SOME GLOBAL THEOREMS ON HYPERSURFACES

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1. Introduction. The purpose of this paper is to establish the following theorems, which were obtained by Hopf and Voss in their joint paper **(2)** for the case where $n = 2$.

THEOREM 1. *Let V^n, V^{*n} be two closed orientable hypersurfaces twice differentiably imbedded in a Euclidean space E^{n+1} of dimension $n + 1 \geqslant 3$. Suppose that there is a differentiable homeomorphism between the two hypersurfaces V^n, V^{*n} such that the orientations of the two hypersurfaces V^n, V^{*n} are preserved and the line joining every pair of corresponding points P, P^* of the two hypersurfaces V^n, V^{*n} is parallel to a fixed direction R, and such that the two hypersurfaces V^n, V^{*n} have equal first mean curvatures at every pair of the points P, P^* but no cylindrical elements whose generators are parallel to the fixed direction R. Then the two hypersurfaces V^n, V^{*n} can be transformed into each other by a translation.*

A closed hypersurface V^n imbedded in a Euclidean space E^{n+1} of dimension $n + 1 \geqslant 2$ is said to be *convex in a given direction*, if no line in this direction intersects the hypersurface V^n at more than two points. It is obvious that a closed hypersurface V^n is convex in the usual sense if it is convex in every direction in the space E^{n+1}.

THEOREM 2. *Let a closed orientable hypersurface V^n twice differentiably imbedded in a Euclidean space E^{n+1} of dimension $n + 1 \geqslant 3$ be convex in a given direction R. If the two first mean curvatures of the hypersurface V^n at every pair of its points of intersection with the lines in the direction R are equal, then the hypersurface V^n has a hyperplane of symmetry perpendicular to the direction R.*

Theorem 2 can easily be deduced from Theorem 1. In fact, let u be a mapping of a hypersurface V^n satisfying the conditions of Theorem 2 onto itself such that the two points of intersection of the hypersurface V^n with any line in the direction R are mapped into each other. In particular, if a line in the direction R is tangent to the hypersurface V^n at a point P, then $uP = P$. Let r be the reflection with respect to an arbitrary hyperplane perpendicular to the direction R, and P any point of the hypersurface V^n. Then the mapping $ruP = P^*$ maps the hypersurface V^n onto the hypersurface $V^{*n} = r(V^n)$ generated by the point P^*, and the two hypersurfaces V^n, V^{*n} satisfy the conditions of Theorem 1 so that $ru = t$ is a translation. Therefore $u = rt$ is a reflection with respect to a hyperplane perpendicular to the direction R, and hence Theorem 2 follows.

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By noting that a closed hypersurface V^n imbedded in a Euclidean space E^{n+1} of dimension $n+1 \geqslant 2$ must be a hypersphere if it has a hyperplane of symmetry perpendicular to every direction in the space E^{n+1}, we arrive readily at the following known result from Theorem 2.

COROLLARY. *A closed convex hypersurface V^n of constant first mean curvature twice differentiably imbedded in a Euclidean space E^{n+1} of dimension $n+1 \geqslant 3$ is a hypersphere.*

THEOREM 3. *Let $V^n(V^{*n})$ be an orientable hypersurface with a closed boundary $V^{n-1}(V^{*n-1})$ of dimension $n-1 \geqslant 1$ twice differentiably imbedded in a Euclidean space E^{n+1} of dimension $n+1$. Suppose that there is a differentiable homeomorphism between the two hypersurfaces V^n, V^{*n} with the same properties as those of the homeomorphism in Theorem 1.*

(i) *If the two boundaries V^{n-1}, V^{*n-1} are coincident, then the two hypersurfaces V^n, V^{*n} are coincident.*

(ii) *If the two normals of the two hypersurfaces V^n, V^{*n} at every pair of corresponding points, under the given homeomorphism, of the two boundaries V^{n-1}, V^{*n-1} are parallel, then the two hypersurfaces V^n, V^{*n} are transformed into each other by a translation.*

2. Preliminaries[1]. In a Euclidean space E^{n+1} of dimension $n+1 \geqslant 3$, let us consider a fixed orthogonal frame $OI_1 \ldots I_{n+1}$ with a point O as the origin. With respect to this orthogonal frame we define the vector product of n vectors $A_1, \ldots, A_n$ in the space E^{n+1} to be the vector A_{n+1}, denoted by $A_1 \times \ldots \times A_n$, satisfying the following conditions:

(a) the vector A_{n+1} is normal to the n-dimensional subspace of E^{n+1} determined by the vectors $A_1, \ldots, A_n$,

(b) the magnitude of the vector A_{n+1} is equal to the volume of the parallelepiped whose edges are the vectors $A_1, \ldots, A_n$,

(c) the two frames $OA_1 \ldots A_n A_{n+1}$ and $OI_1 \ldots I_{n+1}$ have the same orientation.

Let σ be a permutation on the n numbers $1, \ldots, n$, then

$$A_{\sigma(1)} \times \ldots \times A_{\sigma(n)} = (\operatorname{sgn} \sigma) A_1 \times \ldots \times A_n, \tag{2.1}$$

where sgn σ is $+1$ or -1 according as the permutation σ is even or odd. Let $i_1, \ldots, i_{n+1}$ be the unit vectors from the origin O in the directions of the vectors $I_1, \ldots, I_{n+1}$ and let A_α^j $(j = 1, \ldots, n+1)$ be the components[2] of the vector A_α $(\alpha = 1, \ldots, n)$ with respect to the frame $OI_1 \ldots I_{n+1}$, then the scalar product of any two vectors A_α and A_β and the vector product of n vectors $A_1, \ldots, A_n$ are, respectively,

[1]For this section see, for instance, (**3**, pp. 287–289).

[2]Throughout this paper all Latin indices take the values 1 to $n+1$ and Greek indices the values 1 to n unless stated otherwise. We shall also follow the convention that repeated indices imply summation.

$$A_\alpha \cdot A_\beta = \sum_{i=1}^{n+1} A_\alpha^i A_\beta^i, \tag{2.2}$$

$$A_1 \times \ldots \times A_n = (-1)^n \begin{vmatrix} i_1 & i_2 & \ldots & i_{n+1} \\ A_1^1 & A_1^2 & \ldots & A_1^{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ A_n^1 & A_n^2 & \ldots & A_n^{n+1} \end{vmatrix}. \tag{2.3}$$

If A_α^j are differentiable functions of n variables $x^1, \ldots, x^n$, then by equation (2.3) and the differentiation of determinants

$$\frac{\partial}{\partial x^\alpha}(A_1 \times \ldots \times A_n) = \sum_{\beta=1}^{n}\left(A_1 \times \ldots \times A_{\beta-1} \times \frac{\partial A_\beta}{\partial x^\alpha} \times A_{\beta+1} \times \ldots \times A_n\right). \tag{2.4}$$

Now we consider a hypersurface V^n twice differentiably imbedded in the space E^{n+1} with a closed boundary V^{n-1} of dimension $n - 1$. Let $(y^1, \ldots, y^{n+1})$ be the coordinates of a point P in the space E^{n+1} with respect to the orthogonal frame $OI_1 \ldots I_{n+1}$. Then the hypersurface V^n can be given by the parametric equations

$$y^i = f^i(x^1, \ldots, x^n) \qquad (i = 1, \ldots, n+1), \tag{2.5}$$

or the vector equation

$$Y = F(x^1, \ldots, x^n), \tag{2.6}$$

where y^i and f^i are respectively the components of the two vectors Y and F, the parameters $x^1, \ldots, x^n$ take values in a simply connected domain D of the n-dimensional real number space, $f^i(x^1, \ldots, x^n)$ are twice differentiable and the Jacobian matrix $||\partial y^i/\partial x^\alpha||$ is of rank n at all points of the domain D. If we denote the vector $\partial Y/\partial x^\alpha$ by Y_α $(\alpha = 1, \ldots, n)$, then the first fundamental form of the hypersurface V^n at the point P is

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \tag{2.7}$$

where

$$g_{\alpha\beta} = Y_\alpha \cdot Y_\beta, \tag{2.8}$$

and the matrix $||g_{\alpha\beta}||$ is positive definite so that the determinant $g = |g_{\alpha\beta}| > 0$.

Let N be the unit normal vector of the hypersurface V^n at the point P, and N_α the vector $\partial N/\partial x^\alpha$, then

$$N_\alpha = -\, b_{\alpha\beta}\, g^{\beta\gamma}\, Y_\gamma, \tag{2.9}$$

where

$$b_{\alpha\beta} = b_{\beta\alpha} = -\, N_\alpha \cdot Y_\beta \tag{2.10}$$

are the coefficients of the second fundamental form of the hypersurface V^n at the point P, and $g^{\beta\gamma}$ denotes the cofactor of $g_{\beta\gamma}$ in g divided by g so that

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$$g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma \tag{2.11}$$

(the Kronecker deltas). The n principal curvatures $\kappa_1, \ldots, \kappa_n$ of the hypersurface V^n at the point P are the roots of the determinant equation

$$|b_{\alpha\beta} - \kappa g_{\alpha\beta}| = 0, \tag{2.12}$$

from which follows immediately the first mean curvature of the hypersurface V^n at the point P:

$$M_1 = \frac{1}{n} \sum_{\alpha=1}^{n} \kappa_\alpha = \frac{1}{n} b_{\alpha\beta} g^{\alpha\beta}. \tag{2.13}$$

The area element of the hypersurface V^n at the point P is

$$dA = g^{\frac{1}{2}} dx^1 \wedge \ldots \wedge dx^n, \tag{2.14}$$

where the operator d is the exterior differentiation, and the wedge denotes the exterior multiplication. Now we choose the direction of the unit normal vector N in such a way that the two frames $PY_1 \ldots Y_n N$ and $OI_1 \ldots I_{n+1}$ have the same orientation. Then from equations (2.3) and (2.14) it follows that

$$g^{\frac{1}{2}} N = Y_1 \times \ldots \times Y_n, \tag{2.15}$$

$$|Y_1, \ldots, Y_n, N| = g^{\frac{1}{2}}, \tag{2.16}$$

where the left side of equation (2.16) is a determinant indicated by writing only a typical row.

3. An integral formula. Let V^n be an orientable hypersurface with a closed boundary V^{n-1} of dimension $n - 1 \geqslant 1$ twice differentiably imbedded in a Euclidean space E^{n+1} of dimension $n + 1$, and suppose that the hypersurface V^n is given by the vector equation (2.6). Let I be the unit vector in a fixed direction R in the space E^{n+1}, and w a twice differentiable function over the hypersurface V^n. Then §2 can be applied to the hypersurface V^n, and we shall use the same symbols with a star for the corresponding quantities for the hypersurface V^{*n} defined by the vector equation

$$Y^* = Y + W, \tag{3.1}$$

where

$$W = wI. \tag{3.2}$$

Let Ω^α $(\alpha = 1, \ldots, n)$ be n vectors in the space E^{n+1}, and suppose that the components of each vector Ω^α with respect to the orthogonal frame $OI_1 \ldots I_{n+1}$ are differentiable functions of the n variables $x^1, \ldots, x^n$. In order to derive an integral formula for the two hypersurfaces V^n, V^{*n} we use the vector product of vectors and the exterior multiplication of differentials to define the vector

$$\begin{aligned}(3.3)\quad & \Omega^1 \otimes \ldots \otimes \Omega^{\alpha-1} \otimes d\Omega^\alpha \otimes \ldots \otimes d\Omega^n \\ &= (\Omega^1 \times \ldots \times \Omega^{\alpha-1} \times \Omega^\alpha_{\beta_\alpha} \times \ldots \times \Omega^n_{\beta_n}) \, dx^{\beta_\alpha} \wedge \ldots \wedge dx^{\beta_n}\end{aligned}$$

for $\alpha = 1, \ldots, n$, where

$$\Omega^{\alpha}_{\beta_\alpha} = \partial\Omega^{\alpha}/\partial x^{\beta_\alpha}.$$

It is obvious that the vector (3.3) is independent of the order of the vectors $d\Omega^{\alpha}, \ldots, d\Omega^{n}$. Thus from equations (2.9), (2.13), (2.14), (2.15) we obtain

$$(3.4)\quad dY \otimes \ldots \otimes dY = n!\,(Y_1 \times \ldots \times Y_n)\, dx^1 \wedge \ldots \wedge dx^n = n!\, N\, dA,$$

$$(3.5)\quad dY \otimes \ldots \otimes dY \otimes dN$$

$$= (n-1)!\left(\sum_{\alpha=1}^{n} Y_1 \times \ldots \times Y_{\alpha-1} \times N_\alpha \times Y_{\alpha+1} \times \ldots \times Y_n\right) dx^1 \wedge \ldots \wedge dx^n$$

$$= -\,n!\, M_1\, N\, dA.$$

Making use of equations (3.1), (3.2), (3.4) and its analogue for the hypersurface V^{*n}, and noting that

$$\underbrace{dW \otimes \ldots \otimes dW}_{(\alpha \text{ factors})} \otimes \underbrace{dY \otimes \ldots \otimes dY}_{(n-\alpha \text{ factors})} = 0,$$

$$\underbrace{dW \otimes \ldots \otimes dW}_{(\alpha \text{ factors})} \otimes \underbrace{dY^* \otimes \ldots \otimes dY^*}_{(n-\alpha \text{ factors})} = 0$$

for $\alpha \geqslant 2$ and

$$|W, Y_1, \ldots, Y_n| = |W, Y_1^*, \ldots, Y_n^*|,$$

we are easily led to

$$(3.6)\quad (n-1)!\,(N^* dA^* - N dA) = dW \otimes dY \otimes \ldots \otimes dY$$

$$= dW \otimes dY^* \otimes \ldots \otimes dY^*,$$

$$(3.7)\quad W\cdot N\, dA = W\cdot N^*\, dA^*,$$

$$(3.8)\quad |W, N^*, Y_1^*, \ldots, Y_{\alpha-1}^*, Y_{\alpha+1}^*, \ldots, Y_n^*|$$

$$= |W, N^*, Y_1, \ldots, Y_{\alpha-1}, Y_{\alpha+1}, \ldots, Y_n| \qquad (\alpha = 1, \ldots, n).$$

From equations (2.3), (3.3), (3.5), (3.6) it follows immediately that

$$(3.9)\quad W\cdot(N \otimes dY \otimes \ldots \otimes dY)$$

$$= (-1)^n (n-1)! \sum_{\alpha=1}^{n} |W, N, Y_1, \ldots, Y_{\alpha-1}, Y_{\alpha+1}, \ldots, Y_n|$$

$$dx^1 \wedge \ldots \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \wedge \ldots \wedge dx^n,$$

$$(3.10)\quad d[W\cdot(N \otimes dY \otimes \ldots \otimes dY)]$$

$$= -\,N\cdot(dW \otimes dY \otimes \ldots \otimes dY) + W\cdot(dN \otimes dY \otimes \ldots \otimes dY)$$

$$= -\,n!\, M_1\, W\cdot N\, dA - (n-1)!\,(N\cdot N^*\, dA^* - dA).$$

Similarly, in consequence of equations (3.6), (3.7), (3.8) and those analogous to equations (3.5), (3.9) by changing the vectors Y, N to the vectors Y^*, N^* respectively, we obtain

$$(3.11) \quad d[W\cdot(N^* \otimes dY \otimes \ldots \otimes dY)] = d[W\cdot(N^* \otimes dY^* \otimes \ldots \otimes dY^*)]$$
$$= - N^*\cdot(dW \otimes dY^* \otimes \ldots \otimes dY^*) + W\cdot(dN^* \otimes dY^* \otimes \ldots \otimes dY^*)$$
$$= - n!\, M_1^* \, W\cdot N\, dA - (n-1)!\,(dA^* - N^*\cdot N\, dA).$$

Thus, from equations (3.9), (3.10), (3.11),

$$(3.12) \quad d\sum_{\alpha=1}^{n} |W, N - N^*, Y_1, \ldots, Y_{\alpha-1}, Y_{\alpha+1}, \ldots, Y_n| \; dx^1 \wedge \ldots \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \wedge \ldots \wedge dx^n$$
$$= \frac{(-1)^n}{(n-1)!} d[W\cdot(N \otimes dY \otimes \ldots \otimes dY) - W\cdot(N^* \otimes dY \otimes \ldots \otimes dY)]$$
$$= (-1)^n\,[n(M_1^* - M_1)\, W\cdot N\, dA + (1 - N\cdot N^*)\,(dA + dA^*)].$$

Integrating equation (3.12) over the hypersurface V^n and applying the Stokes' theorem to the left side of the equation, we then arrive at the integral formula

$$(3.13) \quad \int_{V^{n-1}} \sum_{\alpha=1}^{n} |W, N - N^*, Y_1, \ldots, Y_{\alpha-1}, Y_{\alpha+1}, \ldots, Y_n| \; dx^1 \wedge \ldots \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \wedge \ldots \wedge dx^n$$
$$= (-1)^n \int_{V^n} [n(M_1^* - M_1)\, W\cdot N\, dA + (1 - N\cdot N^*)\,(dA + dA^*)].$$

In particular, when the hypersurface V^n is closed and orientable, the integral on the left side of equation (3.13) vanishes and hence

$$(3.14) \quad n\int_{V^n} (M_1^* - M_1)\, W\cdot N\, dA + \int_{V^n} (1 - N\cdot N^*)\,(dA + dA^*) = 0.$$

4. Proof of Theorems 1 and 3. It is easily seen that we can apply the results in §3 to two hypersurfaces V^n, V^{*n} satisfying the assumptions of Theorem 1. Since $M_1^* = M_1$ at every pair of corresponding points of the two hypersurfaces V^n, V^{*n}, the formula (3.14) becomes

$$(4.1) \quad \int_{V^n} (1 - N\cdot N^*)\,(dA + dA^*) = 0.$$

But $dA > 0$, $dA^* > 0$ and $1 - N\cdot N^* \geqslant 0$ due to the fact that N and N^* are unit vectors. Thus the integrand of equation (4.1) is non-negative, and therefore equation (4.1) holds when and only when $1 - N\cdot N^* = 0$, which implies that

$$(4.2) \quad N^* = N.$$

Now in the space E^{n+1} we choose the orthogonal frame $OI_1 \ldots I_{n+1}$, with respect to which a point in the space E^{n+1} has coordinates $y^1, \ldots, y^{n+1}$, in such a way that the unit vector I_{n+1} is the fixed unit vector I. Since the hypersurface V^n has no cylindrical elements whose generators are parallel to the fixed vector I, the closed set M of all points of the hypersurface V^n, at each of which the y^{n+1}-component of the unit normal vector N of the hypersurface V^n is zero, has no inner points and therefore the open set $V^n - M$ is everywhere dense over V^n. Thus, in neighborhoods of any point of the set $V^n - M$ and its corresponding point on the hypersurface V^{*n}, $y^1, \ldots, y^n$ are regular parameters of the two hypersurfaces V^n, V^{*n} so that the hypersurfaces V^n, V^{*n} can be represented respectively by the equations

$$(4.3)\qquad \begin{aligned} y^{n+1} &= y^{n+1}(y^1, \ldots, y^n), \\ y^{n+1} &= y^{*n+1}(y^1, \ldots, y^n) = y^{n+1}(y^1, \ldots, y^n) + w(y^1, \ldots, y^n). \end{aligned}$$

By means of equations (2.15), (4.3) we obtain the unit normal vectors N, N^* of the hypersurfaces V^n, V^{*n}:

$$(4.4)\quad N = -g^{-\frac{1}{2}}\left(\sum_{\alpha=1}^{n} \frac{\partial y^{n+1}}{\partial y^\alpha} i_\alpha - i_{n+1}\right), N^* = -g^{*-\frac{1}{2}}\left(\sum_{\alpha=1}^{n} \frac{\partial y^{*n+1}}{\partial y^\alpha} i_\alpha - i_{n+1}\right),$$

from which and equations (4.2), (4.3) it follows immediately that in a neighborhood of any point of the set $V^n - M$,

$$\partial y^{*n+1}/\partial y^\alpha = \partial y^{n+1}/\partial y^\alpha \qquad (\alpha = 1, \ldots, n)$$

and the function w is constant. Thus $\partial w/\partial y^\alpha$ $(\alpha = 1, \ldots, n)$ are zero in the everywhere dense set $V^n - M$ and therefore on the whole hypersurface V^n by continuity. Hence the function w is constant on the whole hypersurface V^n, and the proof of Theorem 1 is complete.

In both parts of Theorem 3 the integral over the boundary V^{n-1} on the left side of the formula (3.13) also vanishes, since over the boundary V^{n-1} $W = 0$ and $N^* = N$ in the two parts respectively. By the same argument as that in the above proof of Theorem 1, we therefore obtain between the two hypersurfaces V^n, V^{*n} a translation, which in part (i) reduces to an identity. Hence Theorem 3 is proved.

Now suppose that in Theorem 3 the fixed direction R is along the vector I_{n+1} and the hypersurfaces V^n, V^{*n} can be represented by equations of the form $y^{n+1} = y^{n+1}\,(y^1, \ldots, y^n)$. Then part (i) of Theorem 3 can be stated as follows: The problem of finding a function $y^{n+1}(y^1, \ldots, y^n)$ over a bounded region in the space $(y^1, \ldots, y^n)$ with given boundary values such that the first mean curvature M_1 of the hypersurface V^n defined by the equation $y^{n+1} = y^{n+1}(y^1, \ldots, y^n)$ is a given function $M_1(y^1, \ldots, y^n)$ admits at most one solution. Making use of equations (2.10), (2.13), (4.4) and

$$\frac{\partial g}{\partial x^\alpha} = g g^{\rho\sigma} \frac{\partial g_{\rho\sigma}}{\partial x^\alpha},$$

we can easily obtain the first mean curvature of the hypersurface V^n, namely,

$$(4.5)\qquad M_1 = n^{-1} g^{-\frac{1}{2}} g^{\alpha\beta} \frac{\partial^2 y^{n+1}}{\partial y^\alpha \partial y^\beta}.$$

Thus the above special case of part (i) of Theorem 3 is a consequence of the well-known uniqueness theorem for the solutions of elliptic differential equations of the second order, since the determinant $|g^{\alpha\beta}| = 1/g > 0$.

5. Connection with symmetrizations. Let $y^1, \ldots, y^{n+1}$ be the coordinates of a point with respect to a fixed orthogonal frame $OI_1 \ldots I_{n+1}$ in a Euclidean space E^{n+1} of dimension $n + 1 \geqslant 3$, and let a closed orientable hypersurface V^n twice differentiably imbedded in the space E^{n+1} be convex in the direction of the vector I_{n+1}. Let P be any point of the hypersurface V^n, and P^* the other point of intersection of the hypersurface V^n by the line l through the point P and in the direction of the vector I_{n+1}. If the line l is tangent to the hypersurface V^n, then the point P^* coincides with the point P. Let y^{n+1}, y^{*n+1} be respectively the $(n + 1)$th coordinates of the points P, P^* with respect to the frame $OI_1 \ldots I_{n+1}$, and M_1^*, N^* the first mean curvature and the unit normal vector of the hypersurface V^n at the point P^*.

The Steiner's symmetrization of the hypersurface V^n with respect to the hyperplane $y^{n+1} = 0$ is a geometric operation by which any point P of the hypersurface V^n goes into a point P' on the line l with

$$y'^{n+1} = \tfrac{1}{2}(y^{n+1} - y^{*n+1}) = y^{n+1} - \tfrac{1}{2}(y^{n+1} + y^{*n+1})$$

as its $(n + 1)$th coordinate with respect to the frame $OI_1 \ldots I_{n+1}$. In the time interval $0 \leqslant t \leqslant 1$, we shift the segment PP^* along its line l into the position $P'P^{*\prime}$ such that the $(n + 1)$th coordinates of the points P', $P^{*\prime}$ with respect to the frame $OI_1 \ldots I_{n+1}$ are respectively given by

$$(5.1) \qquad T_t\colon\ y'^{n+1} = y^{n+1} - \frac{t}{2}(y^{n+1} + y^{*n+1}),\ y^{*\prime n+1} = y^{*n+1} - \frac{t}{2}(y^{n+1} + y^{*n+1}).$$

That is, the segment PP^* is shifted with uniform velocity into the position where it is bisected by the hyperplane $y^{n+1} = 0$. This transformation T_t is called the continuous symmetrization of Steiner.[3] T_0 is the identity and T_1 results in a complete symmetrization. It is obvious that the transformation T_t leaves the volume of the hypersurface V^n unchanged.

Now let us consider a neighboring hypersurface $V^n{}_{(\epsilon)}$ of the hypersurface V^n defined by the vector equation

$$(5.2) \qquad Y^{(\epsilon)} = Y + \epsilon(W \cdot N)\, N,$$

where ϵ is an infinitesimal, Y is the position vector of the point P of the hypersurface V^n with respect to the frame $OI_1 \ldots I_{n+1}$, and

$$(5.3) \qquad W = wI_{n+1}, \quad w = -\, y^{n+1} - y^{*n+1}.$$

An elementary calculation and the use of equations (5.2), (2.8), (2.9) yield the coefficients of the first fundamental form of the hypersurface $V^n{}_{(\epsilon)}$:

[3]For the continuous symmetrization of Steiner in a Euclidean space E^n of dimension $n = 2, 3$ see (**1**, pp. 249–251; **4**, pp. 200–202).

$$g_{\alpha\beta}^{(\epsilon)} = g_{\alpha\beta} - 2\epsilon(W\cdot N)\, b_{\alpha\beta} + (O)(\epsilon^2), \tag{5.4}$$

and therefore

$$g^{(\epsilon)} = |g_{\alpha\beta}^{(\epsilon)}| = g - 2n\,\epsilon(W\cdot N)\, M_1\, g + \ldots, \tag{5.5}$$

where the omitted terms are of degrees $\geqslant 2$ in ϵ. From equations (5.5), (2.14) follows immediately the area of the hypersurface $V^n{}_{(\epsilon)}$:

$$A^{(\epsilon)} = \int_{V^n} \sqrt{g^{(\epsilon)}}\, dx^1 \wedge \ldots \wedge dx^n = A - n\epsilon \int_{V^n} M_1(W\cdot N)dA + \ldots. \tag{5.6}$$

Thus we obtain

$$\left(\frac{\partial A^{(\epsilon)}}{\partial \epsilon}\right)_{\epsilon=0} = -\, n \int_{V^n} M_1(W\cdot N)dA. \tag{5.7}$$

Similarly, replacing equation (5.2) by $Y^{(\epsilon)} = Y^* + \epsilon(W^* \cdot N^*)\, N^*$ gives

$$\left(\frac{\partial A^{(\epsilon)}}{\partial \epsilon}\right)_{\epsilon=0} = -\, n \int_{V^n} M_1^*\, (W^*\cdot N^*)dA^*. \tag{5.8}$$

Noting that $y^{*n+1} = -\, y^{n+1} - w$, $W^* = W$ and making use of equation (3.7), we obtain immediately

$$W^* \cdot N^*\, dA^* = -\, W \cdot N\, dA, \tag{5.9}$$

and therefore equation (5.8) becomes

$$\left(\frac{\partial A^{(\epsilon)}}{\partial \epsilon}\right)_{\epsilon=0} = n \int_{V^n} M_1^*\, (W\cdot N)dA. \tag{5.10}$$

Thus the addition of equations (5.7), (5.10) gives

$$\left(\frac{\partial A^{(\epsilon)}}{\partial \epsilon}\right)_{\epsilon=0} = \frac{n}{2} \int_{V^n} (M_1^* - M_1)\, W\cdot N\, dA. \tag{5.11}$$

As in the proof of Theorem 2 in §1, we consider the reflection r with respect to the hyperplane $y^{n+1} = 0$. By this reflection r the point P^* of the hypersurface V^n goes into the point $\bar{P}^*$ defined by

$$\bar{Y}^* = Y + W, \tag{5.12}$$

which generates a hypersurface $\bar{V}^{*n}$. If equation (5.12) is used instead of equation (3.1), then the formula (3.14) becomes

$$n \int_{V^n} (M_1^* - M_1)\, W\cdot N\, dA + \int_{V^n} (1 - N\cdot\bar{N}^*)\, (dA + d\bar{A}^*) = 0, \tag{5.13}$$

where $\bar{N}^*$ and $d\bar{A}^*$ are respectively the unit normal vector and the area element of the hypersurface $\bar{V}^{*n}$ at the point $\bar{P}^*$. By interchanging the corresponding quantities of the two hypersurfaces V^n, $\bar{V}^{*n}$ at the two points P^*, $\bar{P}^*$ respectively it is easily seen that

$$\int_{V^n} (1 - N\cdot\bar{N}^*)d\bar{A}^* = \int_{V^n} (1 - \bar{N}^*\cdot N)dA. \tag{5.14}$$

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By means of equation (5.14), equation (5.13) reduces to

$$\frac{n}{2}\int_{V^n} (M_1^* - M_1)\, W\cdot N\, dA = - \int_{V^n} (1 - N\cdot \bar{N}^*) dA, \tag{5.15}$$

from which and equation (5.11) we therefore obtain

$$\left(\frac{\partial A^{(\epsilon)}}{\partial \epsilon}\right)_{\epsilon=0} = - \int_{V^n} (1 - N\cdot \bar{N}^*) dA. \tag{5.16}$$

Making use of equations (5.11), (5.15), (5.16) we can easily reach the following conclusion:

If $M_1^* = M_1$ at every point P of the hypersurface V^n, then $(\partial A^{(\epsilon)}/\partial\epsilon)_{\epsilon=0} = 0$ and the hypersurface V^n is symmetric with respect to a hyperplane. If the hypersurface V^n is not symmetric with respect to a hyperplane and $\bar{N}^* \not\equiv N$ at every point P of the hypersurface V^n, then $(\partial A^{(\epsilon)}/\partial\epsilon)_{\epsilon=0} < 0$.

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