A NOTE ON LOCALLY QUASI-UNIFORM SPACES

BY

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ABSTRACT. Locally quasi-uniform spaces are studied, and it is shown that a topological space \((X, t)\) admits exactly one compatible locally quasi-uniform structure if and only if \(t\) is finite.

1. Introduction. Topological spaces with a unique compatible uniform structure have been characterized by R. Doss [2]. In [3], P. Fletcher initiated the study of spaces with a unique compatible quasi-uniform structure, and he conjectured that \((X, t)\) admits exactly one compatible quasi-uniform structure if and only if \(t\) is finite. C. Barnhill and P. Fletcher [1] showed that if \(t\) is finite, then \((X, t)\) is uniquely quasi-uniformizable. In [6] and [7], W. Lindgren gave examples where \((X, t)\) is uniquely quasi-uniformizable with \(t\) infinite, and showed that the conjecture holds for \(R_t\) spaces. The concept of locally quasi-uniform spaces was defined for \(T_x\) spaces in [5], and it was shown that \((X, t)\) admits a local quasi-uniformity with a countable base if and only if it is a \(\gamma\) space if and only if it is a Nagata first countable space.

A general introduction to quasi-uniform spaces may be found in [8].

2. Locally quasi-uniform spaces.

DEFINITION 1. Let \(X\) be a non-empty set and let \(\mathcal{U}\) be a filter on \(X \times X\) such that:

(i) \(\Delta \subseteq U\) for every \(U \in \mathcal{U}\), where \(\Delta = \{(x, x) : x \in X\}\).
(ii) For each \(x \in X\) and \(U \in \mathcal{U}\), there exists \(V(x, U) = V \in \mathcal{U}\) such that \((V \circ V)[x] \subseteq U[x]\). Then \(\mathcal{U}\) is called a locally quasi-uniform structure for \(X\).

\(\mathcal{U}\) gives a topology

\[ t_\mathcal{U} = \{A \subseteq X : \text{for every } x \in A \text{ there exists } U \in \mathcal{U} \text{ such that } U[x] \subseteq A\}. \]

It is clear that every quasi-uniform structure is a locally quasi-uniform structure. If we use a term without defining it, we are using the quasi-uniform space definition. We say that \((X, \mathcal{U})\) is strongly complete if every Cauchy filter converges.

LEMMA 1. Let \((X, t)\) be a topological space and let

\[ \mathcal{B} = \{U : U \supseteq \Delta \text{ and for every } x \in X, U[x] \in t\}. \]

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Then $B$ is a base for a locally quasi-uniform structure $FL$, and $FL$ is the finest compatible structure.

**Proof.** If $V \in B$ and $x \in X$, put

$$U = [(X - V[x]) \times X] \cup (V[x] \times V[x]).$$

If $y \in X$, $U[y] = X$ or $V[x]$. Thus $U \in B$. Also, $(U \circ U)[x] = V[x]$. Hence $B$ is a base for a structure $FL$. If $U$ is a compatible structure and $U \in U$, $U \supseteq W$, the interior of $U$ in $t^{-1} \times t$. $W[x] \subseteq t$ for every $x \in X$ gives $W \in FL$. Thus $U \subseteq FL$.

**Definition 2.** [4] A locally quasi-uniform space $(X, U)$ has the Lebesgue property provided that if $U$ is a $t_{U}$-open cover of $X$, then there is a $U \in U$ such that $\{U[x]: x \in X\}$ refines $U$.

**Theorem 1.** Let $(X, U)$ be a locally quasi-uniform space.

1. If $(X, U)$ has the Lebesgue property, then $(X, U)$ is strongly complete.
2. $FL$ is a compatible strongly complete locally quasi-uniform structure.
3. $U$ is pre-compact if and only if every ultrafilter on $X$ is $U$-Cauchy.
4. If $t_{U}$ is compact, every Cauchy filter converges.
5. $(X, t_{U})$ is compact if and only if $U$ is strongly complete and pre-compact.
6. $(X, t_{U})$ is pre-compact if and only if $FL$ is strongly complete.
7. Suppose $FL$ is pre-compact. By (2), $FL$ is a compatible strongly complete structure. By (5), $t_{U} = t_{FL}$ is compact.

**Proof.** For (1), we note that the proof in [4] for quasi-uniform spaces carries over.

For (2), we show that $FL$ has the Lebesgue property and apply (1). If $U$ is an open cover and $x \in X$, then there exists $C_{x} \in U$ such that $x \in C_{x}$. Let

$$U = \bigcup \{\{x\} \times C_{x}: x \in X\}.$$ 

$U \in FL$ and $\{U[x]: x \in X\}$ refines $U$.

For (3) and (4), we note that the standard quasi-uniform space arguments hold.

(5) follows from (3) and (4).

(6) follows from (5) and the fact that the Pervin structure is pre-compact.

(7) Suppose $FL$ is pre-compact. By (2), $FL$ is a compatible strongly complete structure. By (5), $t_{U} = t_{FL}$ is compact.

**Lemma 2.** Suppose $(X, t)$ has a finite compatible locally quasi-uniform structure $U$. Then $FL = U$, and therefore $t$ has only one compatible locally quasi-uniform structure.

**Proof.** $U = \bigcap \{V: V \in U\} \in U$ gives $U = \{V: V \supseteq U\}$. Clearly, $U[x]$ is the smallest open set containing $x$. Also, $U \subseteq FL$ by Lemma 1. If $A \in FL$, $A \supseteq V$

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where \( V(x) \in t \) for every \( x \in X \). Thus \( V(x) \supseteq U(x) \) for every \( x \in X \), and therefore \( A \supseteq V \supseteq U \) gives \( A \in \mathcal{U} \) or \( \mathcal{F} \subseteq \mathcal{U} \).

**Corollary.** If \( t \) is finite, \((X, t)\) has only one compatible locally quasi-uniform structure.

**Proof.** If \( t \) is finite the Pervin structure is a finite compatible quasi-uniform structure.

**Definition 3.** If \( \{G_n\} \) is a sequence of open sets and \( G_1 \subset G_2 \subset G_3 \subset \ldots \), it is called an *ascending sequence* of open sets.

**Lemma 3.** Let \((X, t)\) be a topological space with \( t \) infinite. There exists an ascending infinite sequence of open sets or there exists a descending infinite sequence of open sets.

**Proof.** Suppose \( t \) contains no ascending infinite sequence of open sets. Then \( t - \{X\} \) has the same property. Thus every \( A \) in \( t - \{X\} \) is contained in a maximal ascending chain in \( t - \{X\} \). Let

\[ \mathcal{M} = \{M : M \text{ is a maximal ascending chain in } t - \{X\}\}. \]

If \( M \in \mathcal{M} \), let \( V_M = \bigcup \{V : V \in M\} \). Then \( V_M \in t - \{X\} \). If \( V_M \neq V_{M_1} \), \( V_{M_1} \cup V_M = X \). For otherwise, \( M_1 \) would not be maximal in \( t - \{X\} \). Let \( \mathcal{V} \) denote the set of distinct \( V_M, M \in \mathcal{M} \).

Case 1. \( \mathcal{V} \) is infinite. We show that \( \mathcal{V} \) has the finite intersection property. If \( \Phi = \bigcap_{i=1}^n V_{M_i}, \) choose \( V_M \neq V_{M_i}, \) \( 1 \leq i \leq n \). Then \( V_M = V_M \cup \Phi = V_M \cup (\bigcap_{i=1}^n V_{M_i}) = \bigcap_{i=1}^n (V_M \cup V_{M_i}) = \bigcap_{i=1}^n X = X \), a contradiction. We show that \( \bigcap_{i=1}^n V_{M_i} \neq \bigcup_{i=1}^n V_{M_i} \). If equality holds, \( \bigcap_{i=1}^n V_{M_i} \subseteq V_{M_i} \). Thus \( V_M = V_{M_i} \cup (\bigcap_{i=1}^n V_{M_i}) = X \), a contradiction. Since \( \mathcal{V} \) is infinite, put \( X_n = \bigcap_{i=1}^n V_{M_i} \) and \( \{X_n\} \) is a descending infinite sequence of open sets.

Case 2. \( \mathcal{V} \) is finite. Since \( t - \{X\} \) is infinite and each \( V \in t - \{X\} \) is contained in \( V_M \) for some \( V_M \in \mathcal{V} \), there exists \( V_M \in \mathcal{V} \) such that an infinite number of members of \( t - \{X\} \) are contained in \( V_M \). Put \( V_M = X_1 \). Let \( t_1 = \{V : V \in t - \{X\} \text{ and } V \subseteq X_1\} \).

\( t_1 \) is an infinite topology for \( X_1 \). Repeat the argument just given for \((X_1, t_1)\) and obtain a topological space \((X_2, t_2)\) such that: (1) \( t_2 \) is infinite, (2) \( X_2 \subset X_1 \subset X \), and (3) \( X_2 \in t - \{X_1\} \subset t - \{X\} \). Using induction we obtain a descending infinite sequence of open sets.

**Theorem 2.** A topological space \((X, t)\) is uniquely locally quasi-uniformizable if and only if \( t \) is finite.

**Proof.** The Corollary gives one half of the theorem. Suppose \( t \) is infinite.
Case 1. \( t \) has a descending infinite sequence of open sets. We obtain a sequence \( \{a_n\} \) of distinct points and a sequence \( \{X_n\} \) of distinct open sets such that \( a_n \in X_n \) and \( a_n \notin X_m \) for \( m > n \). If \((X, t)\) is uniquely locally quasi-uniformizable, \( \mathcal{F} \) is the Pervin structure, and therefore \( \mathcal{F} \) is totally bounded. Let

\[
U = \bigcup_{i=1}^{n} (\{a_i\} \times X_i) \cup \left( X - \bigcup_{i=1}^{\infty} \{a_i\} \right) \times X.
\]

\( U \in \mathcal{F} \) so there exists \( A_1, \ldots, A_n \) such that \( \bigcup_{i=1}^{n} A_i = X \) and \( A_i \times A_j \subseteq U \). There exists \( j, 1 \leq j \leq n \), such that \( A_j \) contains infinitely many elements of \( \{a_n\} \). Choose \( m > n \) such that \( a_n, a_m \in A_j \). We have \( (a_m, a_n) \in A_j \times A_j \subseteq U \) or \( a_n \in U[a_m] = X_m \), a contradiction.

Case 2. \( t \) has an ascending infinite sequence of open sets. We obtain a sequence \( \{a_n\} \) of distinct points and a sequence \( \{X_n\} \) of distinct open sets such that \( a_n \in X_n \) and \( a_n \notin X_m \) for \( m < n \). Now use the argument in Case 1.

**Remark.** After looking at Lemma 3, one might wonder when \((X, t)\) has an ascending infinite sequence of open sets. In [6], it is shown that every subset of \((X, t)\) is compact if and only if every ascending open sequence is finite.

**References**