega_1), \varphi(\omega_2))| < K|v_1(\omega_1, \omega_2)|.$$

PROPOSITION 4. *Condition* 4 *implies condition* 3.

Proof. We fix i and produce $A_i(\varepsilon)$ from A_i, without using the other A_j's. We do not assume $T_1^{e_i}$ is ergodic, so first decompose μ_1 into $T_1^{e_i}$ ergodic components $\mu_{\{\omega\}}$,

$$\mu_1(A) = \int \mu_{\{\omega\}}(A)\, d\mu,$$

where $\{\omega\}$ is the ergodic component of ω. Let

$$B_i = \{\omega : \mu_{\{\omega\}}(A_i) > \mu(A_i)/2\},$$

a $T_1^{e_i}$-invariant set, and

$$\bar{A}_i = A_i \cap B_i.$$

Take a finite collection of vectors $v_1, \ldots, v_l$ so that

$$\mu_1\left(\bigcup_{j=1}^{l} T_1^{v_j}(B_i)\right) > 1 - \frac{\varepsilon}{2}.$$

Now let

$$\bar{\bar{A}}_i = \bigcup_{j=1}^{l} T_1^{v_j}(\bar{A}_i)$$

and write this as a disjoint union,

$$\bar{\bar{A}}_i = \bar{A}_{i,0} \cup \bar{A}_{i,1} \cup \cdots \cup \bar{A}_{i,l}$$

where $\bar{A}_{i,j} = T_1^{v_j}(\bar{A}_i) - \bigcup_{k=0}^{j-1} \bar{A}_{i,k}$.

Each $\bar{A}_{i,j}$ is contained in a union of ergodic components $B_{i,j}$, which are pairwise disjoint. Choose M so that for all $j = 1, \ldots, l$,

$$\mu_1(\bar{\bar{A}}_{i,j}) = \mu_1\left(\bigcup_{k=0}^{M-1} T_1^{ke_i}(\bar{A}_{i,j})\right) > (1 - (\varepsilon/4))\mu(B_{i,j}).$$

Pick $\bar{N}(\varepsilon) > M$ so large that for a subset $\bar{\bar{\bar{A}}}_{i,j} \subset \bar{A}_{i,j}$, $\mu(\bar{\bar{\bar{A}}}_{i,j}) > (1 - \varepsilon/2)\mu(B_{i,j})$, if $\omega_1, \omega_2 \in \bar{\bar{\bar{A}}}_{i,j}$, and

$$v_1(\omega_1, \omega_2) \leq M + \sup_{i=1,\ldots,l} (|v_i|)$$

then

$$|v_2(\varphi(\omega_1), \varphi(\omega_2))| \leq \bar{N}(\varepsilon).$$

Set $A_i(\varepsilon) = \bigcup_{j=1}^{l} \bar{\bar{\bar{A}}}_{i,j}$, (a disjoint union), $N(\varepsilon) = 2\bar{N}(\varepsilon) + K$ and $\bar{K} = K + 2$. Clearly $\mu_1(A_i(\varepsilon)) > 1 - \varepsilon$.

If $\omega_1, \omega_2 \in A_i(\varepsilon)$ and $v_1(\omega_1, \omega_2) = Ne_i$, $N \geq N(\varepsilon)$, then $\omega_1, \omega_2 \in \bar{\bar{\bar{A}}}_{i,j}$ for some j, as the $B_{i,j}$ are $T_1^{e_i}$ invariant. Thus

$$\omega_1 = T_1^{n_1 e_i}(\bar{\omega}_1), \qquad \omega_2 = T_1^{n_2 e_i}(\bar{\omega}_2)$$

where $\bar{\omega}_1, \bar{\omega}_2 \in \bar{\bar{A}}_{i,j}$, $0 \le n_1, n_2 < N$. Now setting

$$\bar{\bar{\omega}}_1 = T^{-v_j}(\bar{\omega}_1), \qquad \bar{\bar{\omega}}_2 = T^{-v_j}(\bar{\omega}_2),$$

$\bar{\bar{\omega}}_1, \bar{\bar{\omega}}_2 \in T^{-v_j}(A_{i,j}) \subset A_i$ and

$$\begin{aligned} |v_1(\bar{\bar{\omega}}_1, \bar{\bar{\omega}}_2)| &= |v_1(\omega_1, \omega_2) + (n_2 - n_1)e_i| \\ &> |v_1(\omega_1, \omega_2)| - M > 2\bar{N}(\varepsilon) + K - N(\varepsilon) > K \end{aligned}$$

so

$$|v_2(\varphi(\bar{\bar{\omega}}_1), \varphi(\bar{\bar{\omega}}_2))| < K(|v_1(\omega_1, \omega_2)|).$$

But

$$\begin{aligned} &|v_2(\varphi(\bar{\bar{\omega}}_1), \varphi(\bar{\bar{\omega}}_2))| \\ &\quad \ge |v_2(\varphi(\omega_1), \varphi(\omega_2))| - |v_2(\varphi(\omega_1), \varphi(\bar{\bar{\omega}}_1))| - |v_2(\varphi(\omega_2), \varphi(\bar{\bar{\omega}}_2))| \\ &\quad \ge |v_2(\varphi(\omega_1), \varphi(\omega_2))| - 2\bar{N}(\varepsilon) \end{aligned}$$

so

$$\begin{aligned} |v_2(\varphi(\omega_1), \varphi(\omega_2))| &\le K(|v_1(\omega_1, \omega_2)|) + 2\bar{N}(\varepsilon) \\ &\le (K+2)|v_1(\omega_1, \omega_2)| = \bar{K}|v_1(\omega_1, \omega_2)|. \end{aligned}$$ □

COROLLARY 5. *Condition* 3 *implies condition* 2.

Proof. Let $A(\varepsilon) = \bigcap_{i=1}^{n} A_i(\varepsilon/n)$. □

PROPOSITION 6. *Condition* 2 *implies condition* 1.

Proof. Without loss of generality we may assume that for each i any ergodic component of $T_1^{e_i}$ intersecting $A(\bar{\varepsilon})$ has component measure at least $\frac{2}{3}$, for any $\bar{\varepsilon} < \varepsilon/2$.

For $\omega \in A(\bar{\varepsilon})$, let $S_i(\omega)$ be the first return under $T_1^{e_i}$ of ω to $A(\bar{\varepsilon})$. Now let

$$f_1^i(\omega) = v_1(\omega, S_i(\omega)) = n(\omega)e_i$$

and

$$f_2^i(\omega) = v_2(\varphi(\omega), \varphi(S_i(\omega))).$$

Certainly f_1^i is a vector-valued $L^1(\mu_1)$-function on $A(\bar{\varepsilon})$. We do not know *a priori* that f_2^i is in $L^1(\mu_1)$, but we do know

$$\left| \sum_{j=0}^{N(\varepsilon)-1} f_2^i(S^j)(\omega)) \right| \le K \sum_{j=0}^{N(\varepsilon)-1} |f_1^i(S^j(\omega))|,$$

so the partial sum from 0 to $N(\varepsilon)$ of f_2^i is in $L^1(\mu_1)$.

It now follows that $(1/kN(\varepsilon)) \sum_{j=0}^{kN(\varepsilon)-1} f_2^i(S^j(\omega))$ converges for almost every ω, hence for a.e. ω, it converges uniformly on the points $\omega, S^j(\omega), \ldots, S^{N(\varepsilon)-1}(\omega)$, and

$$\frac{1}{n} \sum_{j=0}^{n-1} f_2^i(S^j(\omega)) = \frac{1}{n}\left(\sum_{j=0}^{t-1} f_2^i(S^j(\omega)) \right) + \frac{kN(\varepsilon)}{n} \frac{1}{kN(\varepsilon)} \sum_{j=0}^{kN(\varepsilon)-1} f_2^i(S^j(S^t(\omega)))$$

where $n = t + kN(\varepsilon)$, $t < N(\varepsilon)$ and so $\lim_{n\to\infty} (1/n) \sum_{j=0}^{n-1} f_2^i(S^j(\omega))$ exists a.e. Let

$$\bar{e}_i(\omega) = \lim_{N\to\infty} \left(\frac{\sum_{j=0}^{N} f_2(S^j(\omega))}{\sum_{j=0}^{N} n(S^j(\omega))|e_i|} \right).$$

Now $\bar{e}_i(\omega)$ is S_i invariant and defined a.e. on $A(\bar{\varepsilon})$. Suppose $\omega_2 = T_1^{v}(\omega_1)$ and ω_1 are both in $A(\bar{\varepsilon})$. As both $\omega_1, \omega_2 \in A(\bar{\varepsilon})$ we know that the density in the $T_1^{e_i}$ orbit of

https://doi.org/10.1017/S0143385700002297 Published online by Cambridge University Press

either point of points in $A(\bar{\varepsilon})$ is at least $\frac{2}{3}$. Hence for a set of density $\frac{1}{4}$ of integers n we have $T_1^{ne_i}(\omega_1)$, $T_1^{ne_i}(\omega_2) \in A(\bar{\varepsilon})$. But now

$$T_1^{ne_i}(\omega_2) = T_1^{v}(T^{ne_i}(\omega_1)).$$

For a.e. ω there is a $K(\omega)$ so that the density in the $T_1^{e_i}$ orbit of ω of points ω' with

$$|v_2(\varphi(\omega'), \varphi(T_1^{v}(\omega'))| < K(\omega)$$

is at least $\frac{7}{8}$. Thus we can find a sequence $n_k \to \infty$ with

$$K(\omega) \geq \limsup_{k\to\infty} (|v_2(\varphi(T_1^{n_k e_i}(\omega_1)), \varphi(T_1^{n_k e_i}(\omega_2))|)$$
$$= \limsup_{k\to\infty} (n_k(\bar{e}_i(\omega_1) - \bar{e}_i(\omega_2)))$$

and

$$\bar{e}_i(\omega_1) = \bar{e}_i(\omega_2),$$

so $\bar{e}_i$ is constant on those parts of any orbit in $A(\bar{\varepsilon})$, hence a constant $\bar{e}_i$.

Define $M(e_i) = \bar{e}_i|e_i|$. For all $\varepsilon > 0$ and $\omega_1 \in A(\bar{\varepsilon})$, by the ergodic theorem applied to $T_1^{e_i}$ there is an $N = N(\omega_1, \varepsilon)$ so that if $\omega_2 = T^{ne_i}(\omega_1) \in A(\bar{\varepsilon})$, $n > N$, then

$$|v_2(\varphi(\omega_1), \varphi(\omega_2)) - M(v_1(\omega_1, \omega_2))| = |v_2(\varphi(\omega_1), \varphi(\omega_2)) - n\bar{e}_i|e_i||$$
$$< \varepsilon n|e_i| = \varepsilon|v_1(\omega_1, \omega_2)|.$$

Pick $\bar{N}(\varepsilon)$ so large that for a subset $\bar{A}(\varepsilon) \subset A(\bar{\varepsilon})$, $\mu_1(\bar{A}(\varepsilon)) > 1-\varepsilon$, if $\omega \in \bar{A}(\varepsilon)$ then $N(\omega, \varepsilon) \leq \bar{N}(\varepsilon)$. □

PROPOSITION 7. *Condition* 1 *implies* $T_1 \overset{M}{\rightsquigarrow} T_2$.

Proof. Assume $\varepsilon < 1/4^n$, and pick $\bar{\varepsilon} < \varepsilon/2n$, and we again assume, without loss of generality, that on any ergodic component of $T_1^{e_i}$ which $A(\bar{\varepsilon})$ intersects, its fibre measure is at least $1-(\varepsilon/2n)$.

We define a sequence of sets $B_0 \supset B_1 \supset \cdots \supset B_n$ and values $N_1, \ldots, N_n$ inductively. $B_0 = A(\bar{\varepsilon})$, and now assume B_i has been defined so that

$$\mu_1(B_i) > 1 - \frac{\varepsilon}{2(n-i)}$$

and on any $T_1^{e_j}$ ergodic component that B_i intersects it has fibre measure $> 1-(\varepsilon/2(n-i))$. Choose N_{i+1} so large that for a subset $B_{i+1} \subset B_i$,

$$\mu_{\{\omega\}}(B_{i+1}) > 1 - \frac{\varepsilon}{2(n-i-1)} \quad \text{or} \quad \mu_{\{\omega\}}(B_{i+1}) = 0$$

and

$$\mu_1(B_{i+1}) > 1 - \frac{\varepsilon}{2(n-i-1)}$$

where $\mu_{\{\omega\}}$ is any ergodic measure for some $T_1^{e_j}$, and if $\omega \in B_{i+1}$,

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_{B_i}(T_1^{je_{i+1}}(\omega)) > 1 - \frac{\varepsilon}{2(n-i-1)} \qquad \text{for } n \geq N_i.$$

Select $\bar{M} > 2\max_i(N_i)$, so large that for a subset $A'(\varepsilon)$, $\mu(A'(\varepsilon)) > 1-(3\varepsilon/4)$ and if $\omega \in A'(\omega)$, for each $\alpha \in \{-1, 1\}^n$, the sets

$$S(\alpha, \omega) = \left\{ T_1^v(\omega): v = \sum_{i=1}^n (\tfrac{3}{2}\bar{M}\alpha_i + a_i)e_i \right\}$$

where $|a_i| \le \bar{M}/2$, must each contain points of B_n. These are large corner blocks in a parallelepiped whose sides are in the e_i directions.

As card $(S(\alpha, \omega)) > (1/4^n)$ card $(\{T_1^v(\omega): v = \sum a_i e_i, |a_i| \le 2\bar{M}\}$, and $\varepsilon < 1/4^n$, again using the pointwise ergodic theorem, for $\bar{M}$ large enough, these corner blocks each must contain points of B_n.

Now select a $\bar{K}$ so large that for a subset $\bar{A}(\varepsilon) \subset A'(\varepsilon)$, $\mu(\bar{A}(\varepsilon)) > 1-\varepsilon$, if $\omega \in \bar{A}(\varepsilon)$, $v = \sum a_i e_i$, $|a_i| \le \bar{M}$, then

$$|v_2(\varphi(\omega), \varphi(T_1^v(\omega)))| < \bar{K}.$$

Set $\bar{N}(\varepsilon) = (\bar{M}n \sup(|e_i|)/\varepsilon, 4n\bar{K}/\varepsilon)$. Now suppose $\omega_1, \omega_2 \in \bar{A}(\varepsilon)$, $|v_1(\omega_1, \omega_2)| \ge \bar{N}(\varepsilon)$. Select elements $\alpha_1, \alpha_2 \in \{-1, 1\}^n$, and points $\bar{\omega}_1 \in S(\alpha_1, \omega_1) \cap B_n$, $\bar{\omega}_2 \in S(\alpha_2, \omega_2) \cap B_n$ so that $v_1(\bar{\omega}_1, \bar{\omega}_2) = \sum_{i-1}^n n_i e_i$ and all $|n_i| \ge \bar{M}$. We know $|v_2(\varphi(\omega_1), \varphi(\bar{\omega}_1))| < \bar{K}$ and $|v_2(\varphi(\omega_2), \varphi(\bar{\omega}_2))| < \bar{K}$.

We construct inductively a series of pairs of points $(C_{j,1}, C_{j,2})$, $j = n+1, n, \ldots, 2$ where $C_{n+1,1} = \bar{\omega}_1$, $C_{n+1,2} = \bar{\omega}_2$, $C_{j,1}, C_{j,2} \in B_j$, and

$$C_{j-1,1} = T_1^{m_{j-1,1}e_{j-1}}(C_{j,1})$$

and

$$C_{j-1,2} = T_1^{-m_{j-1,2}e_{j-1}}(C_{j,2})$$

where

$$m_{j-1,1} + m_{j-1,2} = n_{j-1}$$

and both m's are of the same sign. We do this as follows. Suppose we have $C_{j,1}$, $C_{j,2}$. Then $T^{v_{1,j}}(C_{j,1}) = C_{j,2}$ where $v_{1,j} = \sum_{i=1}^{j-1} n_i e_i$. Assuming $j > k$, $n_{j-1} \ge \bar{M}$, and as $C_{j,1}, C_{j,2} \in B_j$,

$$\sum_{k=0}^{n_{j-1}} \chi_{B_{j-1}}(T^{ke_{j-1}}(C_{j,1}))$$

and

$$\sum_{k'=0}^{-n_{j-1}} \chi_{B_{j-1}}(T^{k'e_{j-1}}(C_{j,2}))$$

are both

$$> 1 - \frac{\varepsilon}{2(n-j+1)} > \frac{7}{8}$$

so there are values $k = m_{j-1,1}$ and $-k' = m_{j-1,2}$ of the same sign, $m_{j-1,1} + m_{j-1,2} = n_{j-1}|k|$ and $|k'|$ are both larger than $n_{j-1}/4 > \bar{M}$ and both $C_{j-1,1} = T^{ke_{j-1}}(C_{j,1})$ and $C_{j-1,2} = T^{-k'e_{j-1}}(C_{j,2})$ belong to B_{j-1}. Notice that all the $C_{j,1}$, $C_{j,2}$ are in $A(\bar{\varepsilon})$ and

$$C_{2,2} = T^{n_1 e_1}(C_{2,1}).$$

Computing

$$\begin{aligned} &v_2(\varphi(\omega_1), \varphi(\omega_2)) \\ &= \sum_{j=n+1}^{2} v_2(\varphi(C_{j,1}), \varphi(C_{j-1,1})) + \sum_{j=n+1}^{2} v_2(\varphi(C_{j,2}), \varphi(C_{j-1,2})) \\ &\quad + v_2(\varphi(\omega_1), \varphi(\bar{\omega}_1)) + v_2(\varphi(\bar{\omega}_2), \varphi(\omega_2)). \end{aligned}$$

As $|m_{j-1,t}| > \bar{M}$, setting $\bar{e}_i = M(e_i)$, we get

$$v_2(\varphi(C_{j,t}), \varphi(C_{j-1,t})) = (m_{j-1,t}\bar{e}_i)\left(1 \pm \frac{\varepsilon}{2}\right).$$

Thus

$$v_2(\varphi(C_{j,1}), \varphi(C_{j-1,1})) + v_2(\varphi(C_{j,2}), \varphi(C_{j-1,2})) = n_{j-1}\bar{e}_i \pm \frac{\varepsilon}{2n}|v_1(\omega_1, \omega_2)|.$$

We conclude

$$v_2(\varphi(\omega_1), \varphi(\omega_2)) = \sum_{j=1}^{n} n_j\bar{e}_j \pm \left(\frac{\varepsilon}{2} v_1(\omega_1, \omega_2) + 2\bar{K}\right)$$

or

$$|v_2(\varphi(\omega_1), \varphi(\omega_2)) - M(v_1(\omega_1, \omega_2))| \le \varepsilon|v_1(\omega_1, \omega_2)|.$$

It follows from the ergodic theorem applied to $\varphi(\Omega_1)$ that

$$\mu_2(\varphi(\bar{A}(\varepsilon))) = \frac{\mu_1(\bar{A}(\varepsilon))}{\det(M)}(1 \pm \varepsilon)$$

and so $|\det(M)| \ge 1$. □

COROLLARY 8. *If $\varphi: \Omega_1 \to \Omega_1$ is of the form $\varphi(\omega) = T_1^{v(\omega)}(\omega)$ and φ is 1–1, then $T_2 \overset{\text{id}}{\rightsquigarrow} T_1$ where $T_2 = \varphi^{-1}T_1\varphi$, i.e., co-boundaries always give $\overset{\text{id}}{\rightsquigarrow}$.*

Proof. Choose a set A with $\mu_1(A) > 0$ where $v(\omega) = v_0$ is constant. For any $\omega_1, \omega_2 \in A$,

$$v_2(\varphi(\omega_1), \varphi(\omega_2)) = v_1(T^{v_0}(\omega_1), T^{v_0}(\omega_2)) = v_1(\omega_1, \omega_2),$$

and we are done as we satisfy condition 4, hence condition 1, but clearly $M = \text{id}$. □

We are now ready to see that $\underset{w}{\overset{M}{\rightsquigarrow}}$ implies $\overset{M}{\rightsquigarrow}$. It is interesting to work with a condition that is actually weaker, and shows another sense in which Kakutani equivalence in $\mathbb{Z}^n$, $n > 1$, has a much richer structure than in one dimension.

Definition 4. We say two $\mathbb{Z}^n$ actions T_1 and T_2 are *cubicly M-equivalent* $T_1 \underset{c}{\overset{M}{\rightsquigarrow}} T_2$ if there is an orbit injection $\varphi: \Omega_1 \to \Omega_2$, with $\mu_2(\varphi(\Omega_1)) = |\det(M)|$ and for a.e. $\omega \in \Omega_1$, for any $\varepsilon > 0$, there is an $N(\omega, \varepsilon)$, so that if $N > (\omega, \varepsilon)$ then

$$\text{card}\left(\{\varphi(T_1^v(\omega)) | v \in [0, N]^n\} - \{T_2^v(\varphi(\omega)) | v \in \text{c.h.}(M([0, N]^n))\}\right) < \varepsilon N^n.$$

PROPOSITION 9. *$T_1 \underset{c}{\overset{M}{\rightsquigarrow}} T_2$ iff there is a permutation of coordinates matrix $\bar{M}$ with $T_1 \overset{M\bar{M}}{\rightsquigarrow} T_2$.*

https://doi.org/10.1017/S0143385700002297 Published online by Cambridge University Press

Proof. The 'if' part follows easily. For the 'only if', we first verify that $T_1 \overset{M}{\underset{c}{\rightsquigarrow}} T_2$ implies φ satisfies condition 4. Let $e_1, \ldots, e_n$ be the standard basis vectors. Fix e_i and select N_i so large that for a set A_i, $\mu_1(A_i) > \frac{1}{2}$, if $\omega \in A_i$ then $N(\omega, 1/4^n) \le N_i < \infty$. Let $K = 2 \sup_i (N_i, n\|M\|)$. Now suppose condition 4 is false in that for $\omega_2 = T^{ke_i}(\omega_1)$, $\omega_1, \omega_2 \in A_i$ and $k > K$ we obtain

$$|v_2(\varphi(\omega_1), \varphi(\omega_2))| > K|v_1(\omega_1, \omega_2)|.$$

Now

(i) $\operatorname{card}(\{\varphi(T_1^v(\omega_1)) | v \in [0, k]^n\} \cap \{T_2^v(\varphi(\omega_1)) | v \in \text{c.h. } (M([0, k]^n))\}^c) \le k^n/4^n$,

(ii) $\operatorname{card}(\{\varphi(T_1^v(\omega_2)) | v \in [0, k]^n\} \cap \{T_2^v(\varphi(\omega_2)) | v \in \text{c.h. } (M([0, k]^n))\}^c) \le k^n/4^n$,

and

(iii) $\operatorname{card}(\{\varphi(T_1^v(\omega_1)) | v \in [0, 2k]^n\} \cap \{T_2^v(\varphi(\omega_1)) | v \in \text{c.h. } (M([0, 2k]^n))\}^c)$ $\le (2k)^n/4^n$.

But now

$$\{\varphi(T_1^v(\omega_2)) | v \in [0, k]^n\} \subset \{\varphi(T_1^v(\omega_1)) | v \in [0, 2k]^n\},$$

and the smaller is a fraction of at least $1/2^n$ of the larger, so

$$\{\varphi(T_1^v(\omega_2)) | v \in [0, k]^n\} \cap \{T_2^v(\varphi(\omega_2)) | v \in \text{c.h. } (M([0, k]^n))\}$$
$$\subset \{\varphi(T_1^v(\omega_1)) | v \in [0, 2k]^n\},$$

and here using (ii) the cardinality of the smaller set is at least $k^n(1-(1/4^n))$. Thus a fraction of at least $(1-(1/4^n))$ of $\{T_2^v(\varphi(\omega_2)) | v \in \text{c.h. } (M([0, k]^n))\}$ is contained in

$$\{\varphi(T_1^v(\omega_1)) | v \in [0, 2k]^n\}.$$

But on the other hand

$$\{T_2^v(\varphi(\omega_2)) | v \in \text{c.h. } (M([0, k]^n))\} = T_2^{v_2(\varphi(\omega_1), \varphi(\omega_2))}(\{T_2^v(\varphi(\omega_1)) | v \in \text{c.h. } (M([0, k]^n))\})$$

and as $|v_2(\varphi(\omega_1), \varphi(\omega_2)| > K|v_1(\omega_1, \omega_2)\} > 2kn\|M\|$, this set is disjoint from

$$\{T_2^v(\varphi(\omega_1)) | v \in \text{c.h. } (M([0, 2k]^n))\}.$$

Thus

$$\operatorname{card}(\{\varphi(T_1^v(\omega_1)) | v \in [0, 2k)^n\} \cap \{T_2^v(\varphi(\omega_1)) | v \in \text{c.h. } (M(0, 2k]^n))\}^c)$$
$$> \frac{1}{2^n}\left(1 - \frac{1}{4^n}\right)(2k)^n,$$

in conflict with (iii). Thus condition 4 holds and φ is an M_1 equivalence for some M_1. It follows that $|\det(M_1)| = |\det(M)|$ and clearly from the definitions of $\overset{M}{\underset{c}{\rightsquigarrow}}$ and $\overset{M}{\rightsquigarrow}$,

$$\text{c.h. } (M_1([0, 1]^n)) \subseteq \text{c.h. } (M([0, 1]^n)).$$

Hence $M_1([0, 1]^n) = M([0, 1]^n)$, and $M_1^{-1} \circ M$ fixes the unit cube, and so must be a permutation of coordinates. □

Notice that $\underset{c}{\rightsquigarrow}$ and $\rightsquigarrow$ were not identical only because of the symmetries of the unit cube, a subgroup of $L(n, \mathbb{R})$. If instead of cubes we had chosen expansions of

https://doi.org/10.1017/S0143385700002297 Published online by Cambridge University Press

a rectangle, no two sides of which were equal, we would get a condition equivalent to $\rightsquigarrow$. It is now obvious that $\overset{}{\underset{w}{\rightsquigarrow}}$ implies $\rightsquigarrow$. We could also, for example, have chosen spheres, increasing the subgroup of symmetries. Notice that what matters is the symmetries, or asymptotic symmetries in $\mathbb{R}^n$. Each such subgroup would lead to a corresponding equivalence relation.

To complete the formal structure of our indexed family of relations $\overset{M}{\rightsquigarrow}$ we need to show:

PROPOSITION 10. *If* $T_1 \overset{M_1}{\rightsquigarrow} T_2$ *and* $T_3 \overset{M_2}{\rightsquigarrow} T_2$ *and* $|\det(M_2^{-1}M_1)| \geq 1$ *then* $T_1 \overset{M_2^{-1}M_1}{\rightsquigarrow} T_3$.

Proof. Let the orbit injections to Ω_2 be φ_1 and φ_2. As

$$\mu_2(\varphi_1(\Omega_1)) = \frac{1}{|\det(M_1)|} \leq \frac{1}{|\det(M_2)|} = \mu_2(\varphi_2(\Omega_3))$$

we can construct a 1–1 map $\psi : \varphi_1(\Omega_1) \to \varphi_2(\Omega_3)$ of the form $\psi(\omega) = T_2^{v(\omega)}(\omega)$. Set

$$\varphi(\omega) = \varphi_2^{-1}(\psi(\varphi_1(\omega))).$$

Let A be a set where $v(\omega) = v_0$ is constant (to discriminate the structures in Ω_1 from Ω_3 we use subscripts, i.e. $A_1(\varepsilon)$, $N_1(\varepsilon)$ and $A_3(\varepsilon)$, $N_3(\varepsilon)$). Fix $\varepsilon > 0$ and select $\bar{\varepsilon}$ so small that $\bar{A}_1$, where

$$\varphi(\bar{A}) - (\varphi_2^{-1} \circ \psi(\varphi_1(A_1(\bar{\varepsilon})) \cap A) \cap A_3(\bar{\varepsilon})),$$

has positive μ_1 measure. Now let $\omega_1, \omega_2 \in \bar{A}$ with

$$|v_1(\omega_1, \omega_2)| > (1 + \bar{\varepsilon}) N_3(\bar{\varepsilon}) \|M_1^{-1}\| + N_2(\bar{\varepsilon}).$$

We check

$$\begin{aligned} v_3(\varphi(\omega_1), \varphi(\omega_2)) &= v_3(\varphi_2^{-1}(\psi(\varphi_1(\omega_1))), \varphi_2^{-1}(\psi(\varphi_1(\omega_2)))) \\ &= M_2^{-1}(v_2(\psi(\varphi_1(\omega_1)), \psi(\varphi_1(\omega_2)))(1 \pm \varepsilon), \end{aligned}$$

as both $\bar{\omega}_1 = \varphi_2^{-1}(\psi(\varphi_1(\omega_1)))$ and $\bar{\omega}_2 = \varphi_2^{-1}(\psi(\varphi_1(\omega_2)))$ are in $A_3(\varepsilon)$ and $|v_3(\bar{\omega}_1, \bar{\omega}_2)| > N_3(\varepsilon)$. Now

$$v_2(\psi(\varphi_1(\omega_1)), \psi(\varphi_1(\omega_2))) = v_2(\varphi_1(\omega_1), \varphi_2(\omega_2))$$

as $\psi = T^{v_0}$ on A, and this is equal to

$$M_1((v_1(\omega_1, \omega_2)(1 \pm \varepsilon)).$$

For $\bar{\varepsilon}$ small enough $\omega_1, \omega_2 \in \bar{A}(\varepsilon)$ then

$$\begin{aligned} &|v_2(\varphi(\omega_1), \varphi(\omega_2)) - M_2^{-1} M_1(v_1(\omega_1, \omega_2))| \\ &\quad \leq \varepsilon \max\{|v_1(\omega_1, \omega_2)|, |M_2^{-1} M_1(v_1(\omega_1, \omega_2))|\}. \end{aligned}$$

This certainly gives condition 4, so $T_1 \overset{\bar{M}}{\rightsquigarrow} T_2$ by φ for some $\bar{M}$. But now once $\varepsilon < \mu_2(A)/|\det(M)|$ is sufficiently small, $A(\varepsilon) \cap \bar{A}(\varepsilon) \neq \emptyset$ and it follows that $\bar{M} = M_2^{-1} M_1$. □

We are now ready to verify our main result.

PROPOSITION 11. *For* T_1 *and* T_2 *ergodic* $\mathbb{Z}^n$ *actions,*

$$T_1 \overset{M}{\rightsquigarrow} T_2 \quad \textit{iff} \quad T_1 \overset{M}{\leadsto} T_2.$$

We first verify a few preliminaries.

LEMMA 12. *If $\bar{T}$, $\bar{\Omega}$ is the suspension of the ergodic $\mathbb{Z}^n$ action T, Ω and $v_2(\omega_1, \omega_2)$ is a barely linear time change of $\bar{T}$, with matrix M, then for any set $A \subset \Omega$, $\mu(A) > 0$, and a.e. ω,*

$$\lim_{N\to\infty} \frac{\operatorname{card}\{\omega_2 \in A \mid v_2(\omega_1, \omega_2) \in [0, N]^n\}}{N^n} = \frac{\mu(A)}{|\det(M)|}.$$

Proof. Suppose we modify $v_2(\omega_1, \omega_2)$ by a coboundary to

$$\bar{v}_2(\omega_1, \omega_2) = v_2(\omega_1, \omega_2) + v(\omega_1) - v(\omega_2)$$

in this expression, and select $K(\varepsilon)$ so that $A(\varepsilon) = \{\omega \in A: |v(\omega)| < K(\varepsilon)\}$ satisfies $\mu(A(\varepsilon)) > (1-\varepsilon)\mu(A)$. Now select N so large that

$$|v(\omega_1)| + K(\varepsilon) < \varepsilon N/4n,$$

and the density in $[0, N-1]^n$ of integer lattice vectors v with $T^v(\omega_1) \in A(\varepsilon)$ is at least $\mu(A)(1-2\varepsilon)$. It follows that points in

$$\{\omega_2 \in A \mid v_2(\omega_1, \omega_2) \in [0, N]^n\} \triangle \{\omega_2 \in A \mid \bar{v}_2(\omega_1, \omega_2) \in [0, N]^n\}$$

either are not in $A(\varepsilon)$ or have $v_2(\omega_1, \omega_2) \in [-\varepsilon N/4n,\ N + (\varepsilon N/4n)]$. Such points have cardinality less than $3\varepsilon N(n)$. Hence the $\overline{\lim}$ and $\underline{\lim}$ of the expression are a.e. unchanged if we modify v_2 by a coboundary. Hence we can replace v_2 by $v_{2,\varepsilon}$ (see definition 1) and now

$$\frac{\partial v_{2,\varepsilon}(\omega_1 T^v(\omega))}{\partial v}(0)$$

is uniformly within ε of M, in which case the $\overline{\lim}$ and $\underline{\lim}$ are both within $n\varepsilon$ of $\mu(A)/|\det(M)|$. Let $\varepsilon \to 0$. □

LEMMA 13. *Suppose $T_1 \overset{M}{\rightsquigarrow} T_2$ and $A_1 \subseteq \Omega_1$, $A_2 \subseteq \Omega_2$ with $\mu_1(A_1) \le \mu_2(A_2)|\det(M)|$ and φ is a barely linear time change achieving $T_1 \overset{M}{\rightsquigarrow} T_2$. For any $\varepsilon > 0$, φ can be modified bv a C^∞, bounded coboundary to φ_ε so that φ_ε agrees with φ on $\varphi(A_1) \cap A_2$, and*

$$\mu_2(\varphi_\varepsilon(A_1) - A_2) < \varepsilon\mu(A_1)/|\det(M)|.$$

Proof. Fix $\varepsilon > 0$. In Ω_2 build a Rochlin tower $\mathcal{T}$ with base B of size $[0, N]^n$. Choose N as follows. For $\bar{\omega} \in B$ let $\mathcal{T}(\bar{\omega}) = \{\bar{T}_2^v(\bar{\omega}) \mid v \in [0, N]^n\}$ be the tower section over $\bar{\omega}$. Choose N so large that we can select B with $\mu_2(\mathcal{T}) > 1 - (\varepsilon/10)$ and for all $\bar{\omega} \in B$,

$$\operatorname{card}\{\omega \in \mathcal{T}(\bar{\omega}) \cap \varphi(A_1)\} = \frac{\mu(A_1)N^n}{|\det(M)|}\left(1 \pm \frac{\varepsilon}{10}\right)$$

and

$$\operatorname{card}\{\omega \in \mathcal{T}(\bar{\omega}) \cap A_2\} = N^n\left(1 \pm \frac{\varepsilon}{10}\right)\mu(A_2)$$

using lemma 12.

Construct a map $\bar{f}_{\bar{\omega}}$, measurably in $\bar{\omega}$, from as large a subset as possible of

$$S_1 = \{\omega \in \mathcal{T}(\bar{\omega}) \cap \varphi(A_1)\}$$

into

$$S_2 = \{\omega \in \mathcal{T}(\bar{\omega}) \cap A_2\}$$

which is the identity if $\omega \in S_1 \cap S_2$. This map is defined on only a finite set of points in the interior of $\mathcal{T}(\bar{\omega})$. Extend $\bar{f}_{\bar{\omega}}$ to all of $\mathcal{T}(\bar{\omega})$, measurably in $\bar{\omega}$, to a C^∞ map of $\mathcal{T}(\omega)$ to itself that agrees with the identity in a neighbourhood of the boundary. Now set

$$\varphi_\varepsilon(\omega) = \begin{cases} \bar{f}_{\bar{\omega}}(\varphi(\omega)) & \text{if } \omega \in \mathcal{T}(\bar{\omega}) \\ \varphi(\omega) & \text{outside } \mathcal{T}. \end{cases}$$

Notice that

$$v_2(\varphi(\omega_1), \varphi(\omega_2)) - v_2(\varphi_\varepsilon(\omega_1), \varphi_\varepsilon(\omega_2))$$
$$= -v_2(\varphi(\omega_1), \bar{f}_{\bar{\omega}_1}(\varphi(\omega_1))) + v_2(\varphi(\omega_2), \bar{f}_{\bar{\omega}_2}(\varphi(\omega_2)))$$

if $\omega_1 \in \mathcal{T}(\bar{\omega}_1)$, $\omega_2 \in \mathcal{T}(\bar{\omega}_2)$. Setting

$$v(\omega) = \begin{cases} 0 & \text{if } \omega \notin \mathcal{T} \\ -v_2(\varphi(\omega_1), \bar{f}_{\bar{\omega}}(\varphi(\omega_1)), \end{cases}$$

as $|v(\omega)| \le N$ we see φ and φ_ε differ by a bounded C^∞ co-boundary. □

LEMMA 14. *If* $T_1 \overset{M}{\rightsquigarrow} T_2$ *then* $T_1 \overset{M}{\to} T_2$.

Proof. Assume $|\det(M)| \ge 1$. In lemma 13, let $A_1 = \Omega_1$ and $A_2 = \Omega_2$. Let φ_0 be an initial barely linear orbit equivalence achieving $\overset{M}{\rightsquigarrow}$. For convenience later assume φ_0 is C^∞ with bounded 1'st partials. Apply lemma 13 successively with $\varphi = \varphi_i$, $\varepsilon = 1/2^i$ to obtain $\varphi_{i+1} = \varphi_\varepsilon$.

The maps φ_i do not necessarily converge as maps from $\bar{\Omega}_1$ to $\bar{\Omega}_2$, but as φ_{i+1}/Ω_1 agrees with φ_i/Ω_1 on that portion of Ω_1 mapped into Ω_2, $\bar{\varphi}_i = \varphi_i|_{\Omega_1}$ does converge to a 1–1 map $\psi\colon \Omega_1 \to \Omega_2$. We need to check that ψ achieves $T_1 \overset{M}{\to} T_2$.

Let $\mathcal{T}_i$ be the Rokhlin tower on which φ_i is constructed, with base B_i and dimensions $[0, N_i]^n$. We know that for a.e. $\omega \in \bar{\Omega}_1$,

$$\lim_{|v|\to\infty} \frac{|v_2(\varphi_0(\omega), \varphi_0(T^v(\omega))) - M(v)|}{|v|} = 0$$

uniformly in v, as φ_0 has bounded 1'st partials. If this is true of one point on an orbit, it is true of all points, not just almost all, on that orbit.

As the coboundary from φ_0 to any φ_i is bounded, for a.e. $\omega \in \bar{\Omega}_1$,

$$\lim_{|v|\to\infty} \frac{|v_2(\varphi_i(\omega)), \varphi_i(\bar{T}^v(\omega))) - M(v)|}{|v|} = 0,$$

again uniformly on full $\bar{T}^v$-orbits, but certainly not uniformly in i.

Let $A_i = \varphi_i(\Omega_i) \cap \Omega_2$, and clearly $\mu_1(A_i) \overset{i}{\to} 1$. For any $\varepsilon > 0$ let $A(\varepsilon)$ be the first A_i with $\mu(A_i) > 1 - \varepsilon$. Now if both $\omega_1, \omega_2 \in A(\varepsilon)$, then $\psi(\omega_j) = \varphi_i(\omega_j)$, $j = 1, 2$, and as $\lim_{|v|\to\infty} |v_2(\varphi_i(\omega), \varphi_i(T^v(\omega)) - M(v)|/|v| = 0$ uniformly, select $N(\varepsilon)$ so that if $|v| > N(\varepsilon)$,

$$|v_2(\varphi_i(\omega), \varphi_i(T^v(\omega))) - M(v)| < \varepsilon |v|.$$

Now if $v_1(\omega_1, \omega_2) > N(\varepsilon)$, then

$$|v_2(\psi(\omega_1), \psi(\omega_2)) - M(v_1(\omega_1, \omega_2))| < \varepsilon |v_1(\omega_1, \omega_2)|$$

and hence ψ achieves $T_1 \overset{M}{\rightsquigarrow} T_2$. □

The following result will complete the proof of proposition 11.

LEMMA 15. *If* $T_1 \overset{M}{\rightsquigarrow} T_2$ *then* $T_1 \overset{M}{\leadsto} T_2$.

Proof. Let φ be the orbit map achieving $\overset{M}{\rightsquigarrow}$. There is then a subset A so that for $\omega_1 \in A$,

$$\lim_{\substack{|v|\to\infty \\ T^v(\omega_1)\in A}} \frac{|v_2(\varphi(\omega_1), \varphi(T^v(\omega_1)) - M(v)|}{|v|} = 0$$

uniformly on A. Select a subset $A_1 \subseteq A$ so that if $\omega_1, \omega_2 \in A_1$, then $v_1(\omega_1, \omega_2)$ is sufficiently large to guarantee

$$|v_2(\varphi(\omega_1), \varphi(\omega_2)) - M(v_1(\omega_1, \omega_2))| < |M(v_1(\omega_1, \omega_2))|/100.$$

To construct the map $\bar{\varphi} : \bar{\Omega}_1 \leftrightarrow \bar{\Omega}_2$ achieving $\overset{M}{\leadsto}$ we use the technique of lemmas 3.12 and 3.13 of [**6**]. Let $\omega \in \bar{\Omega}_1$. For every $\bar{\omega} \in A_1$, on the $\bar{T}_1$ orbit of ω, let $C(\bar{\omega}, \omega)$ be a cube on the $\bar{T}_2$ orbit of $\varphi(\bar{\omega})$, of points ω' centred at $\bar{T}_2^{M(v_1(\bar{\omega},\omega))}(\varphi(\bar{\omega}))$, of side length $0.02|M(v_1(\bar{\omega}, \omega))|$, with faces oriented with the coordinate axes. Let

$$P(\omega) = \bigcap_{\substack{\bar{\omega}\in A_1 \\ \text{and on the} \\ \text{orbit of } \omega}} C(\bar{\omega}, \omega).$$

Any two of these cubes $C(\bar{\omega}_1, \omega)$, $C(\bar{\omega}_2, \omega)$ must intersect as

$$\begin{aligned} &|v_2(T_2^{M(v_1(\bar{\omega}_1,\omega))}(\varphi(\bar{\omega}_1)), T_2^{M(v_1(\bar{\omega}_2,\omega))}(\varphi(\bar{\omega}_2)))| \\ &\quad = |M(v_1(\bar{\omega}_1, \omega)) - M(v_1(\bar{\omega}_2, \omega)) - v_2(\varphi(\bar{\omega}_1), \varphi(\bar{\omega}_2))| \\ &\quad = |v_2(\varphi(\bar{\omega}_1), \varphi(\bar{\omega}_2)) - M(v_1(\bar{\omega}_1, \bar{\omega}_2))| \\ &\quad \le 0.01|M(v_1(\bar{\omega}_1, \bar{\omega}_2))| \\ &\quad \le 0.01(|M(v_1(\bar{\omega}_1, \omega))| + |M(v_1(\bar{\omega}_2, \omega))|). \end{aligned}$$

It follows that $P(\omega)$ is a non-empty parallelepiped with faces oriented with the coordinate axes (it may be degenerate, i.e. lower dimensional).

Let $\bar{\varphi}(\omega)$ be the midpoint of $P(\omega)$. Arguing as in lemma 3.13 of [**6**] we can conclude

$$|v_2(\bar{\varphi}(\omega_1), \bar{\varphi}(\omega_2)) - M(v_1(\omega_1, \omega_2)| \le 0.01|M(v_1(\omega_1, \omega_2))|$$

and hence $\bar{\varphi}$ is a homeomorphism on orbits.

The above also guarantees that $\bar{\varphi}$ achieves $\bar{T}_1 \overset{M}{\leadsto} \bar{T}_2$ for some $\bar{M}$, ([**6**, theorem 3.4]).

To see that $\bar{M} = M$, perhaps the easiest approach is to apply lemma 14 to get $T_1 \overset{\bar{M}}{\rightsquigarrow} T_2$. But the two orbit maps achieving $T_1 \overset{M}{\rightsquigarrow} T_2$ and $T_1 \overset{\bar{M}}{\rightsquigarrow} T_2$ by construction map a given T_1 orbit into the same T_2 orbit, and arguing precisely as in corollary 8, $M = M'$. □

This completes our proof that $T_1 \overset{M}{\rightsquigarrow} T_2$ and $T_1 \overset{\bar{M}}{\leadsto} T_2$ are identical.

There are two natural directions to extend the arguments given here. One can ask under what conditions can natural discrete cross-section actions be constructed for a continuous Lie group action, satisfying some 'barely linear' or 'barely automorphic' condition allowing for a parallel development. On the other hand, one can remain within the context of $\mathbb{Z}^n$ actions and, as with $\overset{M}{\underset{c}{\rightsquigarrow}}$, look for other relations between maps that arise from some restriction on orbit equivalence. Even for $\overset{M}{\underset{c}{\rightsquigarrow}}$, there remains some work to be done, as it easily implies a higher dimensional $\bar{f}$ metric ([**1**], [**5**]) and hence a notion of Loosely Bernoulli. The zero entropy 'Equivalence Theorem' will now follow from the work of Feldman and Nadler [**2**]. The positive entropy theory remains unexplored. From [**4**] we know that if $T_1 \overset{M}{\rightsquigarrow} T_2$, then $h(T_1) = |\det(M)| h(T_2)$, ($h(T)$ = entropy of T). Another aspect of $\overset{M}{\rightsquigarrow}$ to be explored is the higher dimensional analogue of Feldman's construction in [**1**]. More interesting questions arise in the higher dimensional case just from the greater structure of $L(n, \mathbb{R})$. For example, one can again define the Schwarz group, $\mathcal{G}(T) = \{M \mid T \overset{M}{\rightsquigarrow} T\}$ and ask what groups can arise. As one simple case, does there exist a T, $n > 1$, with $\mathcal{G}(T) = \{\text{id}\}$?

A. del Junco was supported in part by NSF Grant MCS80-03038. D. J. Rudolph was supported in part by NSF Grant MCS82-04024.

REFERENCES

[**1**] J. Feldman. Non-Bernoulli K-automorphisms and a problem of Kakutani. *Israel J. Math.* **24** (1976), 16–38.
[**2**] J. Feldman & D. Nadler. Reparametrization of N-flows of zero entropy. Preprint.
[**3**] A. Katok. The special representation theorem for multi-dimensional group actions. Dynamical Systems, I. Warsaw, *Asterisque* **49** (1977), 117–140.
[**4**] D. Nadler. Abramov's formula for reparametrization of N-flows. Preprint.
[**5**] D. Ornstein, D. Rudolph & B. Weiss. *Equivalence of Measure Preserving Transformations*, Mem. Amer. Math. Soc. **37**, No. 262 (1982).
[**6**] D. Rudolph. Inner and barely linear time changes of ergodic $\mathbb{R}^k$ actions. *Contemporary Mathematics, Conf. in Mod. Anal. and Prob.* **26** (1984), 351–372.