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A NOTE ON GROUP RINGS OF CERTAIN TORSION-FREE GROUPS

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ABSTRACT. As a step towards characterizing *ID*-groups (i.e., groups G such that, for every ring R without zero-divisors, the group ring RG has no zero-divisors), Rudin and Schneider defined Ω-groups, a possibly wider class than that of right-orderable groups, and proved that if every non-trivial finitely generated subgroup of a group G has a non-trivial Ω-group as an epimorphic image, then G is an *ID*-group. We prove that such groups are even Ω-groups and obtain the analogous result for right-orderable groups.

Rudin and Schneider [8] define a group to be an *ID-group* if, for every ring R without zero-divisors, the group ring RG has no zero-divisors. They find a large class of groups (called Ω-groups) which are *ID*-groups. A group G is said to be an *Ω-group* if for every ordered pair of nonempty finite subsets A, B of G, there is at least one pair $(a, b) \in A \times B$ such that $ab \neq a_1b_1$ for any other pair $(a_1, b_1) \in A \times B$. This definition generalizes that of orderable groups and LaGrange and Rhemtulla [7] have observed that even right-orderable groups are Ω-groups: a group G is defined to be a *right-orderable group* (briefly *RO*-group) if there exists a full order $\leq$ on the carrier of G such that, whenever $a \leq b$ then $ag \leq bg$ for all $g \in G$. If we add the requirement that $ga \leq gb$ for all $g \in G$, we obtain the definition of an orderable group (*O*-group). It is well known (see [3]) that nilpotent torsion-free groups are *O*-groups; also that if a group is locally an *O*-group then it is an *O*-group, and that Cartesian and free products of *O*-groups are again *O*-groups. It is easy to see that the latter remarks hold true for Ω-groups and for *RO*-groups. (For example a free product of Ω-groups is an extension of a free group (the Cartesian) by the direct product, and is therefore (by the remarks following) an Ω-group. This argument is valid also for *RO*-groups.) However, an extension of an *O*-group by an *O*-group need not be an *O*-group (see e.g. [2]) whereas the classes of Ω-groups and *RO*-groups are closed under forming extensions ([8], [2], or Corollary 1 below). If O, RO, Ω, ID, TF denote the classes of *O*-groups, *RO*-groups, Ω-groups, *ID*-groups and torsion-free groups respectively, then it is not too difficult to show (see [7], [8]) that

$$O \subset RO \subseteq \Omega \subseteq ID \subseteq TF.$$

Here we shall be concerned with the following definition, applied to the classes *RO* and Ω. Let **X** denote a class of groups closed under forming isomorphic images.

We define a group to be *locally indicable by groups in* **X** (or briefly *locally* **X**-*indicable*) if every finitely generated nontrivial subgroup can be mapped homomorphically onto a nontrivial group in **X**. This terminology is derived essentially from that of Higman [4] who proves that all locally **Z**-indicable groups are in ID, where here **Z** denotes the class of infinite cyclic groups. Rudin and Schneider [8, Theorem 6.3] use Higman's method to prove the conceivably stronger result that a locally Ω-indicable group is an ID-group. However, Higman's argument can be made to yield the following possibly stronger theorem.

THEOREM 1. *If a group is locally Ω-indicable then it is in Ω.*

COROLLARY 1. (Rudin and Schneider [8].) *If a group G has a normal subgroup N such that N and G/N are Ω-groups, then G is an Ω-group.*

Proof. Let H be a finitely generated nontrivial subgroup of G. If $H \leq N$ then H is in Ω. If $H \nleq N$ then HN/N is a nontrivial Ω-group. Thus G is locally Ω-indicable, and therefore in Ω by Theorem 1.

We shall prove by a similar method the following theorem.

THEOREM 2. *If a group is locally RO-indicable then it is an RO-group.*

The following corollary is immediate.

COROLLARY 2. *A locally* **Z**-*indicable group is right-orderable.*

Before proving these theorems we make two remarks. The first remark gives some indication of the size of the class of locally **X**-indicable groups as compared with **X**. A *subnormal system* $\mathscr{S}$ of subgroups of a group G (see Kurosh [6, p. 171]) is a set of subgroups which contains G and the identity subgroup, which is fully ordered by inclusion and closed under intersections and unions of subsets, and which has the further property that whenever $H, K \in \mathscr{S}$ are such that $K < H$ and no subgroup in $\mathscr{S}$ lies properly between K and H, then K is normal in H. The factor groups H/K are called factors of $\mathscr{S}$. The proof of the following is not difficult and we omit it.

THEOREM 3. *Let* **X** *be a class of groups closed under taking isomorphic images and subgroups. If G is a group possessing a subnormal system all of whose factors lie in* **X**, *then G is locally* **X**-*indicable.*

In particular if $\mathbf{X}=\Omega$, this together with Theorem 1 gives a generalization of Corollary 1.

We do not know if the converse is true. However it seems likely that at least Higman's class of locally **Z**-indicable groups coincides with the class of groups possessing a subnormal system with torsion-free abelian factors. Note that the latter class properly contains the class O [3, p. 51] and by Theorems 2, 3, is contained in RO. (We do not know if the latter containment is proper (see [2]).) It would be interesting to know if the class of SN-groups (Kurosh [6, p. 182]) coincides with the class of locally **A**-indicable groups, where **A** is the class of all abelian groups. An affirmative answer would generalize the result that if a group is locally an SN-group then it is an SN-group (Cf. [6, p. 183]).

Secondly, J. Poland has pointed out that the group G presented as

$$\langle x, y \mid x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2}\rangle$$

(which occurs in Karrass and Solitar [5]) is torsion-free metabelian and is not in RO. For suppose $\leq$ is a right order on G. Since any of the four mappings $x \to x^{\pm 1}$, $y \to y^{\pm 1}$ determines an automorphism of G, we may assume that $x<1$, $y<1$. Then $xy<1$, $yx<1$, whence $(xy)^2<1$, $(yx)^2<1$. However $(yx)^2=(xy)^{-2}$, a contradiction. It is unknown whether or not G is an Ω-group.(1)

Proof of Theorem 1. Suppose G is locally Ω-indicable but is not an Ω-group. Let A, B be two nonempty finite subsets of G such that for every pair $(a, b) \in A\times B$ there is at least one distinct pair $(a_1, b_1) \in A\times B$ such that $ab=a_1b_1$. Suppose further that $|A|+|B|$ is minimal with respect to this property. We may assume also that $1 \in A$ and $1 \in B$ since replacement of A, B by gA, Bg_1 respectively, where g, g_1 are arbitrary elements of G, does not affect the above properties. Write $G_1=\text{sgp}\{A, B\}$; clearly G_1 is nontrivial. Let K be a normal subgroup of G_1 such that G_1/K is a nontrivial Ω-group and let $\varphi: G_1 \to G_1/K$, be the natural homomorphism. Then $A\varphi$, $B\varphi$ are finite nonempty subsets of G_1/K and therefore contain elements Ka, Kb say, where $a \in A$, $b \in B$, such that $KaKb=Ka_1Kb_1$ (with $Ka_1 \in A\varphi$, $Kb_1 \in B\varphi$) if and only if $Ka=Ka_1$, $Kb=Kb_1$. Write $A_1=Ka \cap A$, $B_1=Kb \cap B$. Then to every pair $(a, b) \in A_1\times B_1$ there corresponds a distinct pair $(a_1, b_1) \in A_1\times B_1$ such that $ab=a_1b_1$. For, A, B have this property, and if either $a_1 \in A\setminus A_1$ or $b_1 \in B\setminus B_1$ then $a_1b_1 \notin KaKb$, whence *a fortiori* $a_1b_1\neq ab$. Further, we cannot have both $A_1=A$ and $B_1=B$; for if $Ka \supseteq A$ and $Kb \supseteq B$ then $Ka=Kb=K$ (since $1 \in A, B$), contradicting the fact that G_1/K is nontrivial. Thus $|A_1|+|B_1|<|A|+|B|$ and we have reached a contradiction.

Proof of Theorem 2. If $x_1, \ldots, x_n$ are elements of a group, we shall denote by $S\{x_1, \ldots, x_n\}$ the subsemigroup generated by these elements. By a result of Conrad

(1) However ZG has no zero-divisors, where Z is the ring of integers. This follows from a result of Jacques Lewin, as yet unpublished, that if G_1 is an amalgamated product of two soluble groups H_1 and H_2 where ZH_1 and ZH_2 have no zero-divisors, then the same is true of ZG_1. The group G is given in [5] as just such an amalgamated product. (Note added in proof.)

[2, Theorem 2.2] a group is right-orderable if and only if for every finite subset $\{x_1, \ldots, x_n\}$ which does not contain 1, there exist $e_i = \pm 1$ $(i=1, \ldots, n)$ such that $1 \notin S\{x_1^{e_1}, \ldots, x_n^{e_n}\}$.

Suppose G is a locally RO-indicable group which is not in RO. By Conrad's criterion there is a subset $T = \{g_1, \ldots, g_k\} \subset G$, of smallest order k, such that $1 \notin T$ and for every choice of $e_i = \pm 1$ $(i=1, \ldots, k)$, we have $1 \in S\{g_1^{e_1}, \ldots, g_k^{e_k}\}$. Let G_1 be the subgroup of G generated by T, and let K be a normal subgroup of G_1 such that G_1/K is a nontrivial RO-group. Thus we cannot have $g_i \in K$ for all $i=1, \ldots, k$. On the other hand if $Kg_i \neq K$ for all i, then for every choice of $e_i = \pm 1$ $(i=1, \ldots, k)$, since $1 \in S\{g_i^{e_i}\}$, we should have $K \in S\{Kg_i^{e_i}\}$, contradicting the fact that $G_1/K \in RO$. Thus we may suppose, by relabelling the elements of T if necessary, that the elements of T outside K are precisely $g_1, \ldots, g_r$, where $0 < r < k$. Since $G_1/K \in RO$, there exist $\delta_i = \pm 1$ $(i=1, \ldots, r)$ such that

$$K \notin S\{Kg_i^{\delta_i} \mid i = 1, \ldots, r\}. \tag{1}$$

For $r < i \leq k$, choose δ_i such that

$$1 \notin S\{g_i^{\delta_i} \mid i = r+1, \ldots, k\}. \tag{2}$$

This is possible by the minimality of T and since $0 < r$. However, by definition of T we have $1 \in S\{g_i^{\delta_i} \mid i=1, \ldots, k\}$; say

$$1 = g_{i(1)}^{n_1\delta_{i(1)}} \cdots g_{i(s)}^{n_s\delta_{l(i_s)}} \tag{3}$$

where the n_j $(j=1, \ldots, s)$ are positive integers and $1 \leq i(j) \leq k$, and where by (2) at least one of the $i(j) \leq r$. From (3) we infer that

$$K = (Kg_{i(1)}^{n_1\delta_{i(1)}}) \cdots (Kg_{i(s)}^{n_s\delta_{i(s)}}),$$

where at least one of the cosets $Kg_{i(j)}$ is distinct from K. This contradicts (1) and completes the proof.

We conclude with a few related remarks. LaGrange and Rhemtulla [7] prove essentially that an RO-group G has the following property: If A, B are any two finite nonempty subsets of G with $|A| + |B| > 2$, then there are two distinct pairs (a_1, b_1), $(a_2, b_2) \in A \times B$ such that $a_1b_1 \neq ab$ for any other pair $(a, b) \in A \times B$, and the same is true for a_2b_2. They show that the group ring of a group with the latter property, over a ring with no zero-divisors, has all its units of the form ug where u is a unit of the ring and g is an element of the group. This generalizes Theorem 13 of Higman [4]. Properties of this type have also been considered by Banaschewski [1].

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