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A POTENTIAL WELL THEORY FOR THE WAVE EQUATION WITH NONLINEAR SOURCE AND BOUNDARY DAMPING TERMS

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Abstract. The paper deals with local existence, blow-up and global existence for the solutions of a wave equation with an internal nonlinear source and a nonlinear boundary damping. The typical problem studied is

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-2}u & \text{in } [0,\infty)\times\Omega,\\ u = 0 & \text{on } [0,\infty)\times\Gamma_0,\\ \frac{\partial u}{\partial \nu} = -\alpha(x)|u_t|^{m-2}u_t & \text{on } [0,\infty)\times\Gamma_1,\\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x) & \text{on } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a regular and bounded domain, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\lambda_{n-1}(\Gamma_0) > 0$, $2 < p \leq 2(n-1)/(n-2)$ (when $n \geq 3$), $m > 1$, $\alpha \in L^\infty(\Gamma_1)$, $\alpha \geq 0$, and the initial data are in the energy space. The results proved extend the potential well theory, which is well known when the nonlinear damping acts in the interior of Ω, to this problem.

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1. Introduction. We study the problem

$$\begin{cases} u_{tt} - \Delta u = f(x,u) & \text{in } [0,\infty)\times\Omega,\\ u = 0 & \text{on } [0,\infty)\times\Gamma_0,\\ \frac{\partial u}{\partial \nu} = -Q(x,u_t) & \text{on } [0,\infty)\times\Gamma_1,\\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{on } \Omega, \end{cases} \tag{1}$$

where $u = u(t,x)$, $t \geq 0$, $x \in \Omega$, Δ denotes the Laplacian operator, with respect to the variable x, Ω is a regular and bounded domain of $\mathbb{R}^n$ ($n \geq 1$), $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, and $\lambda_{n-1}(\Gamma_0) > 0$ (λ_{n-1} denoting the $(n-1)$-dimensional Lebesgue measure on $\partial\Omega$). The initial data are in the energy space, that is $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$, with the compatibility condition $u_0 = 0$ on Γ_0. Moreover Q represents a nonlinear boundary damping, i.e. $Q(x,v)v \geq 0$, and f is an internal nonlinear source, i.e. $f(x,u)u \geq 0$.

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There is a wide literature on problem (1), when $f(x, u)u \leq 0$ (and $Q(x, v)v \geq 0$). In this case, under various assumptions on Q and f, global existence and decay estimates were proved for arbitrarily large initial data. See for example [**5**], [**6**], [**7**], [**8**], [**16**], [**17**], [**18**], [**20**] and [**38**].

Much less is known when f is a source term. The only paper in this setting, in the author's knowledge, is [**19**], where global existence and exponential decay for small initial data is proved for nonlinearly perturbed wave equations.

In the particular case $\Gamma_1 = \emptyset$ (the same arguments work also when $Q \equiv 0$), when $f(x, u) = |u|^{p-2}u$, $p > 2$, it is well known that the source term causes blow-up of solutions in finite time for sufficiently large initial data. See for example [**2**], [**12**], [**13**], [**14**], [**15**], [**21**], [**22**], [**29**], [**34**]. We also refer to the related papers [**23**] and [**24**], dealing with boundary source terms.

In [**29**] some sharper results, known as a "potential well theory", were proved when $2 < p < 2^*$, where as usual 2^* is the Sobolev critical exponent $2n/(n-2)$ when $n \geq 3$, $2^* = \infty$ when $n = 1, 2$. To illustrate this theory briefly let us introduce some notations: $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, the functional

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, \qquad u \in H_0^1(\Omega), \tag{2}$$

the energy function

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)), \tag{3}$$

associated to a solution u of (1), and the (positive) number

$$d = \inf_{u \in H_0^1(\Omega), u \neq 0} \sup_{\lambda < 0} J(\lambda u). \tag{4}$$

Moreover the value d is shown to be the Mountain Pass level associated to the underlying Dirichlet problem $-\Delta u = |u|^{p-2}u$ in Ω, $u = 0$ on $\partial\Omega$.

In [**29**] it is proved that, if $E(0) < d$, the solution is global when $\nabla J(u_0)u_0 > 0$, while[1] it blows-up in finite time when $\nabla J(u_0)u_0 < 0$ (note the case $\nabla J(u_0)u_0 = 0$ is impossible since $E(0) < d$). One can easily see (the simple proof can be founded in [**36**]) that

$$E(u(0)) < d, \quad \nabla J(u_0)u_0 < 0 \iff (\|\nabla u_0\|_2, E(0) \in A,$$

$$E(u(0)) < d, \quad \nabla J(u_0)u_0 < 0 \iff (\|\nabla u_0\|_2, E(0) \in B,$$

where A and B are defined (see Figure 1) by

$$A = \{(\lambda, E) \in [0, \infty) \times \mathbb{R} : g(\lambda) \leq E < d, \quad \lambda < \lambda_1\},$$

$$B = \{(\lambda, E)\} \in [0, \infty) \times \mathbb{R} : g(\lambda) \leq E < d, \quad \lambda > \lambda_1\},$$

where $g(\lambda) = \frac{1}{2}\lambda^2 - B_1^p\lambda^p/p, \lambda > 0, B_1$ is the optimal constant of the Sobolev embedding, λ_1 is the absolute maximum point of g, and finally $d = g(\lambda_1) > 0$.

[1]here ∇J denotes the gradient vector associated to the Fréchet differential of J on $H_0^1(\Omega)$, that is $\nabla J(u_0)u_0 = \|\nabla u_0\|_2^2 - \|u_0\|_p^p$.

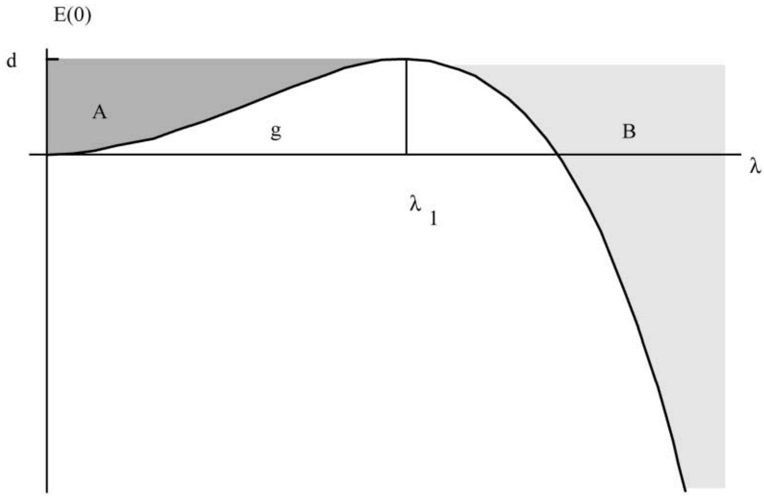

Figure 1. The sets A and B in the plane $(\lambda, E(0))$, where $\lambda = \|\nabla u_0\|_2$.

Blow-up for large initial data was proved in [**11**] and [**25**] when the dissipation arises from an internal nonlinear damping term, that is for solutions of the problem

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-2}u_t = |u|^{p-2}u & \text{in } [0,\infty)\times\Omega, \\ u = 0 & \text{on } [0,\infty)\times\partial\Omega, \\ u(0,x) = u_o(x), \quad u_t(0,x) = u_1(x) & \text{on } \Omega, \end{cases} \tag{5}$$

when $2 < p \leq 1 + 2^*/2$ and $m < p$ (in the case $p \leq \min\{m, 1 + 2^*/2\}$ global existence for arbitrary large initial data is proved in [**11**]). The extension to (5) of the potential well theory illustrated above was established in [**30**] when $m = 2$, and in [**35**] when $m < 1$.

These results known for the case $\Gamma_1 = \emptyset$ motivate us to study (1) when, roughly, $f(x,u) \simeq |u|^{p-2}u, p > 2$ or $f(x,u) \simeq |u|^{p-2}u - |u|^{q-2}u$, $1 < q < p, p > 2$, where the second nonlinear term plays the role of a nonlinear perturbation acting against blow-up. The case $f(x,u) = |u|^{p-2}u, p > 2$ and $Q(x,v) = \alpha(x)v,\ \alpha \in L^\infty(\Gamma_1)$, was studied in [**36**], where the global nonexistence part of the potential well theory was established. The classical concavity argument used there, however, is no longer applicable when the damping Q is nonlinear. Indeed, also local existence of the solutions of (1) must be proved when Q is nonlinear, since classical nonlinear semigroup theory cannot be applied directly in this situation (see [**28**]).

The assumptions on Q allow, for instance, to consider homogenous damping terms of the type $Q(x,v) = \alpha(x)|v|^{m-2}v, m > 1, \alpha \in L^\infty(\Gamma_1), \alpha \geq 0$. The motivation to consider $\alpha(x) \not\equiv 1$, and in particular $\inf_{\Gamma_1} a = 0$, comes from the case $f \equiv 0$ (or $f(x,u)u \leq 0$), studied in [**5**], [**6**], [**7**], [**8**], [**16**], [**17**], [**18**], [**20**] and [**38**]. Moreover, in these papers, a damping term of the form $Q(x, \mathrm{v}) = (\mathrm{x} - \mathrm{x}_0)\nu(\mathrm{x})\mathrm{g}(\mathrm{v})$, where x_0 is a fixed point of $\mathbb{R}^n$, ν is the outward normal vector field on $\partial\Omega$ and the partition $\partial\Omega = \Gamma_0 \cup \Gamma_1$ is given by

$$\Gamma_0 = \{x \in \partial\Omega : (x - x_0)\nu(x) \leq 0\}, \quad \Gamma_1 = \{x \in \partial\Omega : (x - x_0)\nu(x) > 0\}$$

was considered; and exponential (when g is linear near 0), or algebraic (when g is superlinear near 0) decay was also proved as $t \to \infty$. When Ω is a simply connected region $\inf_{\Gamma_1}(x - x_0)\nu(x) = 0$. This shows the interest, in studying blow-up phenomenam for damping terms of this type, since they produce boundary stabilization in absence of source.

Our study also includes the case $Q(x, v) = \alpha(x)(|v|^{m-2}v + v), \alpha \in L^\infty(\Gamma_1), m > 1$, which describes a more realistic dissipation rate, linear for small v and superlinear for large v (see for example [**9**]). Roughly, to include both cases, we shall consider $Q(x, v) \simeq \alpha(x)(|v|^{\mu-2}v + |v|^{\mathrm{m}-2}v), 1 < \mu \leq \mathrm{m}$.

To illustrate in a simple way our results we consider the case $f(x, u) = |u|^{p-2}u$ and $u = 2$, i.e.

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-2}u & \text{in } [0, \infty) \times \Omega, \\ u = 0 & \text{on } [0, \infty) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -\alpha(x)(|u_t|^{m-2}u_t + \beta u_t) & \text{on } [0, \infty) \times \Gamma_1, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{on } \Omega, \end{cases} \tag{6}$$

where $m > 1, \beta \geq 0, \alpha$ is a measurable nonnegative function on Γ_1. Again to a solution u of (6) we associate the energy function given by (3). We denote by $L^m(\Gamma_1, \alpha)$ the L^m space on Γ_1 associated to the measure μ_α defined by $\mu_\alpha(A) = \int_A \alpha(x)dx$ for any measurable subset A of Γ_1, and $L^m(\Gamma_1, 1)$ denotes the standard L^m space associated to λ_{n-1}, that is $L^m(\Gamma_1) = L^m(\Gamma_1, 1)$. The analogous convention will be adopted on $(0, T) \times \Gamma_1$, for $T > 0$. Moreover

$$H^1_{\Gamma_0}(\Omega) = \left\{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\right\}$$

(where $u|_{\Gamma_0}$ is intended in trace sense), equipped with the norm $||\nabla u||_2$, which is equivalent, by the Poincaré inequality (see [**37**]), to the standard one since $\lambda_{n-1}(\Gamma_0) > 0$.

Our first result concerns local existence for solutions of (6).

Theorem 1. *Suppose that* $2 < p \leq 1 + 2^*/2$, *and that* $u_0 \in H^1_{\Gamma_0}(\Omega), u_1 \in L^2(\Omega)$. *Then there is* $T > 0$ *and a unique weak solution* u *of* (6) *in* $(0, T) \times \Omega$, *such that*

$$u \in C([0, T]; H^1_{\Gamma_0}(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \tag{7}$$

$$u_t \in L^m((0, T) \times \Gamma_1, \alpha), \tag{8}$$

$$u_t \in L^2((0, T) \times \Gamma_1, \alpha), \quad \textit{if } \beta > 0, \tag{9}$$

and u *satisfies the energy identity*

$$E(t) - E(s) = -\int_s^t \int_{\Gamma_1} \alpha(x)(|u_t|^m + \beta|u_t|^2) \tag{10}$$

for $0 \leq s \leq t$.

Remark 1. The regularity given by (8) and (eventually) (9) is not a consequence of the trace theorem, but is a further regularity property of u insured by the presence of the damping term.

The global existence theorem for initial data u_0, u_1 such that the corresponding couple $(\|\nabla u_0\|_2, E(0))$ is in the region A.

THEOREM 2. *If the assumptions of Theorem 1 hold and* $(\|\nabla u_0\|_2, E(0)) \in A$ *then* u *can be extended to* $[0, \infty)$*, and* $(\|\nabla u(t)\|_2, E(t)) \in A$ *for all* $t \geq 0$.

REMARK 2. Theorem 1 characterizes in a variational way the neighborhood of stability of the null solution in which global existence holds, as shown in [**19**].

Our final result concerning (6) extends to the case $\Gamma_1 \neq \emptyset$ the blow-up theorem of [**29**].

THEOREM 3. *Assume that the hypotheses of Theorem 1 hold, that* $\alpha \in L^\infty(\Gamma_1)$ *and* $(\|\nabla u_0\|_2, E(0)) \in B$. *Suppose moreover that*

$$m < m_0(p) = \frac{2(n+1)p - 4(n-1)}{n(p-2)+4}. \tag{11}$$

Then there is $T_{max} < 0$ *such that* u *blows-up in the* L^p *norm of* Ω *as* $t \to T_{max}^-$.

REMARK 3. Assumption $\lambda_{n-1}(\Gamma_0) > 0$, together with the Poincaré inequality, yields that $\|\nabla u(t)\|_2 \to \infty$ as $t \to T_{max}^-$, and of course the Hölder inequality implies also the blow-up in the L^∞ norm of Ω.

Assumption (11) strongly reduces the applicability of Theorem 3, see Figure 2, which illustrate the set of the couples (p, m) satisfying the assumptions of Theorem 3. As $m_0(p) > 2$ for $p > 2$, the result is rather sharp in the sublinear case $1 < m \leq 2$, while (11) and $p \leq 1 + 2^*/2$ force that $m < 4$ when $n = 1$, $m < 3$ when $n = 2$ and $m < 2 + 2/(3n-4)$ when $n \geq 3$. This is due to the difficulty in comparing the effect of high order dissipation, which is related to the L^m norm on Γ_1, with the effect of the source, related to the L^p norm on Ω.

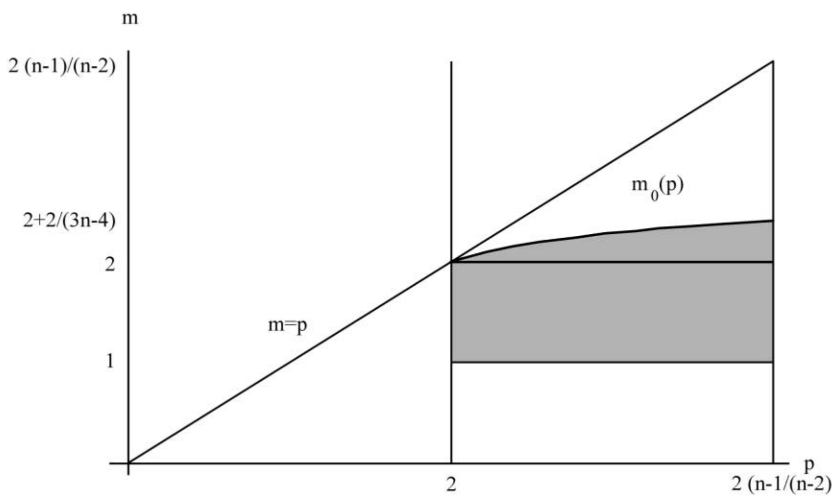

Figure 2. The shaded region is the set of the (p, m) couples for which the assumptions of Theorem 3 hold, when $n \geq 3$. The picture is made in the case $n = 3$.

In this paper we shall consider nonlinearities f and Q more general than those in (6), as we indicated above. In particular also the problem

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-2}u - |u|^{q-2}u & \text{in } [0,\infty) \times \Omega, \\ u = 0 & \text{on } [0,\infty) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -\alpha(x)\big(|u_t|^{\mu-2}u_t + |u_t|^{m-2}u_t\big) & \text{on } [0,\infty) \times \Gamma_1, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & \text{in } \Omega \end{cases}$$

where $1 < \mu \le m$, $1 < q < p$, can be treated. When $1 < q < 2$ the use of a different argument in the proof of local existence is required. The precise assumptions are given later, since they become more shrinking passing from local existence to blow-up through global existence results. The organization of the paper is simple: section 2 is concerned with local existence, while section 3 deals with global existence and blow-up results.

2. Local existence. We prove a local existence result for the solutions of (1) which includes Theorem 1 as a particular case. Our assumptions on Q are the following

(Q1) Q is a Carathéodory real function in $\Gamma_1 \times \mathbb{R}$, and there are a measurable nonnegative function α on Γ_1 and an exponent $m > 1$ such that, if $m \ge 2$,

$$(Q(x,v) - Q(x,w))(v-w) \ge \alpha(x)|v-w|^m$$

for all $x \in \Gamma_1$, $v, w \in \mathbb{R}$, while, if $1 < m < 2$,

$$(Q(x,v) - Q(x,w))(v-w) \ge \alpha(x)\big||v|^{m-2}v - |w|^{m-2}w\big|^{m'}$$

for all $x \in \Gamma_1$, $v, w \in \mathbb{R}$, where $1/m + 1/m' = 1$;

(Q2) there are $1 < \mu \le m$ and $c_1 > 0$ such that

$$|Q(x,v)| \le c_1\alpha(x)\big(|v|^{\mu-1} + |v|^{m-1}\big)$$

for all $x \in \Gamma_1$, $v \in \mathbb{R}$.

It is clear that

$$Q_0(x,v) = \alpha(x)\big(|v|^{\mu-2}v + |v|^{m-2}v\big), \qquad 1 < \mu \le m, \tag{12}$$

satisfies the assumptions we made on Q. Indeed,

$$(Q_0(x,v) - Q_0(x,w))(v-w) \ge \alpha(x)(|v|^{m-2}v - |v|^{m-2}v)(v-w).$$

Hence, when $m \ge 2$, (Q1) immediately follows from the elementary inequality

$$(|v|^{m-2}v - |w|^{m-2}w)(v-w) \ge \text{Const.}|v-w|^m, \qquad v, w \in \mathbb{R}, \tag{13}$$

where the constant depends only on m. When $1 < m < 2$ the condition (Q1) holds since $m' > 2$, again with an application of (13) to $|v|^{m-2}v$ and $|w|^{m-2}w$.

We also remark for a future use some consequences of (Q1)–(Q2). First of all it follows that

$$Q(x, v)v \geq \alpha(x)|v|^m \tag{14}$$

for all $x \in \Gamma_1$, $v \in \mathbb{R}$, and, moreover, that $Q(x, \cdot)$ is increasing for all $x \in \Gamma_1$, and $Q(\cdot, 0) \equiv 0$ by the continuity of Q in the second variable. Then, if we set

$$\Phi(x, u) = \int_0^u Q(x, s)\, ds, \tag{15}$$

we obtain

$$\Phi(x, u) \geq \frac{\alpha(x)}{m}|v|^m \qquad \text{for all } x \in \Gamma_1, v \in \mathbb{R}. \tag{16}$$

Our assumption concerning f is the following

(F1) $f(x, 0) = 0$ and there are $p > 2$ and $c_2 > 0$ such that

$$|f(x, u_1) - f(x, u_2)| \leq c_2|u_1 - u_2|(1 + |u_1|^{p-2} + |u_2|^{p-2})$$

for all $x \in \Omega$, $u_1, u_2 \in \mathbb{R}$.

It is easy to see that

$$f_0(x, u) = a|u|^{q-2}u + b|u|^{p-2}u, \qquad 2 \leq q < p, \qquad a, b \in \mathbb{R}, \tag{17}$$

satisfies (F1).

We shall also consider later the nonlinearity f_0 when $1 < q < 2$. In this case (F1) is not satisfied and the proof of local existence requires the use of a different argument.

THEOREM 4. *Suppose that* (Q1)–(Q2) *and* (F1) *hold, that* $2 < p \leq 1 + 2^*/2$, *and* $u_0 \in H^1_{\Gamma_0}(\Omega)$, $u_1 \in L^2(\Omega)$. *Then there is* $T > 0$ *and a unique weak solution of* (1) *in* $(0, T) \times \Omega$ *such that* (7)–(8) *hold, with the energy identity*

$$E(t) - E(s) = -\int_s^t \int_{\Gamma_1} Q(\cdot, u_t)u_t \tag{18}$$

for $0 \leq s \leq t$, *where*

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 - \int_\Omega F(\cdot, u(t)), \tag{19}$$

and

$$F(x, u) = \int_0^u f(x, s)ds \quad \textit{for} \;\; x \in \Omega, u \in \mathbb{R}. \tag{20}$$

REMARK 4. By a weak solution of (1) we mean a function u (with the required regularity) such that

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$$\int_\Omega u_t\varphi\Big|_0^t = \int_0^t\int_\Omega u_t\varphi_t - \nabla u\nabla\varphi + f(\cdot, u)\varphi - \int_0^t\int_{\Gamma_1} Q(\cdot, u_t)\varphi \tag{21}$$

for all $\varphi \in C([0, T]; H^1_{\Gamma_0}(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap L^m((0, T) \times \Gamma_1, \alpha)$ and $t \in (0, T]$.

An important tool in the proof is the study of the simpler problem

$$\begin{cases} u_{tt} - \Delta u = g(t, x) & \text{in } [0, T) \times \Omega, \\ u = 0 & \text{on } [0, T) \times \Gamma_0, \\ \frac{\partial u}{\partial \nu} = -Q(x, u_t) & \text{on } [0, T) \times \Gamma_1, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) & \text{on } \Omega, \end{cases} \tag{22}$$

where $T > 0$ and g is a fixed forcing term on $(0, T) \times \Omega$.

LEMMA 1. *Suppose that* (Q1)–(Q2) *hold, that* $g \in W^{1,1}(0, T; L^2(\Omega))$, u_0, $u_1 \in H^1_{\Gamma_0}(\Omega)$,

$$\Delta u_0 \in L^2(\Omega), \qquad \frac{\partial u_0}{\partial \nu} = -Q(\cdot, u_1) \quad \text{on } \Gamma_1, \tag{23}$$

and $u_1 \in L^m(\Gamma_1, \alpha)$. *Then* (22) *has a unique global solution in* $(0, T) \times \Omega$ *such that*

$$u \in W^{1,\infty}(0, T; H^1_{\Gamma_0}(\Omega)) \cap W^{2,\infty}(0, T; L^2(\Omega)), \tag{24}$$

$$u_t \in L^m((0, T) \times \Gamma_1, \alpha). \tag{25}$$

REMARK 5. In (23) the Laplacian of u_0 is taken in distributional sense, and $\frac{\partial u_0}{\partial \nu}\big|_{\Gamma_1}$ is not in general well defined by trace theorem, but again in distributional sense, that is $(23)_2$ means that

$$-\int_\Omega \Delta u_0\varphi = \int_\Omega \nabla u_0\nabla\varphi + \int_{\Gamma_1} Q(x, u_1)\varphi$$

for all $\varphi \in H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1, \alpha)$ (with a slight abuse of notation). See also [**38**].

Moreover, by a solution of (22) we mean a function u satisfying (24)–(25) and such that

$$\int_\Omega u_{tt}\varphi + \int_\Omega \nabla u\nabla\varphi + \int_{\Gamma_1} Q(\cdot, u_t)\varphi = \int_\Omega g\varphi \tag{26}$$

for all $\varphi \in H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1, \alpha)$.

Proof. We apply nonlinear semigroup theory. Let $H = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, endowed with the scalar product

$$\left\langle \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\rangle_H = \int_\Omega (\nabla u_1\nabla u_2 + v_1 v_2),$$

and $A : \mathcal{D}(A) \subset H \to H$ the operator given by

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$$A\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -v \\ -\Delta u \end{bmatrix},$$

with

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in [H^1_{\Gamma_0}(\Omega)]^2 : \Delta u \in L^2(\Omega), \quad v \in L^m(\Gamma_1, \alpha), \quad \frac{\partial u}{\partial \nu} = -Q(\cdot, v) \quad \text{on } \Gamma_1 \right\}.$$

Then (22) can be written as

$$\begin{bmatrix} u \\ u_t \end{bmatrix}_t = A\begin{bmatrix} u \\ u_t \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}, \qquad \begin{bmatrix} u \\ u_t \end{bmatrix} \in \mathcal{D}(A),$$

and the existence of u satisfying (22) and (24) can be proved applying [**3**, Theorem 2.2, p. 131], provided that A is a maximal monotone operator in H. The monotonicity of A immediately follows by (Q1) as

$$\left\langle A\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - A\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\rangle_H = \int_{\Gamma_1} (Q(\cdot, v_1) - Q(\cdot, v_2))(v_1 - v_2).$$

To prove that A is maximal we shall equivalently prove, by the nonlinear version of Minty's Theorem (see [**3**, pp. 73–74]), that the equation

$$A\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = h$$

has a solution $\begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{D}(A)$ for every $h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in H$, i.e. that for every $h_1 \in H^1_{\Gamma_0}(\Omega)$ and $h_2 \in L^2(\Omega)$ there are $u, v \in H^1_{\Gamma_0}(\Omega)$ such that $\Delta u \in L^2(\Omega)$ and $v \in L^m(\Gamma_1, \alpha)$, which solve

$$\begin{cases} -v + u = h_1 & \text{in } \Omega, \\ -\Delta u + v = h_2 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = -Q(\cdot, v) & \text{on } \Gamma_1. \end{cases} \tag{27}$$

This is equivalent to find $v \in X = H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1, \alpha)$ such that

$$\int_\Omega (\nabla v \nabla \varphi + v\varphi) + \int_{\Gamma_1} Q(\cdot, v)\varphi = \int_\Omega h_2\varphi - \nabla h_1 \nabla \varphi \tag{28}$$

for all $\varphi \in X$. Indeed, if $\begin{bmatrix} u \\ v \end{bmatrix}$ solves (27), then (28) holds, and, vice–versa, if (28) holds then, taking $\varphi \in C^\infty_c(\Omega)$, one first obtains that $-\Delta(v + h_1) + \ v = h_2$, so $u = v + h_1$ has Laplacian in $L^2(\Omega)$ and then (27) follows in a standard way.

The functional

$$I(v) = \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{2}\|v\|_2^2 + \int_{\Gamma_1} \Phi(\cdot, v) - \int_\Omega (h_2 v - \nabla h_1 \nabla v)$$

(we recall that Φ was defined on (15)) is of class $C^1(X, \mathbb{R})$ by (Q2) and coercive by (16). By the direct method of the Calculus of Variations (see for example [**33**]), there is a critical point of I, namely a solution of (28).

We proved the existence of u satisfying (22) and (24), so to complete the proof we have only to verify (25), which is not a direct consequence of (24) or of the property that $u_t(t) \in L^m(\Gamma_1, \alpha)$ for all $t \in [0, T]$. By (24) and the trace theorem one obtains in particular that $u_t \in L^2(0, T; L^2(\Gamma_1))$; hence , by the Riesz Theorem, u_t has a representative in $L^2((0, T) \times \Gamma_1)$. Since Q is a Carathéodory function also $Q(\cdot, u_t)$ is measurable on $(0, T) \times \Gamma_1$. By (24) we can derive from (26) the energy identity

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t(t)\|_2^2\right) + \int_{\Gamma_1} Q(\cdot, u_t(t))u_t(t) = \int_\Omega g(t)u_t(t). \tag{29}$$

Integrating (29) and using (14) and (24) we get $v \in L^m((0, T) \times \Gamma_1)$, concluding the proof. □

We are now ready to give the

Proof of Theorem 4. We adapt the ideas of [**11**, Section 2], using Lemma 1 instead of [**26**, Theorem 3.1, Chapter 1]. The only changes arise in proving the analogous of [**11**, Proposition 2.1], which can be formulated for (1) as follows. Given any

$$u \in X_T = C([0, T]; H^1_{\Gamma_0}(\Omega)) \cap C^1([0, T]; L^2(\Omega))$$

there is a unique weak solution $v \in X_T$ (in the sense of Remark 4) of

$$\begin{cases} v_{tt} - \Delta v = f(x, u) & \text{in } [0, T \times \Omega, \\ v = 0 & \text{on } [0, T) \times \Gamma_0, \\ \frac{\partial v}{\partial \nu} = -Q(x, v_t) & \text{on } [0, T) \times \Gamma_1, \\ v(0, x) = u_0(x), \qquad v_t(0, x) = u_1(x) & \text{on } \Omega, \end{cases} \tag{30}$$

such that $v_t \in L^m((0, T) \times \Gamma_1, \alpha)$ and the energy identity

$$\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2\Big|_s^t + \int_s^t \int_{\Gamma_1} Q(\cdot, v_t)v_t = \int_s^t \int_\Omega f(\cdot, u)v_t$$

holds for $0 \le s \le t$.

We start approximating the data u_0 and u_1, respectively in $H^1_{\Gamma_0}(\Omega)$ and $L^2(\Omega)$, with more regular (enough to apply Lemma 1) data u_0^k, u_1^k, $k \in \mathbb{N}$, verifying the compatibility condition $(23)_2$.

We simply approximate u_1 in $L^2(\Omega)$ with $u_1^k \in C_c^\infty(\Omega)$, so that, to satisfy also (23) we must find $u_0^k \in H^1_{\Gamma_0}(\Omega)$ such that $\Delta u_0^k \in L^2(\Omega)$, $\frac{\partial u_0^k}{\partial \nu} = 0$ on Γ_1 and $u_0^k \to u_0$ in $H^1_{\Gamma_0}(\Omega)$. This is equivalent to prove that the linear subspace

$$X = \left\{ v \in H^1_{\Gamma_0}(\Omega) : \Delta v \in L^2(\Omega), \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \right\}$$

is dense in $H^1_{\Gamma_0}(\Omega)$, i.e., thanks to the Hahn–Banach theorem, that for any given $u \in H^1_{\Gamma_0}(\Omega)$ such that

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$$\langle u, v\rangle_{H^1_{\Gamma_0}(\Omega)} = 0 \qquad \text{for all } \ v \in X, \tag{31}$$

implies that $u = 0$. This fact can be proved with the following simple argument: given any such u, take v such that

$$\langle v, \phi\rangle_{H^1_{\Gamma_0}(\Omega)} = \langle u, \phi\rangle_{L^2(\Omega)} \qquad \text{for all } \ \phi \in \mathrm{H}^1_{\Gamma_0}(\Omega), \tag{32}$$

which exists by Lax–Milgram theorem (see [**4**]). Then $\Delta v = u \in L^2(\Omega)$ and $\frac{\partial v}{\partial \nu} = 0$ (in the sense of Remark 5), so $v \in X$. Putting $\phi = u$ in (32) and using (31) we obtain that $u = 0$.

Next we approximate u in $C([0, T]; H^1_{\Gamma_0}(\Omega)) \cap C^1([0, T]; L^2(\Omega))$, endowed with the standard norm $\|u\| = \max_{t\in[0,T]} \|u_t(t)\|_2 + \|u(t)\|_{H^1(\Omega)}$, with $u^k \in C^\infty([0, T] \times \overline{\Omega})$, by standard convolution arguments (see [**4**]). We note that $f(x, \cdot) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ and

$$\left|\frac{\partial f}{\partial u}(x, u)\right| \leq 2c_2(1 + |u|^{p-2}). \tag{33}$$

by (F1). Consequently $f(\cdot, u^k) \in W^{1,1}(0, T; L^2(\Omega))$ by (F1)and (33). Then, applying Lemma 1, for any $k \in \mathbb{N}$ there is

$$v^k \in W^{1,\infty}(0, T; H^1_{\Gamma_0}(\Omega)) \cap W^{2,\infty}(0, T; L^2(\Omega)), \qquad v^k_t \in L^m((0, T) \times \Gamma_1, \alpha),$$

solution of

$$\begin{cases} v^k_{tt} - \Delta v^k = f(x, u^k) & \text{in } [0, T) \times \Omega, \\ v^k = 0 & \text{on } [0, T) \times \Gamma_0, \\ \frac{\partial v^k}{\partial \nu} = -Q(x, v^k_t) & \text{on } [0, T) \times \Gamma_1, \\ v^k(0, x) = u^k_0(x), \qquad v^k_t(0, x) = u^k_1(x) & \text{on } \Omega. \end{cases} \tag{34}$$

Now we verify that v^k is a Cauchy sequence in X_T, reproducing verbatim the proof of [**11**, Proposition 2.1] and using (F1) and (Q1).

To verify that (up to a subsequence) v^k_t is a Cauchy sequence in $L^m(\Gamma_1, \alpha)$, we note that, using the same arguments of [**11**], one can deduce that

$$\int_0^T \int_{\Gamma_1} \big(Q(x, v^j_t) - Q(x, v^k_t)\big)(v^j_t - v^k_t) \to 0 \qquad \text{as } j, k \to \infty.$$

When $m \geq 2$, (Q1) immediately yields our claim. When $1 < m < 2$ from (Q1) it follows that

$$\left\| |v^j_t|^{m-2} v^j_t - |v^k_t|^{m-2} v^k_t \right\|_{L^{m'}(\Gamma_1,\alpha)} \to 0 \qquad \text{as } j, k \to \infty,$$

so, using [**4**, Théorème IV.9], there is $\chi \in L^{m'}(\Gamma_1, \alpha)$ such that (up to a subsequence) $|v^k_t|^{m-1} \leq \chi$ on Γ_1 and v^k_t is μ_α–a.e. convergent on Γ_1. Then, by the Lebesgue dominated convergence theorem, v^k_t is convergent in $L^m(\Gamma_1, \alpha)$.

Passing to the limit as $k \to 0$ in (34), and using (F1) and (Q2), we find a unique solution of (30) exactly as in [**11**], concluding the proof. □

As mentioned above, assumption (F1) is not satisfied by f_0 given in (17) when $1 < q < 2$ (unless $a = 0$). The rest of this section is devoted to this case. Indeed we shall consider nonlinearities f satisfying the more general assumption

(F2) $f(x, 0) = 0$ and there are $1 < q < 2 < p$ and $c_2 > 0$ such that

$$|f(x, u_1) - f(x, u_2)| \leq c_2\big[|u_1 - u_2|(1 + |u_1|^{p-2} + |u_2|^{p-2}) + |u_1 - u_2|^{q-1}\big]$$

for all $x \in \Omega$, $u_1, u_2 \in \mathbb{R}$.

This case requires a different technique in order to prove local existence of the solution, that is the application of the Schauder fixed point theorem instead that the contraction principle as in [**11**]. For this reason we cannot consider the case $p = 1 + 2^*/2$ and we renounce to any claim of uniqueness. This is the content of

Theorem 5. *Suppose that* (Q1)–(Q2) *and* (F2) *hold, and* $2 < p < 1 + 2^*/2$. *Then the statement of Theorem 4 (except for the uniqueness of the solution founded) holds.*

Proof. We indicate the modifications of the proof of Theorem 4 which are needed. Certainly we cannot directly use Lemma 1, since $f(\cdot, u^k)$ does not belong in general to $W^{1,1}(0, T; L^2(\Omega))$, in order to prove that (34) has a solution for any $k \in \mathbb{N}$. We shall use a further simple approximation process. Indeed by (F2) $f(\cdot, u^k) \in L^2((0, T) \times \Omega)$, and then we can choose $\chi^{k,\varepsilon} \in C_c^\infty((0, T) \times \Omega)$, $\varepsilon > 0$, such that $\chi^{k,\varepsilon} \to f(\cdot, u^k)$ in $L^2((0, T) \times \Omega)$ as $\varepsilon \to 0^+$. By Lemma 1 there is $v^{k,\varepsilon} \in W^{1,\infty}(0, T; H^1_{\Gamma_0}(\Omega)) \cap W^{2,\infty}(0, T; L^2(\Omega))$ solution of (22) with $g = \chi^{k,\varepsilon}$. Then $w = v^{k,\varepsilon} - v^{k,\delta}$, $\varepsilon, \delta > 0$, is a solution of

$$\begin{cases} w_{tt} - \Delta w = \chi^{k,\varepsilon} - \chi^{k,\delta} & \text{in } [0, T) \times \Omega, \\ w = 0 & \text{on } [0, T \times \Gamma_0, \\ \frac{\partial w}{\partial \nu} = -Q(x, v_t^{k,\varepsilon}) + Q(x, v_t^{k,\delta}) & \text{on } [0, T) \times \Gamma_1, \\ w(0, x) = w_t(0, x) = 0 & \text{on } \Omega, \end{cases}$$

which verifies the energy identity

$$\begin{aligned} &\frac{1}{2}\|w_t(t)\|_2^2 + \frac{1}{2}\|\nabla w(t)\|_2^2 + \int_0^t \int_{\Gamma_1} (Q(x, v_t^{k,\varepsilon}) - Q(x, v_t^{k,\delta}))(v_t^{k,\varepsilon} - v_t^{k,\delta}) \\ &= \int_0^t \int_\Omega (\chi^{k,\varepsilon} - \chi^{k,\delta}) w_t. \end{aligned} \tag{35}$$

Then, by (Q1), a standard use of the Gronwall lemma shows that $(v^{k,\varepsilon})_\varepsilon$ is a Cauchy sequence in $C([0, T], H^1_{\Gamma_0}(\Omega)) \cap C^1([0, T]; L^2(\Omega))$. Hence, using the same arguments of the proof of Theorem 4, we prove that $(v_t^{k,\varepsilon})_\varepsilon$ is a Cauchy sequence in $L^m((0, T \times \Gamma_1, \alpha)$ and the limit v^k (as $\varepsilon \to 0$) solves (34). Moreover, starting from the energy identities for $v^{k,\varepsilon}$ and for $v^{j,\varepsilon} - v^{k,\varepsilon}$ one obtains that v^k and $v^j - v^k$ satisfy the energy identity.

To show here that v^k is a Cauchy sequence in X_T we have to modify the previous arguments first given in [**11**], and use a more general a priori estimate. Indeed, we shall establish an estimate which is stronger than what is immediately needed here. By (F2)

$$I_0 := \left| \int_\Omega (f(\cdot, u_1) - f(\cdot, u_2))(v_1 - v_2) \right|$$
$$\leq c_2 \int_\Omega \left[|u_1 - u_2|(1 + |u_1|^{p-2} + |u_2|^{p-2}) + |u_1 - u_2|^{q-1} \right] |v_1 - v_2|$$

for any $u_1, u_2 \in H^1_{\Gamma_0}(\Omega)$, $v_1, v_2 \in L^2(\Omega)$. Fix a number $r \in (2p-2, 2^*)$, so that $p < 1 + r/2 < 1 + 2^*/2$. As $2(q-1) < 2 < r$, using Hölder inequality

$$I_0 \leq C \Big[\|u_1 - u_2\|_r \Big(1 + \|u_1\|^{p-2}_{s(p-2)} + \|u_2\|^{p-2}_{s(p-2)} \Big) + \|u_1 - u_2\|^{q-1}_r \Big] \|v_1 - v_2\|_2$$

where

$$\frac{1}{r} + \frac{1}{2} + \frac{1}{s} = 1$$

(here and in the sequel of the proof $C = C(p, q, \Omega, T, u_0, u_1)$ will denote positive constants, possibly different). Then, since $s(p-2) \leq r$, using Sobolev embedding, we obtain

$$I_0 \leq C \Big[\|u_1 - u_2\|_r \Big(1 + \|\nabla u_1\|^{p-2}_2 + \|\nabla u_2\|^{p-2}_2 \Big) + \|u_1 - u_2\|^{q-1}_r \Big] \|v_1 - v_2\|_2. \tag{36}$$

Actually, to prove that v^k is a Cauchy sequence in X_T it is enough to apply (36) in the weaker form (since $r < 2^*$)

$$I_0 \leq C \Big[\|\nabla u_1 - \nabla u_2\|_2 \Big(1 + \|\nabla u_1\|^{p-2}_2 + \|\nabla u_2\|^{p-2}_2 \Big) + \|\nabla u_1 - \nabla u_2\|^{q-1}_2 \Big] \|v_1 - v_2\|_2 \tag{37}$$

to $u_1 = u^j$, $u_2 = u^k$, $v_1 = v^j_t$, $v_2 = v^k_t$ and argue as in [**11**]. Then we can conclude the proof of the existence of a solution of (30) as in Theorem 4.

The third and main modification in the proof of Theorem 4 is that, once we proved the analogous of [**11**, Proposition 2.1], we cannot follow verbatim [**11**, Proof of Theorem 2.1]. Indeed, this was exactly the application of the contraction principle which, as mentioned above, is no longer appropriate here.

We argue as follows. Define

$$Y_T = \{u \in X_T : u(0) = u_0, \ u_t(0) = u_1\},$$

and $\Psi : Y_T \to Y_T$ by $\Psi(u) = v$ where v solves (30). We can still easily prove that, for R sufficiently large and T sufficiently small, Ψ maps the ball B_R of Y_T in itself. Indeed (37) yields that, for $u \in B_R$ and $R \geq 1$ (since $q < 2 < p$)

$$\left| \int_\Omega f(\cdot, u) v_t \right| \leq C \Big(\|\nabla u\|_2 + \|\nabla u\|^{p-1}_2 + \|\nabla u\|^{q-1}_2 \Big) \|v_t\|_2 \leq 3CR^{p-1} \|v_t\|_2,$$

so that we can argue as in [**11**] using the energy identity.

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Next we apply to Ψ the Schauder fixed point theorem (see [**10**, Corollary 3.6.2]). We argue as follows: given any sequence u^k in B_R, denoting $w = u^k - u^j$ for $k, j \in \mathbb{N}$, w solves the problem

$$\begin{cases} w_{tt} - \Delta w = f(x, u^j) - f(x, u^k) & \text{in } [0, T) \times \Omega, \\ w = 0 & \text{on } [0, T \times \Gamma_0, \\ \frac{\partial w}{\partial \nu} = -Q(x, v_t^j) + Q(x, v_t^k) & \text{on } [0, T) \times \Gamma_1, \\ w(0, x) = u_0^j(x) - u_0^k(x), \quad w_t(0, x) = u_1^j(x) - u_1^k(x) & \text{on } \Omega. \end{cases}$$

Hence, by the energy identity and (Q1)

$$I_1 := \frac{1}{2}\|w_t(t)\|_2^2 + \frac{1}{2}\|\nabla w(t)\|_2^2 \le \frac{1}{2}\|w_t(0)\|_2^2 + \frac{1}{2}\|\nabla w(0)\|_2^2 + \int_0^t \left| \int_\Omega (f(\cdot, u^j) - f(\cdot, u^k)(v_t^j - v_t^k) \right|.$$

By (36), which essentially depends on the assumption $p < 1 + 2^*/2$, again with $R \ge 1$

$$\begin{aligned} I_1 &\le \frac{1}{2}\|u_1^j - u_1^k\|_2^2 + \frac{1}{2}\|\nabla u_0^j - \nabla u_0^k\|_2^2 \\ &\quad + C\big[3R^{p-2}\|u^j(t) - u^k(t)\|_r + \|u^j(t) - u^k(t)\|_r^{q-1}\big]\|w_t(t)\|_2 \\ &\le \frac{1}{2}\|u_1^j - u_1^k\|_2^2 + \frac{1}{2}\|\nabla u_0^j - \nabla u_0^k\|_2^2 \\ &\quad + C\Big(3R^{p-2}\|u^j - u^k\|_{C([0,T];L^r(\Omega))} + \|u^j - u^k\|_{C([0,T];L^r(\Omega))}^{q-1}\Big)\|w_t(t)\|_2. \end{aligned} \tag{38}$$

Now we claim that, up to a subsequence,

$$\|u^j - u^k\|_{C([0,T];L^r(\Omega))} \to 0 \qquad \text{as j, k} \to \infty. \tag{39}$$

Indeed $\{u^k(t)\}_k$ is relatively compact in $L^r(\Omega)$ for all $t \in [0, T]$ by the Rellich–Kondrachov theorem. Moreover, for all $s, t \in [0, T]$, using the interpolation inequality, the Sobolev embedding and the boundness of u^k in Y_T

$$\|u^k(s) - u^k(t)\|_r \le \|u^k(s) - u^k(t)\|_2^\theta \|u^k(s) - u^k(t)\|_{2^*}^{1-\theta} \le \text{Const.}\|u^k(s) - u^k(t)\|_2^\theta$$

where (consider for simplicity the case $n \ge 3$)

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{2^*}$$

so $(u^k)_k$ is equicontinuous in $C([0, T]; L^r(\Omega))$. The claim then follows by Ascoli's theorem (see [**31**]). By using (38) and (39) and a standard argument we prove that v^k converges in B_R, namely the compactness of Ψ. At this point we complete the proof as in [**11**]. □

3. Global existence and blow-up. This section is devoted to proving our main global existence and blow-up theorems, which include Theorems 2 and 3 as particular cases. For this purpose we use a further assumption on f; that is

(F3) there is $c_3 > 0$ such that

$$F(x, u) \leq \frac{c_3}{p}|u|^p$$

for all $x \in \Omega$ and $u \in \mathbb{R}$, where F is the primitive of f defined in (20).

It is clear that f_0 given in (17) satisfies (F1)–(F3) when $1 < q < p$, $p > 2$, $a \leq 0$ and $b \in \mathbb{R}$.

We set

$$K_0 = \sup_{u \in H^1_{\Gamma_0}(\Omega),\ u \neq 0} \frac{\int_\Omega F(\cdot, u)}{\|\nabla u\|_2^p}. \tag{40}$$

We can suppose without loss of generality that $K_0 > 0$. If this is not the case it is easy to see that $F(x, u) \leq 0$ a.e. in $(0, T) \times \mathbb{R}$, and in this case global existence for arbitrary initial data is an immediate consequence of the local existence, the energy identity and the standard continuation principle. By (F3) one can estimate $K_0 \leq (c_3 B_1/p)^{1/p}$, where B_1 is the optimal constant of the Sobolev embedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^p(\Omega)$.

We denote

$$\lambda_1 = (p/K_0)^{1/(p-2)}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\lambda_1^2.$$

REMARK 6. When $f(x, u) = |u|^{p-2}u$ (see [**36**]) the number E_1 above is equal to d defined in (4) and also equal to the Mountain Pass level associated to the elliptic problem

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \end{cases}$$

that is E_1 is equal to the number $\inf_{\gamma \in \Lambda} \sup_{t \in [0,1]} J(\gamma(t))$ (see (2) for the definition of J), where

$$\Lambda = \{\gamma \in C([0, 1]; H^1_{\Gamma_0}(\Omega) : \gamma(0) = 0, \quad J(\gamma(1)) < 0\}.$$

We begin with a lemma, which illustrates the two situations in which we have information about the time behavior of the solutions of (1).

LEMMA 2. *Suppose that the assumptions of Theorem 4 (or of Theorem 5) hold, together with* (F3), *and let u be a solution of* (1). *Assume moreover that* $E(0) < E_1$.

(i) *If* $\|\nabla u_0\|_2 < \lambda_1$ *then* $\|\nabla u(t)\|_2 < \lambda_1$ *for all* $t \in [0, T]$.
(ii) *If* $\|\nabla u_0\|_2 > \lambda_1$ *then there is* $\lambda_2 > \lambda_1$ *such that* $\|\nabla u(t)\|_2 \geq \lambda_2$ *and* $\|u(t)\|_p \geq (pK_0/c_3)^{1/p}\lambda_2$ *for all* $t \in [0, T]$.

Proof. We shall omit for simplicity the explicit dependence on time. We first note that, by (40),

$$E(t) \geq \frac{1}{2}\|\nabla u\|_2^2 - K_0\|\nabla u\|_2^p = g(\|\nabla u\|), \tag{41}$$

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where $g(\lambda) = \lambda^2/2 - K_0\lambda^p$, $\lambda > 0$. Of course, g is increasing for $0 < \lambda < \lambda_1$, decreasing for $\lambda > \lambda_1$, $g(\lambda_1) = E_1$ and $g(\lambda) \to -\infty$ as $\lambda \to \infty$. Then, since $E_0 := E(0) < E_1$, there are $\lambda_2' < \lambda_1 < \lambda_2$ such that $g(\lambda_2) = g(\lambda_2') = E_0$. Denote $\lambda_0 = \|\nabla u_0\|_2$. Also, by (18),

$$E(t) \leq E_0 \qquad \text{for all } \ t \in [0, T]. \tag{42}$$

Consider now the case (i), that is $\lambda_0 < \lambda_1$. Then, as by (41) we have $g(\lambda_0) \leq E_0$, it follows that $\lambda_0 \leq \lambda_2'$. We claim that $\|\nabla u(t)\|_2 \leq \lambda_2'$ for all $t \in [0, T]$, so that (i) follows. Suppose, by contradiction, that $\|\nabla u(t_0)\|_2 > \lambda_2'$ for some $t_0 > 0$. By the continuity of $\|\nabla u(\cdot)\|_2$ we can suppose that $\|\nabla u(t_0)\|_2 < \lambda_1$. Then, by (41), $E(t_0) \geq g\left(\|\nabla u(t_0)\|_2\right) > g(\lambda_2') = E_0$, in contradiction with (42).

To prove (ii) we deduce as before that $\lambda_0 > \lambda_1$ implies $\|\nabla u(t)\|_2 \geq \lambda_2$ for $t \in [0, T]$. Hence, by (19), (42) and (F3)

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 \leq E_0 + \frac{c_3}{p}\|u\|_p^p \tag{43}$$

and then

$$\frac{c_3}{p}\|u\|_p^p \geq \frac{1}{2}\|\nabla u\|_2^2 - E_0 \geq \frac{1}{2}\lambda_2^2 - g(\lambda_2) = K_0\lambda_2^p,$$

by definition of g. Finally, $\|u\|_p \geq (pK_0/c_3)^{1/p}\lambda_2$, which concludes the proof. □

Our global existence theorem is the following

Theorem 6. *Suppose that the assumptions of Theorem 4 (or of Theorem 5) hold, together with* (F3), *and that*

$$\|\nabla u_0\|_2 < \lambda_1, \qquad E(0) < E_1.$$

Then the solution of (1) *is global on* $[0, \infty)$, *and* $\|\nabla u(t)\|_2 \leq \lambda_1$ *for all* $t \geq 0$.

Proof. It is sufficient to combine the local existence of the solution with the standard continuation principle (see [**32**]) and the estimate given in Lemma 2–(i). □

Now we turn to our blow-up result. We need two further assumptions on the nonlinearities involved:

(Q3) there is $c_4 > 0$ such that

$$Q(x, v)v \geq c_4\alpha(x)(|v|^\mu + |v|^m), \qquad 1 < \mu \leq m,$$

for all $x \in \Gamma_1$, $v \in \mathbb{R}$;

(F4) there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ there exists $c_5 = c_5(\varepsilon) > 0$ such that

$$f(x, u)u - (p - \varepsilon)F(x, u) \geq c_5|u|^p$$

for all $x \in \Omega$, $u \in \mathbb{R}$.

REMARK 7. Clearly Q_0 given in (12) satisfies (Q3) with $c_4 = 1$, and f_0 given in (17) satisfies (F4) when $a \leq 0$ and $b > 0$, with $\varepsilon_0 = p - q > 0$ and $c_5(\varepsilon) = b\varepsilon/p$. Moreover (Q3) immediately follows from (14) when $m = \mu$, while is not a consequence of (Q1)–(Q2) when $\mu < m$. Next (F4) implies the standard growth condition

$$f(x, u)u \geq pF(x, u) \qquad \text{for all } \ x \in \Omega, u \in \mathbb{R}. \tag{44}$$

Clearly (F1)–(F3) and (44) are not sufficient to prove a blow-up theorem, since the case $f \equiv 0$ is included and did not produce blow-up.

THEOREM 7. *Suppose that the assumptions of Theorem 4 (Theorem 5) hold, with* (F3)–(F4) *and* (Q3), *that* $\alpha \in L^\infty(\Gamma_1)$ *and that* (11) *holds. Let u be the solution of* (1) *given in Theorem 4 (Theorem 5). Then, if*

$$\|\nabla u_0\|_2 > \lambda_1, \qquad E(0) < E_1,$$

there is $T_{\max} > 0$ *such that* $\|u(t)\|_p \to \infty$ *(and so also* $\|u(t)\|_\infty \to \infty$ *and* $\|\nabla u(t)\|_2 \to \infty$) as $t \to T_{\max}^-$.

Proof. We appropriately modify the proof of [**35**, Theorem 2] (see also [**11**] and [**25**]). It is enough to prove that no global solution in $[0, \infty)$ can exist. Indeed, the local existence of solutions together with the standard continuation principle (see [**32**]) yields the blow-up of $\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2$, which, by (43), gives the theorem. Hence assume by contradiction that u be defined on the whole $[0, \infty)$.

Keeping the notations of Lemma 2, let $E_2 \in (E_0, E_1)$ be fixed, and set $\mathcal{H}(t) = E_2 - E(t)$. Being $\mathcal{H}$ increasing by (18), it follows that

$$\mathcal{H}(t) \geq \mathcal{H}_0 := \mathcal{H}(0) = E_2 - E_0 > 0. \tag{45}$$

We shall omit, for simplicity, explicit dependence on time on the notations. By Lemma 2–(ii),

$$\mathcal{H}(t) \leq E_2 - \frac{1}{2}\|\nabla u\|_2^2 + \int_\Omega F(\cdot, u) \leq E_1 - \frac{1}{2}\lambda_1^2 + \int_\Omega F(\cdot, u)$$

and then, by the definition of E_1 and (F4),

$$\mathcal{H}(t) \leq \int_\Omega F(\cdot, u) \leq \frac{c_3}{p}\|u\|_p^p. \tag{46}$$

Now we claim that there are positive constants c_6 and c_7 such that

$$I_2 := \frac{d}{dt}\int_\Omega u_t u \geq 2\|u_t\|_2^2 + c_6\|u\|_p^p + c_7\|\nabla u\|_2^2 + 2\mathcal{H} - \int_{\Gamma_1} Q(\cdot, u_t)u \tag{47}$$

on $[0, \infty)$. Taking $\varphi = u$ in (21) we obtain

$$I_2 = \|u_t\|_2^2 - \|\nabla u\|_2^2 + \int_\Omega f(\cdot, u)u - \int_{\Gamma_1} Q(\cdot, u_t)u$$

and then, using (F4), for $0 < \varepsilon < \min\{\varepsilon_0, p - 2\}$,

$$I_2 \geq 2\|u_t\|_2^2 + \int_\Omega [f(\cdot, u)u - (p-\varepsilon)F(\cdot, u)] + \frac{1}{2}(p-\varepsilon-2)\|\nabla u\|_2^2 - (p-\varepsilon)E_2$$
$$+ (p-\varepsilon)\mathcal{H} - \int_{\Gamma_1} Q(\cdot, u_t)u$$
$$\geq 2\|u_t\|_2^2 + c_5\|u\|_p^p + \frac{1}{2}(p-\varepsilon-2)\|\nabla u\|_2^2 - (p-\varepsilon)E_2 + 2\mathcal{H} - \int_{\Gamma_1} Q(\cdot, u_t)u.$$

By Lemma 2–(ii)

$$\frac{1}{2}(p-\varepsilon-2)\|\nabla u\|_2^2 - (p-\varepsilon)E_2 \geq c_7\|\nabla u\|_2^2 + c_8,$$

where

$$c_7 = c_7(\varepsilon) = \frac{1}{2}(p-\varepsilon-2)(1-\lambda_1^2/\lambda_2^2), \qquad c_8 = c_8(\varepsilon) = \frac{1}{2}(p-\varepsilon-2)\lambda_1^2 - (p-\varepsilon)E_2.$$

Clearly $c_7 > 0$ and, as $\varepsilon \to 0^+$,

$$c_8(\varepsilon) \to \frac{1}{2}(p-2)\lambda_1^2 - pE_2 > \frac{1}{2}(p-2)\lambda_1^2 - pE_1 = 0,$$

so also $c_8(\varepsilon) > 0$ for ε sufficiently small. Fixing a sufficiently small ε and setting $c_6 = c_5(\varepsilon)$ we conclude the proof of (47).

Now, in order to use the technique of [**11**] and [**25**], we estimate the last term in (47). We denote

$$\|\cdot\|_{l,\Gamma_1} = \|\cdot\|_{L^l(\Gamma_1)}, \qquad \|\cdot\|_{l,\Gamma_1,\alpha} = \|\cdot\|_{L^l(\Gamma_{1},\alpha)}, \qquad \text{for any } l > 1.$$

Using (Q2), the Hölder inequality (with respect to μ_α), and the fact that $\alpha \in L^\infty(\Gamma_1)$,

$$I_3 := \left| \int_{\Gamma_1} Q(\cdot, u_t)u \right| \leq C_1\left(\|u_t\|_{\mu,\Gamma_1,\alpha}^{\mu-1}\|u\|_{\mu,\Gamma_1} + \|u_t\|_{m,\Gamma_1,\alpha}^{m-1}\|u\|_{m,\Gamma_1} \right)$$

with $C_1 = c_1(\|\alpha\|_\infty^{1/\mu} + 1)$. Applying the Hölder inequality again, since $\mu \leq m$ we obtain

$$I_3 \leq C_2\left(\|u_t\|_{\mu,\Gamma_1,\alpha}^{\mu-1} + \|u_t\|_{m,\Gamma_1,\alpha}^{m-1} \right)\|u\|_{m,\Gamma_1} \tag{48}$$

with $C_2 = C_2(\mu, m, c_1, \|\alpha\|_\infty, \Omega) > 0$.

To estimate the $L^m(\Gamma_1)$ norm of u we introduce the Sobolev space of fractional order $H^s(\Omega)$, where $s > 0$ is a free parameter, to be chosen later. We recall the embedding (see [**1**, Theorem 5.8])

$$\|u\|_{l,\Gamma_1} \leq C_3\|u\|_{H^s(\Omega)}$$

with $C_3 = C_3(l, s, \Omega) > 0$, which holds for $l \geq 1$ and

$$s = \frac{n}{2} - \frac{n-1}{l} > 0.$$

Then, using the Hölder inequality on Γ_1,

$$\|u\|_{m,\Gamma_1} \le C_4 \|u\|_{H^s(\Omega)},$$

$C_4 = C_4(m, s, \Omega) > 0$, for

$$0 < s < 1, \qquad s \ge \frac{n}{2} - \frac{n-1}{m}. \tag{49}$$

Next, by the interpolation (see [**27**, p. 49]) and Poincaré inequalities (see [**37**])

$$\|u\|_{H^s(\Omega)} \le C_5 \|u\|_2^{1-s} \|\nabla u\|_2^s,$$

$C_5 = C_5(s, \Omega) > 0$. Then

$$\|u\|_{m,\Gamma_1} \le C_6 \|u\|_2^{1-s} \|\nabla u\|_2^s, \tag{50}$$

for some $C_6 = C_6(m, s, \Omega) > 0$. By (48) and (50), using the Hölder inequality and the fact that $p > 2$,

$$I_3 \le C_7 \Big(\|u_t\|_{\mu,\Gamma_1,\alpha}^{\mu-1} + \|u_t\|_{m,\Gamma_1,\alpha}^{m-1} \Big) \|u\|_p^{1-s} \|\nabla u\|_2^s,$$

with $C_7 = C_7(\mu, m, p, s, c_1, \|\alpha\|_\infty, \Omega) > 0$. By Young's inequality, if

$$s < \frac{2}{m}, \tag{51}$$

then, for any $\delta > 0$,

$$\begin{aligned} I_3 \le\ & C_7 \Big(C(\delta) \|u_t\|_{\mu,\Gamma_1,\alpha}^{\mu} + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p \Big) \|u\|_p^{1-s-p(1/\mu - s/2)} \\ & + C_7 (C(\delta) \|u_t\|_{m,\Gamma_1,\alpha}^{m} + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p) \|u\|_p^{1-s-p(1/m-s/2)} \end{aligned}$$

for some $C(\delta) = C(\delta, \mu, m) > 0$. By (45)–(46) we have $\|u\|_p \ge (\mathcal{H}_0 p / c_3)^{1/p}$, hence from $1 - s - p\big(\frac{1}{\mu} - \frac{s}{2}\big) \le 1 - s - p\big(\frac{1}{m} - \frac{s}{2}\big)$,

$$I_3 \le C_8 \Big[C(\delta) \Big(\|u_t\|_{\mu,\Gamma_1,\alpha}^{\mu} + \|u_t\|_{m,\Gamma_1,\alpha}^{m} \Big) + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p \Big] \|u\|_p^{1-s-p(1/m-s/2)},$$

with $C_8 = C_8(\mu, m, p, s, c_1, c_3, \|\alpha\|_\infty, \Omega, \mathcal{H}_0) > 0$. Next, if $1 - s - p\big(\frac{1}{m} - \frac{s}{2}\big) < 0$, that is if

$$s < \Big(\frac{p}{m} - 1\Big) \Big/ \Big(\frac{p}{2} - 1\Big), \tag{52}$$

using (46) and setting $\overline{\alpha_s} = -\big[1 - s - p\big(\frac{1}{m} - \frac{s}{2}\big)\big]/p > 0$, we obtain

$$I_3 \le C_8 \Big[C(\delta) \Big(\|u_t\|_{\mu,\Gamma_1,\alpha}^{\mu} + \|u_t\|_{m,\Gamma_1,\alpha}^{m} \Big) + \delta \|\nabla u\|_2^2 + \delta \|u\|_p^p \Big] \mathcal{H}^{-\overline{\alpha_s}},$$

(with a possibly different value of C_8). Then, by (Q3) and (45), for $0 < \alpha_s < \overline{\alpha_s}$,

$$I_3 \le C_8\Big[c_4 C(\delta)\mathcal{H}' + \delta\|\nabla u\|_2^2 + \delta\|u\|_p^p\Big]\mathcal{H}^{-\overline{\alpha}_s}$$
$$\le C_9\Big[C(\delta)\mathcal{H}'\mathcal{H}^{-\alpha_s} + \delta\|\nabla u\|_2^2 + \delta\|u\|_p^p\Big]$$

where $C_9 = C_9(\mu, m, p, s, \alpha_s, c_1, c_3, c_4, \|\alpha\|_\infty, \Omega, \mathcal{H}_0) > 0$. Choosing $\delta \le \min\{c_7/2C_9, c_6/2C_9\}$, this estimate, combined with (47), gives

$$I_2 \ge 2\|u_t\|_2^2 + \frac{c_6}{2}\|u\|_p^p + \frac{c_7}{2}\|\nabla u\|_2^2 + 2\mathcal{H} - C_{10}\mathcal{H}'\mathcal{H}^{-\alpha_s}$$

where $C_{10} = C_9 C(\delta)$. From this point the proof can be completed as in [**11**] or [**25**], provided we can show the existence of a value of the parameter s satisfying (49), (51) and (52). When $1 < m \le 2$ it is enough to verify that $\frac{n}{2} - \frac{n-1}{m} \le 1$, which is clearly true. When $m > 2$, since

$$\left(\frac{p}{m} - 1\right)\Big/\left(\frac{p}{2} - 1\right) \le \frac{2}{m} \le 1,$$

it is enough to verify that $m < p$ and

$$\frac{n}{2} - \frac{n-1}{m} < \left(\frac{p}{m} - 1\right)\Big/\left(\frac{p}{2} - 1\right) \tag{43}$$

which is exactly (11). An easy calculation shows that $m_0(p) < p$ for $p > 2$, concluding the proof. □

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