BANACH ALGEBRAS WITH ONE DIMENSIONAL RADICAL

LAWRENCE STEDMAN

A Banach algebra $A$ with radical $R$ is said to have property (S) if the natural mapping from the algebraic tensor product $A \otimes A$ onto $A^2$ is open, when $A \otimes A$ is given the projective norm. The purpose of this note is to provide a counterexample to Zinde's claim that when $A$ is commutative and $R$ is one dimensional the fulfillment of property (S) in $A$ implies its fulfillment in the quotient algebra $A/R$.

Let $A$ be a Banach algebra with radical $R$ and let $A^2$ denote the linear span of products of elements of $A$. $A$ is said to have property (S) if the natural map $\pi$ from the algebraic tensor product $A \otimes A$ onto $A^2$ is open, when $A \otimes A$ is given the projective norm.

Thus $A$ will have (S) if there is a constant $K$ such that

$$\|z\|_\pi = \inf \left\{ \sum \|x_i\| \cdot \|y_i\| : \sum x_i y_i = z \right\} \leq K\|z\|$$

whenever $z \in A^2$.

In [2] Zinde proved that if $\dim R = 1$ then property (S) will hold in $A$ if it holds in the quotient algebra $A/R$, and stated the converse as obvious. However Loy [7] showed that if $\dim R = 1$ and $A/R$ has (S) then $R \cap A^2 = 0$ implies $R \cap \overline{A^2} = 0$.

We provide an example of a commutative separable Banach algebra $A$

Received 21 September 1982.
with one dimensional radical $R$ such that $A$ has (S) and $\overline{R \cap A^2} = 0$ while $R \cap A^2 \neq 0$, thus showing that the converse to Zinde's result does not hold.

Let $A_0$ be the complex commutative algebra generated by the formal symbols $\{r, a_i, x_i, z_i : i \in \mathbb{N}\}$ subject to

$$r^2 = r x_i = r z_i = r a_i = 0 \quad \text{for all } i,$$

$$x_i y_i = a_i a_j = a_i x_j = 0 \quad \text{whenever } i \neq j,$$

$$x_i^2 = r + z_i \quad \text{for all } i,$$

$$x_i^2 - x_{i+1}^2 = a_i^2 \quad \text{for all } i.$$ 

Thus an element $y \in A_0$ may be uniquely expressed as

\begin{equation}
(1) \quad y = r + \sum \lambda_i a_i + \sum \beta_i x_i + \sum \gamma_i j a_i a_j + \sum \delta_i a_i x_i + \sum \nu_{ij} x_i x_j + \sum \pi_{ij} a_i z_j
\end{equation}

where $\lambda, \alpha_i, \nu_i, \gamma_{ij}, \delta_i, \mu_{ij}, \nu_{ij}, \pi_{ij} \in \mathbb{C}$ for all $i, j$ and the sums are finite.

Define a norm on $A_0$ by

$$\|y\| = |\lambda| + 2 \sum |a_i|^2 - i + 2 \sum \beta_i + \sum |\gamma_{ij}|2^{-2i-j} + \sum |\delta_i|2^{-i} + \sum |\nu_{ij}|2^{-i(2j+1)} + \sum |\pi_{ij}|2^{-i(2j+1)} + \sum |\nu_{ij}|2^{-i(2j+1)} + \sum |\pi_{ij}|2^{-i(2j+1)}$$

It is easily checked that this norm is submultiplicative. Let $A$ be the completion of $A_0$ with respect to $\|\cdot\|$, then $A$ is commutative and separable and each element of $A$ is uniquely expressible as in (1) with possibly infinite sums.

Clearly $R = \text{Rad} A = \mathbb{C} r$, $R \cap A^2 = 0$ and, since $\lim z_i = 0$, $R \cap \overline{A^2} = R$. 

Downloaded from https://www.cambridge.org/core. IP address: 54.191.40.80, on 09 Sep 2017 at 16:18:22, subject to the Cambridge Core terms of use, available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0004972700011539
To show that $A$ has (S) we first consider $z \in A^2 \cap A_0$, so

$$z = \sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{j=2}^{n} \beta_{ij} z_i^j + \sum \gamma_{ij} x_i x_i^j + \sum \delta_i a_i x_i + \sum \pi_{ij} a_i z_i^j + \sum \nu_{ij} a_i x_i z_i^j$$

where the sums are finite. Now

$$\sum_{i=1}^{n} a_i x_i^2 = \sum_{k=1}^{n-1} \left( \sum_{i=1}^{n} \alpha_i \right) \left( x_i^2 - x_i^{2+1} \right) + \left( \sum_{i=1}^{n} \alpha_i \right) x_n^2$$

$$= \sum_{k=1}^{n-1} \left( \sum_{i=1}^{n} \alpha_i \right) a_i^2 + \left( \sum_{i=1}^{n} \alpha_i \right) x_n^2 ,$$

so that

$$\|z\|_n \leq 4 \sum_{i=1}^{n-1} \left| \sum_{k=1}^{n} \alpha_i \right| 2^{-2i} + 4 \left| \sum_{i=1}^{n} \alpha_i \right| + \sum_{j=2}^{n} \left| \beta_{ij} \right| \left| z_i^{j-1} \right| \|x_i\| + \sum_{i=1}^{n} \left| \gamma_{ij} \right| \|x_i\| + \sum_{i=1}^{n} \left| \delta_i \right| \|a_i\| \|x_i\| + \sum_{i=1}^{n} \left| \pi_{ij} \right| \|a_i\| \|x_i\| + \sum_{i=1}^{n} \left| \nu_{ij} \right| \|a_i\| \|x_i\|$$

$$\leq 8 \left( \sum_{i=1}^{n} \left| a_i \right| 2^{-2i} + \left| \sum_{i=1}^{n} \alpha_i \right| + \sum_{j=2}^{n} \left| \beta_{ij} \right| 2^{-2i} \right)$$

$$\leq 8 \left( \sum_{i=1}^{n} \left| a_i \right| 2^{-2i} + \left| \sum_{i=1}^{n} \alpha_i \right| + \sum_{j=2}^{n} \left| \beta_{ij} \right| 2^{-2i} \right)$$

$$\leq 8 \|z\| .$$

If $y \in A$ is written as in (1) with infinite sums, we denote by $y_k$ the element of $A_0$ obtained by summing all indices from 1 to $k$ only.

Now consider an arbitrary $a = \sum_{i=1}^{n} t_i s_i \in A^2$. Then

$$t_i = (t_i)_k + \delta_{tik}, \quad s_i = (s_i)_k + \delta_{sik},$$

where the sums are finite.
where \( \delta_{tik}, \delta_{sik} \rightarrow 0 \) as \( k \rightarrow \infty \). So given any \( p \in \mathbb{N} \) we may choose \( k \) sufficiently large to ensure that
\[
\max_{i=1,\ldots,n} \{ \| \delta_{tik} \|, \| \delta_{sik} \| \} < \frac{1}{p}.
\]
Then \( a = a_k + \Delta_k \) where
\[
a_k = \sum_{i=1}^{n} (t_{ik}) \cdot (s_{ik}) \cdot k,
\]
\[
\Delta_k = \sum_{i=1}^{n} (\delta_{tik} \cdot \delta_{sik} + \delta_{tik} \cdot (s_{ik}) \cdot k + \delta_{sik} \cdot (t_{ik}) \cdot k).
\]
Then \( \| \Delta_k \|_{\pi} \leq n \{ p^{-2} + (M+N)p^{-1} \} \) where
\[
M = \max_{i=1,\ldots,n} \| s_{ik} \|, \quad N = \max_{i=1,\ldots,n} \| t_{ik} \|,
\]
and so \( \| \Delta_k \|_{\pi} \rightarrow 0 \) as \( k \rightarrow \infty \). Now
\[
\| a \|_{\pi} \leq \| a_k \|_{\pi} + \| \Delta_k \|_{\pi}
\]
\[
\leq 8 \| a_k \| + \| \Delta_k \|_{\pi},
\]
since \( a_k \in A^2 \cap A_0 \). Letting \( k \rightarrow \infty \) we obtain
\[
\| a \|_{\pi} \leq 8 \| a \|
\]
whenever \( a \in A^2 \), so that \( A \) has property (S).

References

Banach algebras with one dimensional radical


Department of Mathematics,
Faculty of Science,
Australian National University,
PO Box 4,
Canberra, ACT 2600,
Australia.