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COMMON SLOTS OF BILINEAR AND QUADRATIC PFISTER FORMS

ADAM CHAPMAN

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Abstract

We show that over any field F of characteristic 2 and 2-rank n, there exist 2^n bilinear n-fold Pfister forms that have no slot in common. This answers a question of Becher ['Triple linkage', *Ann. K-Theory*, to appear] in the negative. We provide an analogous result also for quadratic Pfister forms.

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1. Introduction

The study of linkage of quadratic or bilinear n-fold Pfister forms and its connections to important field invariants, for example, the u-invariant and the cohomological 2-dimension, has been the focus of several interesting papers in the last five decades. The first significant result was obtained in [14] where it was shown for nonreal fields F with $\mathrm{char}(F) \neq 2$ that if I^nF is linked (that is, every pair of anisotropic n-fold Pfister forms have an $(n-1)$-fold Pfister form as a common factor) then $I^{n+2}F = 0$, and it was concluded that if F is linked (that is, I^2F is linked) then $u(F)$ can be either $0, 1, 2, 4$ or 8. The analogous result for I_q^nF when $\mathrm{char}(F) = 2$ was given in [7] based on preliminary results obtained in [15].

There is an intrinsic complication with quadratic forms when $\mathrm{char}(F) = 2$: there exist two kinds of quadratic field extensions—separable and inseparable—which means that a maximal subfield shared by two given quaternion division algebras can be either a separable or inseparable extension of the centre, and two quadratic n-fold Pfister forms can share either a quadratic or bilinear $(n-1)$-fold Pfister form as a common factor. We specify the terms 'separable' and 'inseparable' linkage accordingly. It was shown that inseparable linkage for I_q^2F implies separable linkage [12] but not vice versa [18]. This fact was generalised to I_q^nF for arbitrary n in [15] and to symbol division p-algebras of arbitrary prime degree in [6].

In [5] the linkage property was extended to larger sets of n-fold Pfister forms: we say that I^nF is m-linked if m anisotropic bilinear m-fold Pfister forms always have an $(n-1)$-fold Pfister form as a common factor. It was shown for nonreal fields F with $\mathrm{char}(F) \neq 2$ that if I^nF is 3-linked then $I^{n+1}F = 0$, and consequently that if I^2F is 3-linked then $u(F) \leqslant 4$. The analogous results for I_q^nF when $\mathrm{char}(F) = 2$ were obtained in [9].

Becher noticed that there exist fields F for which I^2F is m-linked for any finite m (such as number fields) and asked the following natural question.

Question 1.1 [5, Question 5.2]. Suppose $I^nF \neq 0$ and I^nF is 3-linked. Does it follow that I^nF is m-linked for every finite $m \geqslant 3$?

In this paper we provide a negative answer to this question when $\mathrm{char}(F) = 2$. We also consider the analogous questions for I_q^nF. We say that I_q^nF is separably (inseparably) m-linked if m anisotropic quadratic n-fold Pfister forms over F always have a quadratic (bilinear) $(n-1)$-fold Pfister form as a common factor. This leads to two analogues of Question 1.1 for I_q^nF.

Question 1.2. Suppose $I_q^nF \neq 0$ and I_q^nF is inseparably 2-linked. Does it follow that I_q^nF is inseparably m-linked for every finite $m \geqslant 2$?

Question 1.3. Suppose $I_q^nF \neq 0$ and I_q^nF is separably 3-linked. Does it follow that I_q^nF is separably m-linked for every finite $m \geqslant 3$?

We answer Question 1.2 in the negative, and likewise Question 1.3 for $n \geqslant 3$. We conjecture that the answer to [5, Question 5.2] is negative also when $\mathrm{char}(F) \neq 2$.

2. Preliminaries

For general reference on symmetric bilinear forms and quadratic forms see [13]. The group $W_qF = I_qF$ is generated by the forms $\varphi(u,v) = \alpha u^2 + uv + \beta v^2$ for $\alpha, \beta \in F$, denoted by $[\alpha, \beta]$. We write $\langle \beta_1, \ldots, \beta_n\rangle_b$ for the diagonal bilinear form

$$B((v_1,\ldots,v_n),(w_1,\ldots,w_n)) = \sum_{i=1}^{n} \beta_i v_i w_i$$

and $\langle \beta_1, \ldots, \beta_n\rangle$ for the diagonal quadratic form $\varphi(v_1,\ldots,v_n) = \sum_{i=1}^{n} \beta_i v_i^2$. We denote by $D(\varphi)$ the set of nonzero values φ represents, that is, $\{\varphi(v) : v \in V, \varphi(v) \neq 0\}$, and by $D(B)$ the set $\{B(v,v) : v \in V, B(v,v) \neq 0\}$.

The bilinear forms $\langle\langle \beta\rangle\rangle_b = \langle 1, \beta\rangle_b$ are called bilinear 1-fold Pfister forms. These forms generate the basic ideal IF of WF. Powers of IF are denoted by I^nF. The tensor products $\langle\langle \beta_1,\ldots,\beta_n\rangle\rangle_b = \langle\langle \beta_1\rangle\rangle_b \otimes \cdots \otimes \langle\langle \beta_n\rangle\rangle_b$ are called bilinear n-fold Pfister forms.

The quadratic form $[1,\alpha]$ is called a quadratic 1-fold Pfister form and denoted by $\langle\langle \alpha]]$. For any quadratic form φ and $\beta_1,\ldots,\beta_n \in F^\times$,

$$\langle \beta_1,\ldots,\beta_n\rangle_b \otimes \varphi = \beta_1\varphi \perp \cdots \perp \beta_n\varphi.$$

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For any integer $n \geqslant 2$, we define the quadratic n-fold Pfister form $\langle\langle\beta_1,\ldots,\beta_{n-1},\alpha]]$ as $\langle\langle\beta_1,\ldots,\beta_{n-1}\rangle\rangle_b \otimes \langle\langle\alpha]]$. A quadratic Pfister form is isotropic if and only if it is hyperbolic, and a bilinear Pfister form is isotropic if and only if it is metabolic. We define $I_q^n F$ to be the group generated by the scalar multiples of quadratic n-fold Pfister forms.

A quadratic n-fold Pfister form $\varphi = \langle\langle\beta_1,\ldots,\beta_{n-1},\alpha]]$ over F decomposes as $\varphi = [1,\alpha] \perp \varphi''$. The quadratic form $\varphi' = \langle 1\rangle \perp \varphi''$ is independent of the choice of presentation of φ and is called the 'pure part' of φ. A bilinear form $B = \langle\langle\beta_1,\ldots,\beta_n\rangle\rangle_b$ over F decomposes as $B = \langle 1\rangle_b \perp B'$ for a unique symmetric bilinear form B' called the 'pure part' of B.

3. Bilinear Pfister forms

Suppose that $\mathrm{char}(F) = 2$. We define the 2-rank of F, denoted by $\mathrm{rank}_2(F)$, to be $\log_2([F : F^2])$. It is known to be an integer. For any finitely generated field extension L/F, we have $\mathrm{rank}_2(L) = \mathrm{rank}_2(F) + \mathrm{tr.deg}(L/F)$ (see [16, Lemma 2.7.2]). By [13, Example 6.5], a given bilinear n-fold Pfister form $\langle\langle\beta_1,\ldots,\beta_n\rangle\rangle_b$ is anisotropic if and only if $\log_2([F^2(\beta_1,\ldots,\beta_n) : F^2]) = n$. As a result, if $\mathrm{rank}_2(F) = r$ then $I^n F \neq 0$ for all $n \leqslant r$ and $I^n F = 0$ for all $n > r$.

Theorem 3.1. *Let F be a field of $\mathrm{char}(F) = 2$ and $\mathrm{rank}_2(F) = n$ for some integer $n \geqslant 2$. Then for any $m \in \{1,\ldots,n-1\}$, every collection of $2^{n-m+1} - 1$ anisotropic bilinear n-fold Pfister forms has a bilinear m-fold Pfister form as a common factor.*

Proof. Write $N = 2^{n-m+1} - 1$. Consider N anisotropic bilinear n-fold Pfister forms $B_1,\ldots,B_N$. Let i be an integer in $\{0,\ldots,m-1\}$. Suppose there exists a bilinear i-fold Pfister form ρ such that $B_\ell = \rho \otimes \pi_\ell$ for some $(n-i)$-fold Pfister forms π_ℓ for all $\ell \in \{1,\ldots,N\}$. For each ℓ, $D(\rho \otimes \pi'_\ell)$ is an F^2-vector subspace of F of dimension $2^n - 2^i$. Since $2^i N \leqslant 2^n - 2^i < 2^n$, the spaces $D(\rho \otimes \pi'_\ell)$ for $\ell \in \{1,\ldots,N\}$ have a nontrivial intersection. Hence, by [13, Proposition 6.15], there exists $\beta \in F^\times$ such that $B_\ell = \rho \otimes \langle\langle\beta\rangle\rangle_b \otimes \psi_\ell$ for some bilinear $(n-i-1)$-fold Pfister forms ψ_ℓ for all $\ell \in \{1,\ldots,N\}$. The statement then follows by induction. □

Corollary 3.2. *Let F be a field of $\mathrm{char}(F) = 2$ with $I^n F \neq 0$ for some $n \geqslant 2$. Then $I^n F$ is 3-linked if and only if $\mathrm{rank}_2(F) = n$.*

Proof. Suppose every triple of anisotropic bilinear n-fold Pfister forms over F have a common $(n-1)$-fold Pfister factor. By [5, Theorem 5.1], $I^{n+1}F = 0$, and therefore $\mathrm{rank}_2(F) \leqslant 2^n$. Since $I^n F \neq 0$, it follows that $\mathrm{rank}_2(F) = n$. The opposite direction is Theorem 3.1 with $m = n-1$. □

If we substitute $m = n-1$ in Theorem 3.1, then it says that when $\mathrm{rank}_2(F) = n \geqslant 2$, every $2^n - 1$ anisotropic bilinear n-fold Pfister forms have a common bilinear 1-fold Pfister factor, that is, a common slot. The following theorem shows that this bound is sharp by providing 2^n bilinear n-fold Pfister forms that do not have a common slot.

Theorem 3.3. *Let F be a field of* $\text{char}(F) = 2$ *with* $\text{rank}_2(F) = n$ *for some* $n \geqslant 2$. *Then there exist* 2^n *anisotropic bilinear* n*-fold Pfister forms with no common slot.*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be a 2-basis of F (meaning that $F = F^2(\alpha_1, \ldots, \alpha_n)$). Write $I = \{0, 1\}^{\times n}$, $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{d} = (d_1, \ldots, d_n)$ for an arbitrary element in I and $\alpha^{\mathbf{d}}$ for $\prod_{i=1}^n \alpha_i^{d_i}$. For every $\mathbf{d} \in I \setminus \{\mathbf{0}\}$, let $B_{\mathbf{d}}$ be $\langle\langle \alpha_1, \ldots, \widehat{\alpha_\ell}, \ldots, \alpha_n \rangle\rangle_b \otimes \langle\langle 1 + \alpha^{\mathbf{d}} \rangle\rangle_b$ where ℓ is the minimal integer in $\{1, \ldots, n\}$ for which $d_\ell \neq 0$. For every $\mathbf{e} \in I \setminus \{\mathbf{0}\}$ with $e_\ell = 0$, both $\alpha^{\mathbf{e}}$ and $\alpha^{\mathbf{e}}(1 + \alpha^{\mathbf{d}}) = \alpha^{\mathbf{e}} + \alpha^{\mathbf{e}+\mathbf{d}}$ are in $D(B'_{\mathbf{d}})$, and so also $\alpha^{\mathbf{e}+\mathbf{d}} \in D(B'_{\mathbf{d}})$. Therefore, the elements $\{\alpha^{\mathbf{e}} : \mathbf{e} \in I \setminus \{\mathbf{0}, \mathbf{d}\}\} \cup \{1 + \alpha^{\mathbf{d}}\}$ are all in $D(B'_{\mathbf{d}})$ and, since they are linearly independent over F^2 and $D(B'_{\mathbf{d}})$ is of dimension $2^n - 1$ over F^2, they form a basis of $D(B'_{\mathbf{d}})$ over F^2.

Let $B_{\mathbf{0}}$ be $\langle\langle \alpha_1, \ldots, \alpha_n \rangle\rangle_b$, so that $D(B'_{\mathbf{0}})$ is spanned over F^2 by $\{\alpha^{\mathbf{e}} : \mathbf{e} \in I \setminus \{\mathbf{0}\}\}$. Since $D(B_d)$ for all $d \in I$ are of dimension 2^n over F^2, they are anisotropic by [13, Example 6.5]. By elementary linear algebra, for any given $\mathbf{d} \in I \setminus \{\mathbf{0}\}$, the intersection $D(B'_{\mathbf{0}}) \bigcap D(B'_{\mathbf{d}})$ is spanned by $\{\alpha^{\mathbf{e}} : \mathbf{e} \in I \setminus \{\mathbf{0}, \mathbf{d}\}\}$ and so the intersection $\bigcap_{\mathbf{d} \in I} D(B'_{\mathbf{d}})$ is trivial. This means the pure parts of the bilinear n-fold Pfister forms $\{B_{\mathbf{d}} : \mathbf{d} \in I\}$ do not represent a common element and so they have no slot in common. □

This means the answer to Question 1.1 is always negative. Fields of 2-rank n are easily provided: take any perfect field F_0 of $\text{char}(F) = 2$, and let F be either the function field $F_0(\alpha_1, \ldots, \alpha_n)$ in n algebraically independent variables over F_0, or the field of iterated Laurent series $F_0((\alpha_1)) \ldots ((\alpha_n))$ in n variables over F_0.

The situation in quadratic forms is more complicated, as we shall see in the next section. This is a good opportunity to point out another surprising difference between quadratic forms and symmetric bilinear forms in characteristic 2.

Proposition 3.4. *Suppose two given anisotropic bilinear* n*-fold Pfister forms* B_1 *and* B_2 *over* F *with* $\text{char}(F) = 2$ *satisfy the following property: for every* $\alpha \in F^\times$, $\langle\langle \alpha \rangle\rangle$ *is a factor of* B_1 *if and only if it is a factor of* B_2. *Then* $B_1 \simeq B_2$.

Proof. For every $\alpha \in F^\times$, $\langle\langle \alpha \rangle\rangle$ is a factor of B_1 if and only if α is represented by B'_1. If α is represented by B'_1 if and only if it is represented by B'_2 for every α, it means that $D(B'_1) = D(B'_2)$, that is, $D(B'_1)$ and $D(B'_2)$ are the same $(2^n - 1)$-dimensional F^2-subspace V of F. Let ρ be a common i-fold factor of B_1 and B_2. Write $B_1 = \rho \otimes \psi_1$ and $B_2 = \rho \otimes \psi_2$. The spaces $D(\rho \otimes \psi'_1)$ and $D(\rho \otimes \psi'_2)$ are $(2^n - 2^i)$-dimensional F^2-subspaces of V. If $i \leqslant n - 1$ then they have a nonzero intersection, because $[V : F^2] = 2^n - 1$. Let β be a nonzero element in the intersection. By [13, Proposition 6.15], $\rho \otimes \langle\langle \beta \rangle\rangle$ is a common factor of B_1 and B_2. This works for every $i \in \{1, \ldots, n-1\}$. Therefore, we obtain by induction that $B_1 \simeq B_2$. □

This is not true for quadratic n-fold Pfister forms, which can share all 1-fold factors (either bilinear or quadratic, or both) without being isomorphic (see [8] for reference).

Remark 3.5. In this section we focused on anisotropic bilinear n-fold Pfister forms. From [1, page 909], an isotropic bilinear n-fold Pfister form B decomposes as $B = \langle\langle \underbrace{1, \ldots, 1}_{k \text{ times}} \rangle\rangle_b \otimes B_1$, where B_1 is an anisotropic bilinear $(n-k)$-fold Pfister form and

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$D(B) = D(B_1)$ for some unique integer k. However, B_1 is not unique and there can certainly be a different anisotropic bilinear Pfister form B_2 such that $B = \langle\langle 1\rangle\rangle_b^k \otimes B_2$ as well. For example, take an anisotropic $B_1 = \langle\langle x\rangle\rangle_b$, $B_2 = \langle\langle x+1\rangle\rangle_b$ and $B = \langle\langle 1, x\rangle\rangle_b$ (see also [1, Proposition A.8]). The situation is therefore more fluid when it comes to isotropic forms. In addition, anisotropic bilinear n-fold Pfister forms represent nonzero classes in I^nF and are mapped to nonzero classes in the Milnor K-groups $K_nF/2K_nF$, while all the isotropic n-fold Pfister forms are trivial in I^nF and mapped to zero by the isomorphism $I^nF/I^{n+1}F \cong K_nF/2K_nF$ from [17], which gives anisotropic forms greater significance in the algebraic theory of bilinear forms, K-theory and in general.

4. Quadratic Pfister forms

In this section we provide a negative answer to Question 1.2, and to Question 1.3 in all cases but $n = 2$. The technique is to study the common quadratic inseparable splitting fields of quadratic n-fold Pfister forms. Given an anisotropic quadratic n-fold Pfister form φ over F and an inseparable quadratic field $K = F[\sqrt{\gamma}]$, φ_K is isotropic if and only if the bilinear 1-fold Pfister form $\langle\langle\gamma\rangle\rangle_b$ is a factor of φ. Given a quadratic form $\varphi : V \to F$, a subform ψ of φ is the restriction of φ to some subspace W of V.

Lemma 4.1. *If $F[\sqrt{\gamma}]$ is a splitting field of an anisotropic quadratic n-fold Pfister form φ over F, then $\langle 1, \gamma\rangle$ is a subform of φ'.*

Proof. This follows from [8, Proposition 3.2]. □

We focus on valued fields with a sufficiently large value group. For general reference on valuation theory see [19].

Lemma 4.2 [10, Lemma 10.1]. *Let $n \geqslant 2$ and F be a field of $\mathrm{char}(F) = 2$ with a valuation $\mathfrak{v}$ onto the totally ordered group Γ. Write $\overline{\mathfrak{v}}$ for the function mapping each $q \in F^\times$ to the class of q in $\Gamma/2\Gamma$. Let $\alpha_1, \dots, \alpha_n$ be elements in $F^\times$ of negative values whose images under $\overline{\mathfrak{v}}$ are linearly independent over $\mathbb{F}_2$, and consider the quadratic n-fold Pfister form $\varphi = \langle\langle \alpha_1, \dots, \alpha_n]]$ with underlying vector space V with basis $\{v_{\mathbf{d}} : \mathbf{d} \in I\}$, where I, $\mathbf{d}$ and $\alpha^{\mathbf{d}}$ are the same as in Theorem 3.3. Then:*

(a) *for every $v = \sum_{\mathbf{d}\in I} c_{\mathbf{d}} v_{\mathbf{d}} \in V$, $\mathfrak{v}(\varphi(v)) = \mathfrak{v}(\varphi(c_{\mathbf{d}} v_{\mathbf{d}}))$ for some specific $\mathbf{d} \in I$ with $c_{\mathbf{d}} \neq 0$;*
(b) *φ is anisotropic;*
(c) *$\overline{\mathfrak{v}}(D(\varphi))$ is the $\mathbb{F}_2$-subspace of $\Gamma/2\Gamma$ spanned by $\{\overline{\mathfrak{v}}(\alpha_i) : i \in \{1, \dots, n\}\}$.*

Corollary 4.3 [10, Corollary 10.2]. *For any two-dimensional subspace U of V, there exists an element in U whose image under $\overline{\mathfrak{v}}$ is nonzero.*

Corollary 4.4. *If $\overline{\mathfrak{v}}(D(\varphi)) = \Gamma/2\Gamma$, then the form φ described in Lemma 4.2 satisfies $\{\overline{\mathfrak{v}}(q) : q \in D(\varphi')\} = \Gamma/2\Gamma \setminus \{\overline{\mathfrak{v}}(\alpha_n)\}$.*

Proof. This follows immediately from the fact that φ' is the restriction of φ to the subspace of V spanned by $\{v_{\mathbf{d}} : \mathbf{d} \in I \setminus \{(0, \dots, 0, 1)\}\}$ and from the linear independence of the images of $\alpha_1, \dots, \alpha_n$ under $\overline{\mathfrak{v}}$ over $\mathbb{F}_2$. □

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THEOREM 4.5. *Let $n \geqslant 2$ be an integer and F be a field of* $\mathrm{char}(F) = 2$ *with a discrete rank-n valuation. Then there exist $2^n - 1$ quadratic n-fold Pfister forms with no common quadratic inseparable splitting field.*

PROOF. Write $\mathfrak{v}$ for the valuation and $\Gamma(\cong \mathbb{Z}^{\times n})$ for the group. Write $\overline{\mathfrak{v}}$ for the function mapping each $q \in F^\times$ to the class of q in $\Gamma/2\Gamma$. Let $\alpha_1, \ldots, \alpha_n$ be elements in $F^\times$ of negative values whose images under $\overline{\mathfrak{v}}$ are linearly independent over $\mathbb{F}_2$. Let I, $\mathbf{0}$, $\mathbf{d}$ and $\alpha^{\mathbf{d}}$ be the same as in Theorem 3.3. For every $\mathbf{d} \in I \backslash \{\mathbf{0}\}$, let $\varphi_{\mathbf{d}}$ be $\langle\langle \alpha_1, \ldots, \widehat{\alpha_\ell}, \ldots, \alpha_n \rangle\rangle \otimes \langle\langle \alpha^{\mathbf{d}}]]$ where ℓ is the minimal integer in $\{1, \ldots, n\}$ for which $d_\ell \neq 0$. We will show that the forms $\{\varphi_{\mathbf{d}} : \mathbf{d} \in I \backslash \{\mathbf{0}\}\}$ do not have a common inseparable quadratic splitting field.

By Corollary 4.4, $\overline{\mathfrak{v}}(D(\varphi'_{\mathbf{d}})) = \Gamma/2\Gamma \backslash \overline{\mathfrak{v}}(\alpha^{\mathbf{d}})$ for each $\mathbf{d} \in I \backslash \{\mathbf{0}\}$. Therefore

$$\bigcap_{\mathbf{d} \in I \backslash \{\mathbf{0}\}} \overline{\mathfrak{v}}(D(\varphi'_{\mathbf{d}})) = \{\overline{\mathbf{0}}\}.$$

However, by Lemma 4.1, if the forms $\varphi_{\mathbf{d}}$ have a common inseparable quadratic splitting field then the forms $\varphi'_{\mathbf{d}}$ have a common two-dimensional subform. All the elements q represented by this two-dimensional subform must satisfy $\overline{\mathfrak{v}}(q) = \overline{\mathbf{0}}$, which contradicts Corollary 4.3. □

Note that the forms appearing in the statement of Theorem 4.5 do not have a bilinear $(n-1)$-fold Pfister form as a common factor, because they do not even share one inseparable quadratic splitting field. When $n \geqslant 3$ these forms do not have a quadratic $(n-1)$-fold Pfister form as a common factor for the same reason.

For the construction of counterexamples for Question 1.2 we need a necessary condition for $I_q^n F$ to be separably 3-linked.

LEMMA 4.6. *Let $\varphi = \langle\langle a_1, \ldots, a_n]]$ be an n-fold quadratic Pfister form over a field F with* $\mathrm{char}(F) = 2$. *Write $\varphi = [a_n, 1] \perp \varphi''$ and consider $d \in D(\varphi)$ such that*

$$d = \varphi(w, x, u_1, \ldots, u_{2^n-2}) = a_n w^2 + wx + x^2 + \varphi''(u_1, \ldots, u_{2^n-2})$$

for some $w, x, u_1, \ldots, u_{2^n-2} \in F$ with $w \neq 0$. Then there exist $b_1, \ldots, b_{n-1} \in F^\times$ such that $\varphi = \langle\langle b_1, \ldots, b_{n-1}, d/w^2]]$.

PROOF. Let v_1 be the vector $(1, x/w, u_1/w, \ldots, u_{2^n-2}/w)$ and v_2 be the vector $(0, 1, 0, \ldots, 0)$. Then the subform $\varphi|_{Fv_1 + Fv_2}$ is isometric to

$$[a_n + x/w + x^2/w^2 + \varphi''(u_1/w, \ldots, u_{2^n-2}/w), 1].$$

By [3, Ch. 4, Lemma 4.1], there exist $b_1, \ldots, b_{n-1} \in F$ such that

$$\varphi = \langle\langle b_1, \ldots, b_{n-1}, a_n + x/w + x^2/w^2 + \varphi''(u_1/w, \ldots, u_{2^n-2}/w)]]. \qquad \square$$

Recall that $u(F)$ is the maximal dimension of an anisotropic nonsingular quadratic form over F (see [13, page 163]).

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Proposition 4.7 (Cf. [5, Corollary 5.4]). *Let $n \geqslant 3$ be an integer and F be a field of* $\mathrm{char}(F) = 2$ *such that the function field $K = F(t)$ in one variable over F has $u(K) \leqslant 2^{n+1}$. Then $I_q^n F$ is separably 3-linked.*

Proof. Let φ_1, φ_2 and φ_3 be three anisotropic quadratic n-fold Pfister forms over F. Write $\varphi_i = [1, \alpha_i] \perp \varphi_i''$ for $i \in \{1, 2, 3\}$. The system of two quadratic equations

$$\alpha_1 w^2 + \varphi_1''(v_1) = \alpha_2 w^2 + w x_2 + x_2^2 + \varphi_2''(v_2),$$
$$\alpha_1 w^2 + \varphi_1''(v_1) = \alpha_3 w^2 + w x_3 + x_3^2 + \varphi_3''(v_3)$$

has a solution over F if and only if the quadratic form

$$\psi : K \times K \times K \times K^{\times(2^n-2)} \times K^{\times(2^n-2)} \times K^{\times(2^n-2)} \to K$$

mapping $(w, x_2, x_3, v_1, v_2, v_3)$ to

$$\alpha_1 w^2 + \varphi_1''(v_1) + \alpha_2 w^2 + w x_2 + x_2^2 + \varphi_2''(v_2)$$
$$+ t(\alpha_1 w^2 + \varphi_1''(v_1) + \alpha_3 w^2 + w x_3 + x_3^2 + \varphi_3''(v_3))$$

is isotropic by [13, Theorem 17.14]. The form ψ is of dimension $3 \cdot (2^n - 1)$, which is greater than 2^{n+1}. Therefore ψ is isotropic ($u(K) \leqslant 2^{n+1}$), and so the system above has a solution over F. If in this solution $w \neq 0$ then by Lemma 4.6 the forms φ_1, φ_2 and φ_3 have a common right slot, that is, $\varphi_i = \rho_i \otimes \langle\langle \alpha]]$ for $i \in \{1, 2, 3\}$ for some $\alpha \in F$ and bilinear $(n-1)$-fold Pfister forms ρ_1, ρ_2 and ρ_3. If the solution has $w = 0$ then, by [2, Lemma 3.5], φ_1, φ_2 and φ_3 have a common bilinear 1-fold Pfister form as a common factor. By [11, Corollary 6.2], since $n \geqslant 3$, the forms φ_1, φ_2 and φ_3 also have a common right slot, so they have a common right slot regardless of w.

Write $\varphi_i = B_i \otimes \rho$, $i \in \{1, 2, 3\}$, for some bilinear $(n-k)$-fold Pfister forms B_1, B_2, B_3 and some quadratic k-fold Pfister form ρ, where k is an integer in $\{1, \ldots, n-2\}$. The system of two equations

$$(B_1' \otimes \rho)(v_1) = (B_2' \otimes \rho)(v_2),$$
$$(B_1' \otimes \rho)(v_1) = (B_3' \otimes \rho)(v_3)$$

has a solution over F if and only if the quadratic form

$$\theta : K^{\times(2^n-2^k)} \times K^{\times(2^n-2^k)} \times K^{\times(2^n-2^k)}$$

mapping (v_1, v_2, v_3) to

$$(B_1' \otimes \rho)(v_1) + (B_2' \otimes \rho)(v_2) + t((B_1' \otimes \rho)(v_1) = (B_3' \otimes \rho)(v_3))$$

is isotropic by [13, Theorem 17.14]. The dimension of θ is $3 \cdot (2^n - 2^k)$, which is greater than 2^{n+1} because $k \leqslant n-2$. Therefore by [2, Lemma 3.5] there exists $\gamma \in F^\times$ such that $\langle\langle \gamma \rangle\rangle \otimes \rho$ is a common factor of φ_1, φ_2 and φ_3. The statement then follows by induction. □

We are now ready to give negative answers to Questions 1.2 and 1.3.

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Example 4.8. Let F_0 be an algebraically closed field of $\mathrm{char}(F_0) = 2$, such as the separable closure of $\mathbb{F}_2$, and let F be either the function field $F_0(\alpha_1, \ldots, \alpha_n)$ in n algebraically independent variables, or the field $F_0((\alpha_1^{-1}))\ldots((\alpha_n^{-1}))$ of iterated Laurent series in n variables over F_0. In these cases the maximal dimension of an anisotropic form in $I_q^n F$ is 2^n, so $I_q^n F$ is inseparably 2-linked. However, F has a discrete rank-n valuation. Therefore there exist $(2^n - 1)$ quadratic n-fold Pfister forms without a common quadratic inseparable splitting field, providing a negative answer to Question 1.2. Moreover, these fields are C_n fields [13, Section 97], and therefore $u(F(t)) = 2^{n+1}$. By Proposition 4.7, when $n \geqslant 3$, $I_q^n F$ is separably 3-linked. However, $I_q^n F$ is not separably $(2^n - 1)$-linked for the reason mentioned above, giving a negative answer to Question 1.3 (when $n \geqslant 3$).

Our ability to answer Question 1.3 when $n \geqslant 3$ relies heavily on the fact that when $n \geqslant 3$, quadratic n-fold Pfister forms with a common quadratic $(n-1)$-fold Pfister factor must have a common inseparable quadratic splitting field. This is certainly not true for $n = 2$, and we leave Question 1.3 open in this case. The existence of inseparable quadratic field extensions is special to the case of $\mathrm{char}(F) = 2$, so our techniques do not apply (at least not in an obvious manner) to the more common case of $\mathrm{char}(F) \neq 2$.

5. Quaternion algebras

Given a field F of $\mathrm{char}(F) = 2$, a quaternion algebra over F is of the form

$$(\beta, \alpha]_{2,F} = F\langle x, y : x^2 + x = \alpha, y^2 = \beta, yxy^{-1} = x + 1\rangle$$

for some $\alpha \in F$ and $\beta \in F^\times$. There is a one-to-one correspondence between quaternion algebras $(\beta, \alpha]_{2,F}$ and their norm forms $\langle\langle \beta, \alpha]]$ which are quadratic 2-fold Pfister forms (see [13, Section 12] and [8, Section 6]). In particular, the splitting fields of the quaternion algebra and its norm form are the same. We therefore obtain the following theorem.

Theorem 5.1. *Let F be a field of $\mathrm{char}(F) = 2$ with a valuation $\mathfrak{v}$ of rank 2 and value group $\Gamma(\cong \mathbb{Z} \times \mathbb{Z})$. Write $\overline{\mathfrak{v}}$ for the function mapping each $q \in F^\times$ to the class of q in $\Gamma/2\Gamma$. Let α, β be elements in $F^\times$ of negative values whose images under $\overline{\mathfrak{v}}$ are linearly independent over $\mathbb{F}_2$. Let $Q_1 = (\beta, \alpha]_{2,F}$, $Q_2 = (\alpha, \beta]_{2,F}$ and $Q_3 = (\beta, \alpha\beta]_{2,F}$. Then Q_1, Q_2 and Q_3 do not have a common inseparable quadratic splitting field.*

Fields F with $u(F) = 4$ are the fields over which every pair of quaternion algebras share an inseparable quadratic splitting field [4, Theorem 3.1]. If every triple of quaternion algebras over F share an inseparable quadratic splitting field, it does not affect the value of $u(F)$. Nevertheless, there exist fields that do not have this property while still having $u(F) = 4$.

Example 5.2. Let F_0 be an algebraically closed field of $\mathrm{char}(F_0) = 2$. Let F be either the function field $F_0(\alpha, \beta)$ in two algebraically independent variables over F_0, or the

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field of iterated Laurent series $F_0((\alpha^{-1}))((\beta^{-1}))$ in two variables over F_0. Then every pair of quaternion algebras over F shares a quadratic inseparable splitting field, but not every triple.

Proof. The field F in both cases is a C_2 field (see [13, Section 97]) with nontrivial quaternion algebras, and so $u(F) = 4$. Therefore every pair of quaternion algebras over F share a quadratic inseparable splitting field. However, by Theorem 5.1 there exist three quaternion algebras that do not share a quadratic inseparable splitting field. □

There are still fields over which every collection of quaternion algebras shares a quadratic inseparable splitting field, as the following example demonstrates. This means that unlike Question 1.1, the answer to Question 1.2 is not always negative.

Example 5.3. Let F_0 be a perfect field of $\mathrm{char}(F_0) = 2$ with nontrivial $\acute{\mathrm{E}}\mathrm{t}_2(F)$ (for example, any finite field). Let F be either the function field $F_0(\alpha)$ in one variable over F_0, or the field of Laurent series $F_0((\alpha))$ over F_0. Then any finite number of quaternion algebras over F share a quadratic inseparable splitting field, because F has a unique quadratic inseparable field extension.

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ADAM CHAPMAN, Department of Computer Science,
Tel-Hai Academic College, Upper Galilee, 12208, Israel
e-mail: adam1chapman@yahoo.com

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