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A RESULT OF MULTIPLICITY OF SOLUTIONS FOR A CLASS OF QUASILINEAR EQUATIONS

CLAUDIANOR O. ALVES[1], GIOVANY M. FIGUEIREDO[1,2] AND UBERLANDIO B. SEVERO[3]

[1] *Unidade Acadêmica de Matemática e Estatística, Universidade Federal de Campina Grande, 58109-970 Campina Grande PB, Brazil* (coalves@dme.ufcg.edu.br)
[2] *Faculdade de Matemática, Universidade Federal do Pará, 66075-110 Belém PA, Brazil* (giovany@ufpa.br)
[3] *Departamento de Matemática, Universidade Federal da Paraíba, 58051-900 João Pessoa PB, Brazil* (uberlandio@mat.ufpb.br)

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Abstract We establish the multiplicity of positive weak solutions for the quasilinear Dirichlet problem $-L_p u + |u|^{p-2}u = h(u)$ in Ω_λ, $u = 0$ on $\partial\Omega_\lambda$, where $\Omega_\lambda = \lambda\Omega$, Ω is a bounded domain in $\mathbb{R}^N$, λ is a positive parameter, $L_p u \doteq \Delta_p u + \Delta_p(u^2)u$ and the nonlinear term $h(u)$ has subcritical growth. We use minimax methods together with the Lyusternik–Schnirelmann category theory to get multiplicity of positive solutions.

Keywords: variational methods; quasilinear equations; standing wave solutions; Lyusternik–Schnirelmann category

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1. Introduction

This paper is concerned with the existence of multiple positive solutions for a quasilinear problem of the form

$$-L_p u + |u|^{p-2}u = h(u), \quad u \in W_0^{1,p}(\Omega_\lambda), \tag{P_λ}$$

where $\Omega_\lambda = \lambda\Omega$, Ω is a bounded domain in $\mathbb{R}^N$, λ is a positive parameter, $2 \leqslant p < N$,

$$L_p u \doteq \Delta_p u + \Delta_p(u^2)u,$$

$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $h\colon \mathbb{R} \to \mathbb{R}$ is a C^1-function verifying the following conditions:

(H_0) $h(s) = 0$ for $s < 0$ and $h(s) = o(|s|^{p-1})$ at the origin;

(H_1) $\lim_{|s|\to\infty} h(s)|s|^{1-q} = 0$ for some $q \in (2p, 2p^*)$, where $p^* = Np/(N-p)$;

(H_2) there exists $\theta > 2p$ such that $0 < \theta H(s) \leqslant sh(s)$ for all $s > 0$, where

$$H(s) = \int_0^s h(t)\,\mathrm{d}t;$$

(H_3) the function $s \to h(s)/s^{2p-1}$ is increasing for $s > 0$.

A typical example of a function satisfying the conditions (H_0)–(H_3) is given by $h(s) = s^{\mu}$ for $s \geqslant 0$, with $2p - 1 < \mu < q - 1$, and $h(s) = 0$ for $s < 0$.

Throughout the paper, a function $u\colon \Omega_\lambda \to \mathbb{R}$ is called a weak solution of (P_λ) if $u \in W_0^{1,p}(\Omega_\lambda) \cap L^{\infty}_{\mathrm{loc}}(\Omega_\lambda)$ and

$$\begin{aligned}\int_{\Omega_\lambda} [(1 + 2^{p-1}|u|^p)|\nabla u|^{p-2}\nabla u \nabla\varphi + 2^{p-1}|\nabla u|^p |u|^{p-2} u\varphi]\mathrm{d}x \\ = \int_{\Omega_\lambda} [h(u) - |u|^{p-2}u]\varphi\,\mathrm{d}x \quad \text{for all } \varphi \in C_0^{\infty}(\Omega_\lambda). \qquad (1.1)\end{aligned}$$

For $p = 2$, the solutions of (P_λ) are related to existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$\mathrm{i}\partial_t \psi = -\Delta\psi + V(x)\psi - \tilde{h}(|\psi|^2)\psi - \kappa\Delta[\rho(|\psi|^2)]\rho'(|\psi|^2)\psi, \qquad (1.2)$$

where $\psi\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $V = V(x)$ is a given potential, κ is a real constant and ρ, $\tilde{h}$ are real functions. Quasilinear equations of the form (1.2) have been studied in relation to some mathematical models in physics. For example, when $\rho(s) = s$, the above equation is

$$\mathrm{i}\partial_t \psi = -\Delta\psi + V(x)\psi - \kappa\Delta[|\psi|^2]\psi - \tilde{h}(|\psi|^2)\psi. \qquad (1.3)$$

It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to (1.3) and numerical results on this equation are obtained in [**7**]. In [**12**], motivated by nanotubes and fullerene-related structures, it was proposed and shown that a discrete system describing the interaction of a two-dimensional hexagonal lattice with an excitation caused by an excess electron can be reduced to (1.3) and numerical results have been found for domains of disc type, cylinder type and sphere type. The superfluid film equation in plasma physics also has the structure (1.2) for $\rho(s) = s$ [**15**].

The general equation (1.2) with various forms of quasilinear terms $\rho(s)$ has been derived as models of several other physical phenomena corresponding to various types of $\rho(s)$. For example, in the case $\rho(s) = (1 + s)^{1/2}$, (1.2) models the self-channelling of a high-power ultra-short laser in matter [**6**, **21**]. Equation (1.2) also appears in fluid mechanics [**14**], in the theory of Heisenberg ferromagnets and magnons [**24**], in dissipative quantum mechanics and in condensed-matter theory [**19**]. The semilinear case corresponding to $\kappa = 0$ in the whole $\mathbb{R}^N$ has been studied extensively in recent years (see, for example, [**5**, **11**, **13**] and references therein).

Setting $\psi(t, x) = \exp(-\mathrm{i}Ft)u(x)$, $F \in \mathbb{R}$, in (1.3), we obtain the corresponding equation

$$-\Delta u - \Delta(u^2)u + V(x)u = h(u) \quad \text{in } \Omega_\lambda, \qquad (1.4)$$

where we have renamed $V(x) - F$ as $V(x)$, $h(u) = \tilde{h}(u^2)u$ and we assume, without loss of generality, that $\kappa = 1$.

The quasilinear equation (1.4) in the whole $\mathbb{R}^N$ has received special attention in the past few years; see, for example, [**8**–**10**, **16**–**18**, **20**] and references therein. These papers present important results on the existence of non-trivial solutions of (1.4) and a good insight into this quasilinear Schrödinger equation. The two main strategies used are the following. The first consists in using a constrained minimization argument, which gives a solution of (1.4) with an unknown Lagrange multiplier λ in front of the nonlinear term (see, for example, [**20**]). The other consists in using a special change of variables to get a new semilinear equation and an appropriate Orlicz space framework (for more details see [**8**, **9**, **17**]). In [**2**], Alves *et al*. showed the existence of multiple solutions, by using the Lyusternik–Schnirelmann category, for the following class of problems:

$$-\varepsilon^p \Delta_p u - \varepsilon^p \Delta_p(u^2)u + V(x)|u|^{p-2}u = h(u), \quad u \in W^{1,p}(\mathbb{R}^N),$$

for ε sufficiently small and the potential $V(x)$ verifying some suitable assumptions.

Since in the literature we find few works where multiple solutions have been established for problems involving the operator $L_p u$ by using the Lyusternik–Schnirelmann category, the present paper aims to show a class of problems involving the operator $L_p u$ where the Lyusternik–Schnirelmann category can be used to get multiple positive solutions. Here we improve the main result proved in [**1**]; Alves established therein the existence of multiple solutions by using the Lyusternik–Schnirelmann category for the following problem:

$$-\Delta_p u + |u|^{p-2}u = h(u), \quad u \in W_0^{1,p}(\Omega_\lambda),$$

where h has subcritical growth and λ is large enough. The main result by Alves completes the study made in [**3**, **4**] for the case $p \geqslant 2$. The presence of the term $\Delta_p(u^2)u$ in the operator $L_p u$ implies that several estimates used in [**1**] cannot be repeated for the functional energy associated to (P_λ). As observed in [**22**, **23**], there are some technical difficulties in applying variational methods directly to the formal functional associated to (P_λ) given by

$$\mathrm{J}_\lambda(u) = \frac{1}{p}\int_{\Omega_\lambda} (1 + 2^{p-1}|u|^p)|\nabla u|^p \,\mathrm{d}x + \frac{1}{p}\int_{\Omega_\lambda} |u|^p \,\mathrm{d}x - \int_{\Omega_\lambda} H(u)\,\mathrm{d}x,$$

where

$$H(s) = \int_0^s h(t)\,\mathrm{d}t.$$

The main difficulty is related to the fact that J_λ is not well defined in all $W_0^{1,p}(\Omega_\lambda)$ for $N > p > 1$. For example, if $u \in C_0^1(\Omega_\lambda \setminus \{0\})$ is defined by

$$u(x) = |x|^{(p-N)/2p} \quad \text{for } x \in \Omega_\lambda \setminus \{0\},$$

we have $u \in W_0^{1,p}(\Omega_\lambda)$, while the function $|u|^p|\nabla u|^p$ does not belong to $L^1(\Omega_\lambda)$. To overcome this difficulty, we use arguments developed in [**22**, **23**] which generalize some arguments found in [**8**, **17**] for the case $p = 2$. More precisely, we make the change of variables $v = f^{-1}(u)$, where f is defined by

$$\left.\begin{aligned} f'(t) &= \frac{1}{(1 + 2^{p-1}|f(t)|^p)^{1/p}} \quad \text{on } [0, +\infty),\\ f(t) &= -f(-t) \qquad\qquad\qquad\quad \text{on } (-\infty, 0].\end{aligned}\right\} \tag{1.5}$$

Therefore, after the change of variables, the functional $\mathrm{J}_\lambda(u)$ can be rewritten as follows:

$$I_\lambda(v) \doteq \mathrm{J}_\lambda(f(v)) = \frac{1}{p}\int_{\Omega_\lambda} |\nabla v|^p \,\mathrm{d}x + \frac{1}{p}\int_{\Omega_\lambda} |f(v)|^p \,\mathrm{d}x - \int_{\Omega_\lambda} H(f(v))\,\mathrm{d}x, \tag{1.6}$$

which is well defined on the Banach space $W_0^{1,p}(\Omega_\lambda)$ endowed with the norm

$$\|v\| = |\nabla v|_p + \inf_{\xi>0}\left[\frac{1}{\xi} + \int_{\Omega_\lambda} |f(\xi v)|^p \,\mathrm{d}x\right].$$

In [**22**, **23**] the reader can find more details about the function f and the proof that $\|\cdot\|$ is a norm in $W_0^{1,p}(\Omega_\lambda)$. For this proof, the fact that the function $|f(s)|^p$ is convex for $p \geqslant 2$ is crucial. A direct computation implies that $\|\cdot\|$ is an equivalent norm to the usual norm of $W_0^{1,p}(\Omega_\lambda)$.

Under the conditions (H_0)–(H_2), a straightforward computation shows that the functional $I_\lambda \colon W_0^{1,p}(\Omega_\lambda) \to \mathbb{R}$ is of class C^1 with

$$\langle I'_\lambda(v), w\rangle = \int_{\Omega_\lambda} [|\nabla v|^{p-2}\nabla v \nabla w + |f(v)|^{p-2} f(v) f'(v) w]\mathrm{d}x - \int_{\Omega_\lambda} h(f(v)) f'(v) w \,\mathrm{d}x$$

for $v, w \in W_0^{1,p}(\Omega_\lambda)$. Thus, the critical points of I_λ correspond exactly to the weak solutions of the quasilinear problem

$$\left.\begin{aligned} -\Delta_p v + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v) \quad &\text{in } \Omega_\lambda,\\ v = 0 \quad &\text{on } \partial\Omega_\lambda.\end{aligned}\right\} \tag{D_λ}$$

The problem above has a closed relation with problem (P_λ), because if $v \in W_0^{1,p}(\Omega_\lambda) \cap L^\infty_{\mathrm{loc}}(\Omega_\lambda)$ is a critical point of the functional I_λ, then $u = f(v)$ is a weak solution of (P_λ). In § 2, more precisely in Corollary 2.6, we shall show that each critical point v of I_λ belongs to $W_0^{1,p}(\Omega_\lambda) \cap L^\infty(\Omega_\lambda)$. Hence, we shall work to find non-trivial critical points of I_λ.

Before stating our main result, we recall that if Y is a closed set of a topological space X, we denote the Lyusternik–Schnirelmann category of Y in X by $\mathrm{cat}_X(Y)$, which is the least number of closed and contractible sets in X that cover Y. Hereafter, $\mathrm{cat}\, X$ denotes $\mathrm{cat}_X(X)$. We are now ready to state the main result of this work.

Theorem 1.1. *Assume that* (H_0)–(H_3) *hold. Then there exists* $\lambda^* > 0$ *such that for* $\lambda > \lambda^*$ (P_λ) *has at least* $\mathrm{cat}\,\Omega_\lambda$ *positive weak solutions.*

2. Preliminary results

In this section, we show some results that are essential in the following sections. We begin by showing some properties of the change of variables $f\colon \mathbb{R} \to \mathbb{R}$ defined in (1.5), which will be used below. The proof of the following lemma can be found in [**22**, **23**].

Lemma 2.1. *The function $f(t)$ and its derivative have the following properties:*

(i) *f is uniquely defined, C^2 and invertible;*

(ii) *$|f'(t)| \leqslant 1$ for all $t \in \mathbb{R}$;*

(iii) *$|f(t)| \leqslant |t|$ for all $t \in \mathbb{R}$;*

(iv) *$f(t)/t \to 1$ as $t \to 0$;*

(v) *$|f(t)| \leqslant 2^{1/2p}|t|^{1/2}$ for all $t \in \mathbb{R}$;*

(vi) *$f(t)/2 < tf'(t) < f(t)$ for all $t > 0$;*

(vii) *$f(t)/\sqrt{t} \to a > 0$ as $t \to +\infty$;*

(viii) *there exists a positive constant C such that*

$$|f(t)| \geqslant \begin{cases} C|t|, & |t| \leqslant 1, \\ C|t|^{1/2}, & |t| \geqslant 1; \end{cases}$$

(ix) *$|f(t)f'(t)| \leqslant 1/2^{(p-1)/p}$ for all $t \in \mathbb{R}$.*

Corollary 2.2. *The following properties involving the functions f and h hold:*

(i) *the function $(f(t))^{p-1}f'(t)t^{1-p}$ is decreasing for $t > 0$;*

(ii) *the function $(f(t))^{2p-1}f'(t)t^{1-p}$ is increasing for $t > 0$;*

(iii) *the function $h(f(t))f'(t)t^{1-p}$ is increasing for $t > 0$.*

Proof. By using Lemma 2.1 (vi), it is easy to see that $f(t)/t$ is non-increasing for $t > 0$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(f(t))^{p-1}f'(t)}{t^{p-1}}\right) = (p-1)\left(\frac{f(t)}{t}\right)^{p-2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(t)}{t}\right)f'(t) + \frac{(f(t))^{p-1}}{t^{p-1}}f''(t) < 0$$

for all $t > 0$, which shows (i).

To prove (ii), we observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\frac{(f(t))^{2p-1}f'(t)}{t^{p-1}}\right) \\ &= \frac{1}{t^{2(p-1)}}((2p-1)(f(t))^{2p-2}(f'(t))^2t^{p-1} \\ &\qquad - 2^{p-1}(f(t))^{3p-2}(f'(t))^{p+2}t^{p-1} - (p-1)(f(t))^{2p-1}f'(t)t^{p-2}).\end{aligned}$$

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Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(f(t))^{2p-1}f'(t)}{t^{p-1}}\right) \geqslant f'(t)(f(t))^{2p-2}t^{p-2}\frac{(2p-1)f'(t)t - f'(t)t - (p-1)f(t)}{t^{2(p-1)}},$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{(f(t))^{2p-1}f'(t)}{t^{p-1}}\right) \geqslant f'(t)(f(t))^{2p-2}t^{p-2}(p-1)\frac{2f'(t)t - f(t)}{t^{2(p-1)}} > 0$$

for all $t > 0$, where we have used (vi) and (ix) in Lemma 2.1. The last inequality proves (ii).

The proof of (iii) follows by using (H_3), (ii) and the equality

$$\frac{h(f(t))f'(t)}{t^{p-1}} = \left[\frac{h(f(t))}{(f(t))^{2p-1}}\right]\left[\frac{(f(t))^{2p-1}f'(t)}{t^{p-1}}\right] \quad \text{for } t > 0.$$

□

The next proposition will be used in the proof of some results later.

Proposition 2.3. *Let A be an open set of $\mathbb{R}^N$, let $B\colon A \to \mathbb{R}$ be a non-negative measurable function and let (v_n) be a sequence in $W_0^{1,p}(A)$ verifying*

$$\int_A B(x)|f(v_n)|^p \to 0 \quad \text{as } n \to \infty.$$

Then,

$$\inf_{\xi>0}\left\{\frac{1}{\xi} + \int_A B(x)|f(\xi v_n)|^p\right\} \to 0 \quad \text{as } n \to \infty.$$

Proof. Since f is odd and $f(t)/t$ is non-increasing for $t > 0$, for each $\xi > 1$, we have

$$\frac{1}{\xi} + \int_A B(x)|f(\xi v_n)|^p \leqslant \frac{1}{\xi} + \xi^p \int_A B(x)|f(v_n)|^p.$$

Hence, for each $\delta > 0$, fixing ξ_* sufficiently large that $1/\xi_* < \delta/2$, we get

$$\inf_{\xi>0}\left\{\frac{1}{\xi} + \int_A B(x)|f(\xi v_n)|^p\right\} \leqslant \tfrac{1}{2}\delta + \xi_*^p \int_A B(x)|f(v_n)|^p.$$

Thus,

$$\limsup_{n\to\infty}\left(\inf_{\xi>0}\left\{\frac{1}{\xi} + \int_A B(x)|f(\xi v_n)|^p\right\}\right) \leqslant \tfrac{1}{2}\delta \quad \text{for all } \delta > 0,$$

which proves the proposition. □

An immediate consequence of this proposition is the following corollary.

Corollary 2.4. *Let A be an open set and let $(v_n) \subset W_0^{1,p}(A)$. Defining*

$$\mathcal{Q}_A(v) := \int_A |\nabla v|^p + \int_A |f(v)|^p,$$

we derive that $\mathcal{Q}_A(v_n) \to 0$ if and only if $\|v_n\| \to 0$.

An important result that we shall use in this work is related to the existence of a positive ground-state solution for the problem

$$\left.\begin{aligned} -\Delta_p v + |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v) \quad \text{in } \mathbb{R}^N, \\ v \in W^{1,p}(\mathbb{R}^N), \\ v(x) > 0 \quad \text{for all } x \in \mathbb{R}^N, \end{aligned}\right\} \tag{P_∞}$$

that is, with the existence of a positive function $w \in W^{1,p}(\mathbb{R}^N)$ verifying

$$I_\infty(w) = c_\infty \quad \text{and} \quad I'_\infty(w) = 0,$$

where

$$I_\infty(v) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla v|^p + \frac{1}{p} \int_{\mathbb{R}^N} |f(v)|^p - \int_{\mathbb{R}^N} H(f(v)),$$

and c_∞ denotes the minimax level of the mountain-pass theorem associated to the functional I_∞. Furthermore, $\mathcal{M}_\infty$ denotes the Nehari manifold associated to I_∞. The theorem below shows the existence of a ground-state solution for (P_∞) and its proof can be found in [**2**].

Theorem 2.5. *Under hypotheses* (H_0)–(H_3), *problem (P_∞) has a positive ground-state solution $v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying $v(x) \to 0$ as $|x| \to \infty$.*

The arguments used in the proof of the above theorem can be repeated to prove the following result.

Corollary 2.6. *If $v \in W_0^{1,p}(\Omega_\lambda)$ is a solution of (D_λ), then $v \in L^\infty(\Omega_\lambda)$. Hence, the function $u = f(v)$ is a solution for (P_λ).*

3. A compactness result

In this section we establish a compactness result on the Nehari manifold involving minimizing sequences. For this, we must first recall some definitions. Let V be a Banach space, let $\mathcal{V}$ be a C^1-manifold of V and let $I\colon V \to \mathbb{R}$ be a C^1-functional. We say that I restricted to $\mathcal{V}$ satisfies the Palais–Smale (PS) condition at level c if any sequence $(u_n) \subset \mathcal{V}$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here, we denote by $\|I'(u)\|_*$ the norm of the derivative of I restricted to $\mathcal{V}$ at the point u [**27**, § 5.3].

Lemma 3.1 (a compactness lemma). *Let $(v_n) \subset \mathcal{M}_\infty$ be a sequence of non-negative functions satisfying $I_\infty(v_n) \to c_\infty$. Then,*

(i) *(v_n) has a subsequence strongly convergent in $W^{1,p}(\mathbb{R}^N)$ or*

(ii) *there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $|y_n| \to +\infty$, $\bar{v}_n(x) := v_n(x + y_n)$ converges strongly in $W^{1,p}(\mathbb{R}^N)$.*

In particular, there exists a minimizer for c_∞.

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Proof. Applying the Ekeland variational principle (see [**27**, Theorem 8.5]), we may suppose that (v_n) is a $(\mathrm{PS})_{c_\infty}$ condition for I_∞ in $\mathcal{M}_\infty$, that is, $I_\infty(v_n) \to c_\infty$ and $\|I'_\infty(v_n)\|_* = o_n(1)$. Then, there exists $(\gamma_n) \subset \mathbb{R}$ such that

$$I'_\infty(v_n) = \gamma_n J'_\infty(v_n) + o_n(1), \tag{3.1}$$

where $J_\infty \colon W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ is given by

$$J_\infty(v) = \int_{\mathbb{R}^N} |\nabla v|^p + \int_{\mathbb{R}^N} |f(v)|^{p-2} f(v) f'(v) v - \int_{\mathbb{R}^N} h(f(v)) f'(v) v.$$

Note that

$$\begin{aligned}\langle J'_\infty(v_n), v_n\rangle = p \int_{\mathbb{R}^N} |\nabla v_n|^p &+ (p-1) \int_{\mathbb{R}^N} |f(v_n)|^{p-2} [f'(v_n)]^2 [v_n]^2 \\ &+ \int_{\mathbb{R}^N} |f(v_n)|^{p-1} f''(v_n)[v_n]^2 + \int_{\mathbb{R}^N} |f(v_n)|^{p-1} f'(v_n) v_n \\ &- \int_{\mathbb{R}^N} h'(f(v_n))[f'(v_n)]^2 [v_n]^2 - \int_{\mathbb{R}^N} h(f(v_n)) f''(v_n)[v_n]^2 \\ &- \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n.\end{aligned}$$

Since $(v_n) \subset \mathcal{M}_\infty$, we derive

$$\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} |f(v_n)|^{p-1} f'(v_n) v_n = \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n. \tag{3.2}$$

Using Lemma 2.1 (vi), it follows that

$$\int_{\mathbb{R}^N} |f(v_n)|^{p-2} [f'(v_n)]^2 [v_n]^2 \leqslant \int_{\mathbb{R}^N} |f(v_n)|^{(p-1)} f'(v_n) v_n.$$

Now, combining (3.2) and the above inequality, we get

$$\begin{aligned}\langle J'_\infty(v_n), v_n\rangle \leqslant \int_{\mathbb{R}^N} |f(v_n)|^{p-1} f''(v_n)[v_n]^2 &+ (p-1) \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n \\ &- \int_{\mathbb{R}^N} h'(f(v_n))[f'(v_n)]^2 [v_n]^2 - \int_{\mathbb{R}^N} h(f(v_n)) f''(v_n)[v_n]^2.\end{aligned}$$

From Lemma 2.2 (iii), we know that $h(f(t))f'(t)t^{1-p}$ is increasing for $t > 0$. Therefore,

$$(p-1) h(f(t)) f'(t) - h'(f(t))(f'(t))^2 t - h(f(t)) f''(t) t \leqslant 0 \quad \text{for } t \geqslant 0,$$

which implies that

$$\langle J'_\infty(v_n), v_n\rangle \leqslant \int_{\mathbb{R}^N} |f(v_n)|^{p-1} f''(v_n)[v_n]^2.$$

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Since $f''(t) = -2^{p-1}(f(t))^{p-1}(f'(t))^{p+2}$ for $t \geqslant 0$, from the latter inequality we have

$$\langle J'_\infty(v_n), v_n\rangle \leqslant -2^{p-1}\int_{\mathbb{R}^N} |[f(v_n)]^2 f'(v_n)|^p.$$

From this, we can assume that $\langle J'_\infty(v_n), v_n\rangle \to l \leqslant 0$. If $l = 0$, the inequality above yields

$$\int_{\mathbb{R}^N} |[f(v_n)]^2 f'(v_n)|^p \to 0. \tag{3.3}$$

On the other hand, from (H_0) and (H_1), for each $\delta > 0$ there exists $C_\delta > 0$ such that

$$0 \leqslant h(t) \leqslant \delta t^{q-1} + C_\delta t^p \quad \text{for } t \geqslant 0$$

and so

$$\begin{aligned}0 &\leqslant h(f(v_n))f'(v_n)v_n\\ &\leqslant \delta|f(v_n)|^{q-1}f'(v_n)v_n + C_\delta|f(v_n)|^{p-2}f^2(v_n)f'(v_n)v_n\\ &\leqslant \delta|f(v_n)|^q + C_\delta|v_n|^{p-1}f^2(v_n)f'(v_n).\end{aligned}$$

This, together with the boundedness of (v_n) in $W^{1,p}(\mathbb{R}^N)$, yields

$$0 \leqslant \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \leqslant \delta C + \hat{C}_\delta\biggl(\int_{\mathbb{R}^N} |[f(v_n)]^2 f'(v_n)|^p\biggr)^{1/p}.$$

Now, using (3.3) we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \leqslant \delta C \quad \text{for all } \delta > 0,$$

showing that

$$\int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \to 0.$$

By (3.2), the last limit implies that $\mathcal{Q}_{\mathbb{R}^N}(v_n) \to 0$, and by Corollary 2.4 it follows that $v_n \to 0$ in $W^{1,p}(\mathbb{R}^N)$. However, it is not difficult to check that there exists $C > 0$ such that

$$C \leqslant \|v\| \quad \text{for } v \in \mathcal{M}_\infty, \tag{3.4}$$

from whence it follows that $\|v_n\| \geqslant C$ for all $n \in \mathbb{N}$; in this way we obtain a contradiction. Thus, $l \neq 0$ and $\gamma_n = o_n(1)$. From (3.1) and by the fact that $\|J'_\infty(v_n)\|$ is bounded, $I'_\infty(v_n) = o_n(1)$. Therefore, (v_n) is a $(\mathrm{PS})_c$ sequence for I_∞. Thus, going to a subsequence if necessary, we have that $v_n \rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}^N)$ and it is standard to show that $I'_\infty(v) = 0$.

If $v \neq 0$, we can use the fact that $[f(t)]^p - [f(t)]^{p-1}f'(t)t$ and $(1/p)h(f(t))f'(t)t - H(f(t))$ are non-negative functions for $t \geqslant 0$ together with Fatou's Lemma to conclude that v is a ground-state solution of the autonomous problem (P_∞), that is, $I_\infty(v) = c_\infty$.

If $v \equiv 0$, applying the same arguments employed in the proof of [**2**, Lemma 3.5], there exists a sequence $(y_n) \subset \mathbb{R}^N$ with $|y_n| \to +\infty$ satisfying

$$\bar{v}_n \rightharpoonup \bar{v} \quad \text{in } W^{1,p}(\mathbb{R}^N),$$

where $\bar{v}_n = v_n(x + y_n)$. Therefore, $\bar{v}_n$ is also a $(\mathrm{PS})_{c_\infty}$ sequence of I_∞ and $\bar{v} \not\equiv 0$, and so $\bar{v}$ is a ground-state solution of the autonomous problem (P_∞). $\square$

Lemma 3.2. *The functional I_λ satisfies the Palais–Smale condition on $W_0^{1,p}(\Omega_\lambda)$.*

Proof. Let $(v_n) \subset W_0^{1,p}(\Omega_\lambda)$ be a sequence such that

$$I_\lambda(v_n) \to c \quad \text{and} \quad I'_\lambda(v_n) \to 0.$$

Thus,

$$\begin{aligned} C_1 + o_n(1)\|v_n\| &\geqslant I_\lambda(v_n) - \frac{2}{\theta}\langle I'_\lambda(v_n), v_n\rangle \\ &\geqslant \left(\frac{1}{p} - \frac{2}{\theta}\right)\int_{\Omega_\lambda} |\nabla v_n|^p + \frac{1}{p}\int_{\Omega_\lambda} |f(v_n)|^p - \int_{\Omega_\lambda} H(f(v_n)) \\ &\qquad - \frac{2}{\theta}\int_{\Omega_\lambda} |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n + \frac{2}{\theta}\int_{\Omega_\lambda} h(f(v_n)) f'(v_n) v_n. \end{aligned}$$

From (H_2) and Lemma 2.1 (vi),

$$C_1 + o_n(1)\|v_n\| \geqslant \left(\frac{1}{p} - \frac{2}{\theta}\right)\int_{\Omega_\lambda} |\nabla v_n|^p + \left(\frac{1}{p} - \frac{2}{\theta}\right)\int_{\Omega_\lambda} |f(v_n)|^p, \tag{3.5}$$

where $o_n(1) \to 0$ as $n \to \infty$. Recalling that $|\nabla v_n|_p \leqslant 1 + |\nabla v_n|_p^p$, we obtain

$$C_1 + o_n(1)\|v_n\| \geqslant C_2\left(|\nabla v_n|_p - 1 + \int_{\Omega_\lambda} |f(v_n)|^p\right) \tag{3.6}$$

and therefore

$$C_3 + o_n(1)\|v_n\| \geqslant C\left(|\nabla v_n|_p + 1 + \int_{\Omega_\lambda} |f(v_n)|^p\right) \geqslant C\|v_n\|, \tag{3.7}$$

which yields that (v_n) is bounded in $W_0^{1,p}(\Omega_\lambda)$. Since $(W_0^{1,p}(\Omega_\lambda), \|\cdot\|)$ is reflexive, there is a subsequence of (v_n), still denoted by (v_n), and $v \in W_0^{1,p}(\Omega_\lambda)$ such that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega_\lambda)$ and $v_n \to v$ in $L^s(\Omega_\lambda)$, with $p \leqslant s < p^*$. Thus,

$$\begin{aligned} 0 &\leqslant \int_{\Omega_\lambda} |\nabla v_n - \nabla v|^p \\ &\leqslant \int_{\Omega_\lambda} \langle |\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v, \nabla v_n - \nabla v\rangle \\ &= \int_{\Omega_\lambda} |\nabla v_n|^p - \int_{\Omega_\lambda} |\nabla v_n|^{p-2}\nabla v_n \nabla v + o_n(1), \end{aligned}$$

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where

$$o_n(1) = \int_{\Omega_\lambda} |\nabla v|^p - \int_{\Omega_\lambda} |\nabla v|^{p-2}\nabla v \nabla v_n.$$

From the definition of I'_λ,

$$0 \leqslant \int_{\Omega_\lambda} |\nabla v_n - \nabla v|^p \leqslant \langle I'_\lambda(v_n), v_n\rangle - \langle I'_\lambda(v_n), v\rangle + R_n + T_n + o_n(1),$$

where

$$R_n = \int_{\Omega_\lambda} |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n - \int_{\Omega_\lambda} |f(v_n)|^{p-2} f(v_n) f'(v_n) v$$

and

$$S_n = \int_{\Omega_\lambda} h(f(v_n)) f'(v_n) v - \int_{\Omega_\lambda} h(f(v_n)) f'(v_n) v_n.$$

Once we have established that $\langle I'_\lambda(v_n), v_n\rangle = \langle I'_\lambda(v_n), v\rangle = o_n(1)$, we may obtain

$$0 \leqslant \int_{\Omega_\lambda} |\nabla v_n - \nabla v|^p \leqslant R_n + T_n + o_n(1).$$

Combining (v) and (vi) in Lemma 2.1 with the subcritical growth of h, Lebesgue's Theorem implies that $R_n = T_n = o_n(1)$. Hence,

$$\int_{\Omega_\lambda} |\nabla v_n - \nabla v|^p = o_n(1).$$

Since the norm $\|\cdot\|$ is equivalent to the usual norm in $W_0^{1,p}(\Omega_\lambda)$, the above equality yields $v_n \to v$ in $W_0^{1,p}(\Omega_\lambda)$. Consequently, I_λ satisfies the Palais–Smale condition. □

Proposition 3.3. *The functional I_λ satisfies the Palais–Smale condition on $\mathcal{M}_\lambda$.*

Proof. Let (v_n) be a $(\mathrm{PS})_c$ sequence for I_λ in $\mathcal{M}_\lambda$. Thus, $I_\lambda(v_n) \to c$ and $\|I'_\lambda(v_n)\|_* = o_n(1)$. Arguing as in Lemma 3.1, we can suppose that (v_n) is a Palais–Smale sequence for I_λ in $W_0^{1,p}(\Omega_\lambda)$ and the result follows from Lemma 3.2. □

Corollary 3.4. *If v is a critical point of I_λ on $\mathcal{M}_\lambda$, then v is a non-trivial critical point of I_λ on $W_0^{1,p}(\Omega_\lambda)$.*

Proof. The proof follows by using arguments similar to those explored in Proposition 3.3. □

4. Behaviour of minimax levels

In this section, we study the behaviour of some minimax levels in relation to the parameter λ. To this end, we need to make some definitions. For each $x \in \mathbb{R}^N$ and $R > r > 0$, let us denote by $A_{R,r,x}$ the following set:

$$A_{R,r,x} = B_R(x) \setminus \bar{B}_r(x);$$

when $x = 0$, let us denote by $A_{R,r}$ the set $A_{R,r,0}$. Following ideas found in [**4**], for $v \in W^{1,p}(\mathbb{R}^N)$, whose positive part $v^+ = \max\{v, 0\}$ is non-zero and has a compact support, we can define the centre of mass of v^+, denoted by $\beta(v^+) \in \mathbb{R}^N$, as follows:

$$\beta(v) = \int_{\mathbb{R}^N} x(v^+)^p \bigg(\int_{\mathbb{R}^N} (v^+)^p \bigg)^{-1}.$$

Moreover, for each $x \in \mathbb{R}^N$, let us denote by $a(R, r, \lambda, x)$ the following number:

$$a(R, r, \lambda, x) = \inf\{\hat{I}_{\lambda,x}(v) \colon v \in \hat{\mathcal{M}}_{\lambda,x} \text{ and } \beta(v) = x\},$$

where

$$\hat{I}_{\lambda,x}(v) = \frac{1}{p} \int_{A_{\lambda R,\lambda r,x}} |\nabla v|^p + \frac{1}{p} \int_{A_{\lambda R,\lambda r,x}} |f(v)|^p - \int_{A_{\lambda R,\lambda r,x}} H(f(v)) \tag{4.1}$$

and

$$\hat{\mathcal{M}}_{\lambda,x} = \{v \in W_0^{1,p}(A_{\lambda R,\lambda r,x}) \setminus \{0\} \colon \langle \hat{I}'_{\lambda,x}(v), v\rangle = 0\}.$$

Next, let us denote by $a(R, r, \lambda)$ the number $a(R, r, \lambda, 0)$, let $\hat{I}_\lambda$ denote the functional $\hat{I}_{\lambda,0}$ and let $\hat{\mathcal{M}}_\lambda$ denote the set $\hat{\mathcal{M}}_{\lambda,0}$.

Proposition 4.1. *The number $a(R, r, \lambda)$ satisfies*

$$\liminf_{\lambda\to\infty} a(R, r, \lambda) > c_\infty.$$

Proof. From the definition of $a(R, r, \lambda)$ and c_∞, we get

$$a(R, r, \lambda) \geqslant c_\infty.$$

Assume, by contradiction, that there exist $\lambda_n \to \infty$ and $v_n \in \hat{\mathcal{M}}_{\lambda_n}$ verifying

$$\beta(v_n) = 0 \quad \text{and} \quad a(R, r, \lambda_n) \to c_\infty.$$

A direct computation shows that we can assume that $v_n \geqslant 0$ for all $n \in \mathbb{N}$. Moreover, since $v_n = 0$ on $\partial A_{\lambda_n R,\lambda_n r}$, we can set $v_n = 0$ on $A^c_{\lambda_n R,\lambda_n r}$. Consequently,

$$v_n \rightharpoonup 0 \quad \text{in } W^{1,p}(\mathbb{R}^N), \qquad I_\infty(v_n) = a(R, r, \lambda_n) \to c_\infty \qquad \text{and} \qquad v_n \in \mathcal{M}_\infty.$$

Recalling that $c_\infty > 0$, we obtain that (v_n) is not strongly convergent. From Lemma 3.1, we reach

$$v_n(x) = w_n(x) + \Psi(x - y_n),$$

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where $(w_n) \subset W^{1,p}(\mathbb{R}^N)$ is a sequence converging strongly to $0 \in W^{1,p}(\mathbb{R}^N)$, $(y_n) \subset \mathbb{R}^N$ is such that $|y_n| \to \infty$ and $\Psi \in W^{1,p}(\mathbb{R}^N)$ is a positive function verifying

$$I_\infty(\Psi) = c_\infty \quad \text{and} \quad I'_\infty(\Psi) = 0.$$

Since I_λ is rotationally invariant, we can assume that

$$y_n = (y_n^1, 0, 0, \dots, 0)$$

and $y_n^1 < 0$.

Now we set

$$M = \int_{\mathbb{R}^N} |\Psi|^p.$$

Clearly, $M > 0$. Since $\|w_n\| \to 0$, it follows that

$$\int_{B_{r\lambda_n/2}(y_n)} |w_n + \Psi(\cdot - y_n)|^p \to M,$$

from which we obtain

$$\int_{\Theta_n} |v_n|^p \to M,$$

where $\Theta_n = B_{r\lambda_n/2}(y_n) \cap [B_{\lambda_n R}(0) \setminus B_{\lambda_n r}(0)]$, and hence

$$\int_{\Upsilon_n} |v_n|^p \to 0, \tag{4.2}$$

where $\Upsilon_n = [B_{\lambda_n R}(0) \setminus B_{\lambda_n r}(0)] \setminus B_{\lambda_n r/2}(y_n)$. Since $\beta(v_n) = 0$, we get

$$0 = \int_{A_{\lambda_n R, \lambda_n r}} x_1 |v_n|^p = \int_{\Theta_n} x_1 |v_n|^p + \int_{\Upsilon_n} x_1 |v_n|^p.$$

Thus,

$$-(\tfrac{1}{2} r\lambda_n)(M + o_n(1)) + R\lambda_n \int_{\Upsilon_n} |v_n|^p \geqslant 0$$

with $o_n(1) \to 0$. Then,

$$\int_{\Upsilon_n} |v_n|^p \geqslant \frac{rM}{2R} - o_n(1)$$

and this contradicts (4.2). □

Henceforth, let us denote by b_λ the minimax level of the mountain-pass theorem of the energy functional $I_{\lambda,B} \colon W_0^{1,p}(B_\lambda) \to \mathbb{R}$ given by

$$I_{\lambda,B}(v) = \frac{1}{p} \int_{B_{\lambda r}} |\nabla v|^p + \frac{1}{p} \int_{B_{\lambda r}} |f(v)|^p - \int_{B_{\lambda r}} H(f(v)),$$

where $B_{\lambda r} = B_{\lambda r}(0)$, and denote by $\mathcal{M}_{\lambda,B}$ the Nehari manifold related to $I_{\lambda,B}$ given by

$$\mathcal{M}_{\lambda,B} = \{v \in W_0^{1,p}(B_{\lambda r}) \setminus \{0\} \colon \langle I'_{\lambda,B}(v), v\rangle = 0\}.$$

Using Corollary 2.2, it is easy to check that

$$b_\lambda = \inf_{v \in \mathcal{M}_{\lambda,B}} I_{\lambda,B}(v).$$

Moreover, c_λ and $\mathcal{M}_\lambda$ denote the minimax level and the Nehari manifold related to the functional I_λ, respectively. From now on, we shall assume without loss of generality that $0 \in \Omega$. Furthermore, let us fix a real number $r > 0$ such that the sets

$$\Omega_+ = \{x \in \mathbb{R}^N;\ d(x, \bar{\Omega}) \leqslant r\}$$

and

$$\Omega_- = \{x \in \Omega;\ d(x, \partial\Omega) \geqslant r\}$$

are homotopically equivalent to Ω.

Proposition 4.2. *The numbers b_λ and c_λ verify the limits*

$$\lim_{\lambda\to\infty} c_\lambda = c_\infty \quad \textit{and} \quad \lim_{\lambda\to\infty} b_\lambda = c_\infty.$$

Proof. Here, we shall prove only the first limit, because the second limit follows from the same type of argument. Let Φ be a function in $C_0^\infty(\mathbb{R}^N)$ defined by $\Phi(x) = 1$ in $B_1(0)$, $\Phi(x) = 0$ in $B_2^c(0)$ and $0 \leqslant \Phi(x) \leqslant 1$ for all $x \in \mathbb{R}^N$. For each $R > 0$, let us consider the functions $\Phi_R(x) = \Phi(x/R)$ and $w_R(x) = \Phi_R(x)w(x)$, where w is a ground-state solution of problem (P_∞). Since $0 \in \Omega$, there exists $\lambda^* > 0$ such that $B_{2R}(0) \subset \Omega_\lambda$ for $\lambda \geqslant \lambda^*$. Let $t_R > 0$ such that

$$I_\lambda(t_R w_R) = \max_{t\geqslant 0} I_\lambda(t w_R) = \max_{t\geqslant 0} I_\infty(t w_R).$$

Thus, $\langle I'_\lambda(t_R w_R), t_R w_R\rangle = 0$, which implies that $t_R w_R \in \mathcal{M}_\lambda$. Then

$$c_\lambda \leqslant I_\lambda(t_R w_R) = I_\infty(t_R w_R) \quad \text{for all } \lambda \geqslant \lambda^*.$$

Once R is proved to be independent of λ, we obtain that t_R is also independent of λ. Hence, taking the limit when $\lambda \to \infty$, we obtain

$$\limsup_{\lambda\to\infty} c_\lambda \leqslant I_\infty(t_R w_R).$$

Now, we shall show that

$$\lim_{R\to\infty} t_R = 1. \tag{4.3}$$

Indeed, from the definition of t_R we obtain

$$\int_{\mathbb{R}^N} |\nabla w_R|^p = \int_{\mathbb{R}^N} [h(f(t_R w_R))f'(t_R w_R)t_R^{1-p} - |f(t_R w_R)|^{p-2} f(t_R w_R) f'(t_R w_R) t_R^{1-p}] w_R.$$

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From Corollary 2.2, the right-hand side in the equality above is non-negative for $t \geqslant 0$, because it is increasing. Thus, for $R > 1$, we derive

$$\int_{\mathbb{R}^N} |\nabla w_R|^p \geqslant \int_{B_1(0)} h(f(t_R a))f'(t_R a)t_R^{1-p} a - \int_{B_1(0)} |f(t_R a)|^{p-2} f(t_R a) f'(t_R a) t_R^{1-p} a,$$

where $a = \min_{|x|\leqslant 1} w_R(x)$. Note that (t_R) is bounded, because if there exists $R_n \to \infty$ with $t_{R_n} \to \infty$, we have

$$\int_{\mathbb{R}^N} |\nabla w_{R_n}|^p \to \infty \quad \text{or} \quad \|w_{R_n}\| \to \infty,$$

which is absurd. Therefore, (t_R) is bounded. Note also that $t_R \nrightarrow 0$, because if there exists $R_n \to \infty$ with $t_{R_n} \to 0$, we have, by (H_1) and (H_2),

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^N} h(f(t_{R_n} w_{R_n}))f'(t_{R_n} w_{R_n}) t_{R_n} w_{R_n} \\ &\leqslant \varepsilon \int_{\mathbb{R}^N} |f(t_{R_n} w_{R_n})|^{p-1} |f'(t_{R_n} w_{R_n})| t_{R_n} w_{R_n} \\ &\qquad + C_\varepsilon \int_{\mathbb{R}^N} |f(t_{R_n} w_{R_n})|^{q-1} |f'(t_{R_n} w_{R_n})| t_{R_n} w_{R_n}. \end{aligned}$$

From (ii), (iii) and (v) in Lemma 2.1, we get

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^N} h(f(t_{R_n} w_{R_n}))f'(t_{R_n} w_{R_n}) t_{R_n} w_{R_n} \\ &\leqslant \varepsilon t_{R_n}^p \int_{\mathbb{R}^N} |w_{R_n}|^p + C_\varepsilon t_{R_n}^{(q+1)/2} \int_{\mathbb{R}^N} |w_{R_n}|^{(q+1)/2}. \end{aligned}$$

Since $w_{R_n} \to w$ in $W^{1,p}(\mathbb{R}^N)$, the above inequality implies that $t_R \nrightarrow 0$. Thus, $t_R \to t_0 > 0$ and

$$\int_{\mathbb{R}^N} |\nabla w|^p = \int_{\mathbb{R}^N} h(f(t_0 w))f'(t_0 w) t_0^{1-p} w - \int_{\mathbb{R}^N} |f(t_0 w)|^{p-2} f(t_0 w) f'(t_0 w) t_0^{1-p} w.$$

By Corollary 2.2, $t_0 = 1$ and $I_\infty(t_R w_R) \to I_\infty(w) = c_\infty$ as $R \to \infty$, and therefore

$$\limsup_{\lambda\to\infty} c_\lambda \leqslant c_\infty.$$

Using the definition of c_λ and c_∞, we reach

$$c_\lambda \geqslant c_\infty \quad \text{for all } \lambda > 0,$$

which implies that

$$\liminf_{\lambda\to\infty} c_\lambda \geqslant c_\infty,$$

from which we conclude that

$$\lim_{\lambda\to\infty} c_\lambda = c_\infty.$$

□

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Proposition 4.3. *There exists $\hat{\lambda} > 0$ such that if $I_\lambda(v) \leqslant b_\lambda$ and $v \in \mathcal{M}_\lambda$, then*

$$\beta(v^+) \in \lambda\Omega_+ \quad \textit{for all } \lambda \geqslant \hat{\lambda}.$$

Proof. Assume that there exist $\lambda_n \to \infty$, $v_n \in \mathcal{M}_{\lambda_n}$ and $I_{\lambda_n}(v_n) \leqslant b_{\lambda_n}$ with

$$x_n = \beta(v_n^+) \notin \lambda_n\Omega_+.$$

Without loss of generality, we can assume that $v_n \geqslant 0$ for all $n \in \mathbb{N}$, and hence

$$x_n = \beta(v_n) \notin \lambda_n\Omega_+.$$

Fixing $R > \operatorname{diam}(\Omega)$, we have that

$$A_{\lambda_n R, \lambda_n r, x_n} \supset \Omega_{\lambda_n},$$

and so,

$$a(R, r, \lambda_n, x_n) \leqslant I_{\lambda_n}(v_n) \leqslant b_{\lambda_n}.$$

Using the fact that $a(R, r, \lambda_n, x_n) = a(R, r, \lambda_n)$ we have

$$a(R, r, \lambda_n) \leqslant b_{\lambda_n}. \tag{4.4}$$

Talking the limit of $n \to \infty$ in (4.4) and using Proposition 4.2, it follows that

$$\limsup_{n\to\infty} a(R, r, \lambda_n) \leqslant c_\infty,$$

which is a contradiction of Proposition 4.1. □

Proposition 4.4. *The problem associated to the functional $I_{\lambda,B}$ has a ground-state solution $v_{\lambda,r}$ that is radially symmetric at the origin.*

Proof. For simplicity, in this proof we denote by I the functional $I_{\lambda,B}$. Repeating the arguments used in the proof of Theorem 2.5, there exists $v \in W_0^{1,p}(B_{\lambda r}(0))$, a non-negative function, such that

$$I(v) = b_\lambda \quad \text{and} \quad I'(v) = 0.$$

If v^* is the Schwarz symmetrization of v, we have that $v^* \in W_0^{1,p}(B_{\lambda r}(0)), v^* \geqslant 0$ and satisfies

$$\int_{B_{\lambda r}(0)} |\nabla v^*|^p \leqslant \int_{B_{\lambda r}(0)} |\nabla v|^p. \tag{4.5}$$

Moreover, since $H \circ f$ and $t \mapsto |f(t)|^p$ are continuous and increasing functions with $(H \circ f)(0) = 0$ and $f(0) = 0$, we derive

$$\int_{B_{\lambda r}(0)} H(f(\alpha v^*)) = \int_{B_{\lambda r}(0)} H(f(\alpha v)) \quad \text{for all } \alpha > 0 \tag{4.6}$$

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and

$$\int_{B_{\lambda r}(0)} |f(\alpha v^*)|^p = \int_{B_{\lambda r}(0)} |f(\alpha v)|^p \qquad \text{for all } \alpha > 0. \tag{4.7}$$

Using the fact that $v \in \mathcal{M}_{\lambda,B}$, we obtain

$$\langle I'(v), v\rangle = 0 \quad \text{and} \quad I(v) = \max_{t \geqslant 0} I(tv).$$

From (H_1) and (H_2), there exists a unique $t^* > 0$ such that $t^* v^* \in \mathcal{M}_{\lambda,B}$. Thus, by (4.5)–(4.7),

$$b_\lambda \leqslant I(t^* v^*) \leqslant I(t^* v) \leqslant \max_{t \geqslant 0} I(tv) = I(v) = b_\lambda,$$

that is,

$$b_\lambda = I(t^* v^*) \quad \text{and} \quad t^* v^* \in \mathcal{M}_{\lambda,B}.$$

From the latter equality, $t^* v^*$ is a critical point of I on $\mathcal{M}_{\lambda,B}$, so $t^* v^*$ is a critical point of I in $W_0^{1,p}(B_{\lambda r})$ and thus

$$I(t^* v^*) = b_\lambda \quad \text{and} \quad I'(t^* v^*) = 0.$$

□

In what follows, we denote by $u_{\lambda,r}$ the ground-state solution $t^* v^*$ given in Proposition 4.4. For $\lambda > 0$ and $r > 0$, we define the operator $\Psi_r \colon \lambda\Omega_- \to W_0^{1,p}(\Omega_\lambda)$ given by

$$[\Psi_r(y)](x) = \begin{cases} u_{\lambda,r}(|x-y|) & \text{for } x \in B_{\lambda r}(y), \\ 0 & \text{for } x \in \Omega_\lambda \setminus B_{\lambda r}(y). \end{cases}$$

Note that for every $y \in \lambda\Omega_-$ we have

$$\beta(\Psi_r(y)) = y.$$

In the next result, we denote by $I_\lambda^{b_\lambda}$ the following set:

$$I_\lambda^{b_\lambda} = \{u \in \mathcal{M}_\lambda \colon I_\lambda(u) \leqslant b_\lambda\}.$$

Proposition 4.5. *For $\lambda \geqslant \hat{\lambda}$, we have*

$$\operatorname{cat} I_\lambda^{b_\lambda} \geqslant \operatorname{cat} \Omega_\lambda.$$

Proof. Assume that $\operatorname{cat} I_\lambda^{b_\lambda} = n$. This means that n is the smallest positive integer such that

$$I_\lambda^{b_\lambda} = A_1 \cup \cdots \cup A_n,$$

where A_j, $j = 1, \dots, n$, is closed and contractible in $I_\lambda^{b_\lambda}$, i.e. there exists $h_j \in C([0,1] \times A_j, I_\lambda^{b_\lambda})$ such that

$$h_j(0,u) = u \quad \text{for all } u \in A_j \qquad \text{and} \qquad h_j(1,u) = w_j \quad \text{for all } u \in A_j,$$

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for some $w_j \in I_\lambda^{b_\lambda}$ fixed. Consider $B_j = \Psi_r^{-1}(A_j)$, $1 \leqslant j \leqslant n$. The sets B_j are closed and

$$\lambda\Omega_- = B_1 \cup \cdots \cup B_n.$$

Using the deformation $g_j \colon [0,1] \times B_j \to \lambda\Omega_+$ given by

$$g_j(t,y) = \beta((h_j(t,\Psi_r(y)))^+),$$

we have that, for all $y \in B_j$,

$$g_j(0,y) = \beta((h_j(0,\Psi_r(y)))^+) = \beta(\Psi_r(y)) = y$$

and

$$g_j(1,y) = \beta((h_j(1,\Psi_r(y)))^+) = \beta(w_j^+)$$

for some $\beta(w_j) \in \lambda\Omega_+$ fixed. From this, we see that B_j is contractible in $\lambda\Omega_+$ for $1 \leqslant j \leqslant n$, which implies that $\operatorname{cat}_{\lambda\Omega_+}(\lambda\Omega_-) \leqslant n$. On the other hand, since Ω_+ and Ω_- are homotopically equivalent to Ω, it follows that $\operatorname{cat}\Omega_\lambda = \operatorname{cat}_{\lambda\Omega_+}(\lambda\Omega_-)$, and so $\operatorname{cat}\Omega_\lambda \leqslant n$. □

Proof of Theorem 1.1. Since I_λ satisfies the Palais–Smale condition on $\mathcal{M}_\lambda$, applying the Lyusternik–Schnirelmann theory and Proposition 4.5, we find that I_λ on $\mathcal{M}_\lambda$ has at least $\operatorname{cat}_{\Omega_\lambda}(\Omega_\lambda)$ critical points whose energy is less than b_λ for $\lambda \geqslant \hat{\lambda}$. Moreover, all solutions obtained are positive by the maximum principle [**25**, **26**]. □

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