NORMAL COMPLEMENTS IN FINITE SOLVABLE GROUPS

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1. Introduction

A well known theorem ([1] page 432) in the study of finite groups states that if \( P \) is a Sylow \( p \)-subgroup of the finite group \( G \), and if \( P_0 \) is a normal subgroup of \( P \) such that whenever two elements, \( \sigma \) and \( \tau \), of \( P \) are conjugate in \( G \), the cosets \( \sigma P_0 \) and \( \tau P_0 \) are conjugate in \( P/P_0 \), then there is a normal subgroup \( K \) of \( G \) such that \( G = KP \) and \( K \cap P = P_0 \). In this note we will extend this result to allow \( P \) to be any Hall subgroup if \( G \) is solvable. More precisely, following theorem will be proved.

**Theorem.** Let \( G \) be a finite solvable group, let \( H \) be a Hall subgroup of \( G \) and let \( H_0 \) be a normal subgroup of \( H \) such that whenever \( \sigma \) and \( \tau \) are elements of \( H \) with \( \sigma \) conjugate to \( \tau \) in \( G \), \( \sigma H_0 \) and \( \tau H_0 \) are conjugate in \( H/H_0 \). Then there exists a normal subgroup \( K \) of \( G \) with \( G = KH \) and \( K \cap H = H_0 \).

When \( H_0 \) is the trivial subgroup we get the following special case.

**Corollary 1.** If \( G \) is a finite solvable group and \( H \) is a Hall subgroup of \( G \) such that for \( \sigma, \tau \in H \), \( \sigma \) is conjugate to \( \tau \) in \( G \) if, and only if, \( \sigma \) is conjugate to \( \tau \) in \( H \), then \( H \) has a normal complement in \( G \).

The following result is a direct consequence of Corollary 1.

**Corollary 2.** If \( H \) is a finite solvable group and \( \sigma \) is an automorphism of \( H \) of order a prime \( p \) which preserves conjugacy classes in \( H \), then \( p \mid |H| \).

The converse of the theorem is, of course, also true and easily seen under more general hypotheses. If \( H \) is any subgroup of \( G \), if \( K \triangleleft G \), \( G = KH \), and \( H_0 = K \cap H \), then \( \sigma H_0 \) and \( \tau H_0 \) are \( H/H_0 \)-conjugate whenever \( \sigma \) and \( \tau \) are \( G \)-conjugate.

Corollary 1 and a fortiori the theorem are false if the assumption of solvability is omitted. This is easily seen by examination of the symmetric group on \( p-1 \) letters, \( S_{p-1} \), as a subgroup of \( S_p \), where \( p \) is any prime greater than 3.
Any two elements of \( S_{p-1} \) that are conjugate in \( S_p \) are already conjugate in \( S_{p-1} \), but \( S_p \) has no normal \( p \)-subgroups. These are the only counterexamples known to the author.

The notation in the paper is standard. The expression \( \sigma \sim \tau \) means that there is an element \( \rho \) in \( G \) such that \( \rho^{-1}\sigma \rho = \tau \). \( O_p(G) \) is the maximal normal \( p \)-subgroup of \( G \), and \( O_{p'}(G) \) is the inverse image in \( G \) of the maximal \( p' \)-subgroup of \( G/O_p(G) \). The author would like to thank Professor E. C. Dade for stimulating interest in the question.

2. Proof of the Theorem

The proof of the theorem will be carried out by induction on the order of \( G \). If \( G \) is a minimal counterexample, then \( \lceil [G:H] \rceil \) is a power of a prime \( q \). For by looking at a Sylow system for \( G \) that contains a Sylow system for \( H \), one can find Sylow subgroups \( Q_1, \ldots, Q_k \) such that \( Q_i H \) is a subgroup of \( G \) properly containing \( H \) for all \( i \). If \( H \) normalizes \( Q_i H_0 \) for each \( i \), then \( Q_1 \cdots Q_k H_0 \) is the required normal subgroup and \( G \) is not a counterexample. Hence there must be a Sylow \( q \)-subgroup \( Q_i \) such that \( H \) does not normalize \( Q_i H_0 \) but \( Q_i H \) is a subgroup. Then \( Q_i H \) is a smaller counterexample, since the conjugation properly is inherited by all subgroups of \( G \) containing \( H \).

From now on, we assume that \( \lceil [G:H] \rceil = q^n \) and \( Q \) is a Sylow \( q \)-subgroup of \( G \).

Suppose that \( Q_1 = O_q(G) \neq 1 \). We shall show that \( \bar{G} = G/Q_1, H = \bar{H} Q_1/Q_1, \bar{H}_0 = H_0 Q_1/Q_1 \) satisfy the hypotheses. Let \( \eta_1, \eta_2 \in H, \bar{\eta}_i = \eta_i Q_1, i = 1, 2 \). Then \( \bar{\eta}_1 \sim G \bar{\eta}_2 \) implies that

\[ \eta_1 \sim G \eta_2 \sigma \sigma \in Q_1. \]

Now \( H \) is a Sylow \( \pi \)-subgroup of \( H Q_1 \). Hence \( \eta_2 \sigma \) is \( H Q_1 \)-conjugate to an element \( \eta_3 \) of \( H \). Since \( Q_1 \) acts trivially on \( \bar{H}, \eta_2 \sim H \eta_3 \). Combining these facts we see that \( \bar{\eta}_1 \sim \bar{G} \eta_2 \). By our assumptions on \( G \), \( \eta_1 H_0 \sim H_0 H_0 \eta_2 H_0 \) and hence

\[ \bar{\eta}_1 H_0 \sim H_0 \eta_1 \bar{H}_0. \]

Thus in \( \bar{G} \), there exists a normal subgroup \( K \) such that \( K \cap \bar{H} = \bar{H}_0, \bar{G} = K \bar{H} \).

The inverse image \( K \) satisfies \( K \cap H = H_0, G = KH \), contradicting our choice of \( G \). Thus from now on we may assume that \( O_q(G) = 1 \), and any minimal normal subgroup of \( G \) lies in \( H \).

Let \( H_1 \) be any normal subgroup of \( G \) contained in \( H \). Let

\[ \eta_1, \eta_2 \in H, \bar{\eta}_i = \eta_i H_1, i = 1, 2, \]

and assume \( \bar{\eta}_1 \sim G \eta_1 \). Then \( \eta_1 \sim G \eta_2 \eta_3 \) where \( \eta_3 \in H_1 \). By assumption

\[ \eta_1 H_0 \sim H_0 \eta_2 H_0 \eta_3 H_0 \quad \text{and} \quad \eta_1 H_1 H_0 \sim H_0 H_1 \eta_2 H_0 H_1. \]

Therefore \( G/H_1, H/H_1, \) and \( H_0 H_1 / H_1 \) satisfy the hypotheses of the theorem.
and $H/H_0H_1$ has a normal complement in $G/H_1$, i.e., $QH_0H_1 \triangleleft G$ for any normal subgroup $H_1$ of $G$ contained in $H$.

Suppose now that $H_1$ is chosen to be a minimal normal subgroup of $G$. Then $H_1 \cap H_0 \triangleleft G$. For $H$ normalizes both $H_1$ and $H_0$ and $G$ normalizes $H_1$. If $\eta \in H_1 \cap H_0$ and $\sigma \in G$, then $\eta^\sigma = \eta_1 \in H_1$. Then $\eta H_0 \sim_{H_1/H_0} \eta_1 H_0$ by assumption, but $\eta H_0 = H_0$. Thus $\eta_1 \in H_0$, and $G$ normalizes $H_1 \cap H_0$. Since $H_1$ is minimal, $H_1 \cap H_0 = 1$ or $H_1$. But if $H_1 \leq H_0$, then from the preceding paragraph $QH_0$ is the required normal complement. Hence we may assume that $H_0 \cap H_1 = 1$ and that $[H_0, H_1] = 1$.

Since $H_1 \triangleleft G$, both $Q$ and $H$ act as automorphisms on $H_1$. $H$ decomposes $H_1$ into $H$-conjugacy classes $C_1, C_2, \ldots, C_k$. $Q$ must preserve these conjugacy classes. For if $\eta \in C_i$ and $\sigma \in Q$, then $\eta^\sigma = \eta_1 \in H_1$. But $\eta H_0 \sim_{H_1/H_0} \eta_1 H_0$ implies that $\eta \sim_H \eta_1$ since $H_0 \cap H_1 = 1$ and $H_0$ centralizes $H_1$. Therefore $\eta_1 \in C_i$. Since the $Q$-orbits of each $C_i$ have length a power of $q$ and their sum must be a divisor of $|H_1|$, in each $C_i$ there must be an element centralized by $Q$. Hence $C_{H_1}(Q) \neq 1$. But

$$C_{H_1}(Q) = \langle H_0H_1Q \rangle \cap H_1$$

which is normal in $G$ since $\langle H_0H_1Q \rangle$ is characteristic in $H_0H_1Q \triangleleft G$. By the minimality of $H_1$, we must have $C_{H_1}(Q) = H_1$. Suppose now that $O_q(G/H_1) = 1$. Then $G/H_1$ must have a minimal normal subgroup $H_2H_1/H_1$ contained in $H_0H_1/H_1$ since

$$QH_0H_1/H_1 \triangleleft G/H_1.$$ 

The $H_2$ is an elementary abelian $r$-group for some prime $r$, and $H_1$ is an elementary abelian $p$-group for a prime $p$. If $p \neq r$ then $H_2$ is characteristic in $H_2H_1$ and hence normal in $G$. But the argument in the preceding paragraph rules out the possibility of any subgroups of $H_0$ being normal in $G$. On the other hand, if $p = r$, then $Q$, a $q$-group with $q \neq p$, must normalize a complement to $H_1$ in $H_1H_2$ which we may assume to be $H_2$ since any complement normalized by $Q$ must lie in $H_0$. But if $\eta \in H_1H_2$ then $\eta_1 \in H_0$ and $H_0 \cap H_1H_2 = H_2$. Therefore $H$ normalizes $H_2$ and again we have a normal subgroup of $G$ contained in $H_0$. Thus we conclude that $O_q(G/H_1) \neq 1$. But $O_q(G/H_1) = Q_1H_1/H_1$ with $Q_1$ characteristic in $Q_1H_1 \triangleleft G$. This contradict $O_q(G) = 1$, and we have completed the proof.

Reference