Generalised Young Tableaux

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Introduction.

The present note contains generalisations and new proofs of certain theorems in the theory of Young Tableaux and Invariant Matrices. For an account of Young Tableaux and their applications, and an introduction to the method of Clebsch-Aronhold symbols, reference should be made to Rutherford [1], and Turnbull [1], respectively. An invariant matrix $T(A)$ of a given square matrix A is, as appears from the context in § 4 below, a matrix of polynomials in the elements of A, regarded as independent variables, such that $T(AB) = T(A)\,T(B)$. Further details, and references to original sources, are given in Wallace [1].

The echelon tableaux, or "skew" tableaux, treated below have been used by G. de B. Robinson (Robinson [1], [2], [3]) in his work on the representations of the symmetric group, and Staal (Staal [1]) has discussed what might be called the arithmetic theory of such tableaux with a view towards applications to modular representation theory.

§ 1. *Echelon Tableaux.*

Let n symbols, not necessarily all distinct, be arranged in rows with λ_1 symbols in the first row, λ_2 in the second, and so on, where

$$\lambda_1+\lambda_2+\ldots+\lambda_r = n$$
$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r,$$

r being the number of rows. If the first element of each row is vertically below that of the first row, then the configuration is known as a Young Tableau corresponding to the partition (λ) of n. But a more general configuration is obtained if the rows from the second downwards are displaced to the left in such a way that the first symbol of each row is vertically above some symbol of the row immediately below. Thus

$$\begin{array}{cccc} & \times & \times & \times \\ & \times & \times & \\ \times & \times & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \times & \times & \times \\ \times & \times & \times & & \\ \times & \times & & & \end{array}$$

are admissible shapes, the crosses marking the positions which are to be occupied by symbols, but the conditions of the definition exclude shapes

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such as

$$\begin{array}{ccccc} & & \times & \times & \times \\ \times & \times & \times & & \\ & \times & \times & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \times & \times & \times \\ & & \times & \times & \\ \times & \times & & & \end{array}$$

because in each case the first symbol of the second row fails to lie above any symbol of the third row (see § 6 below). The more general shape so defined will be called an echelon tableau, while in the special case where the first column contains the first symbol of each row the tableau will be called quadrantal—the latter type is simply the ordinary Young tableau.

Let a standard order be preassigned for the n symbols. An echelon tableau will be said to be standard if no column contains any symbol twice, and if the symbols appear in standard order reading along any row from left to right or down any column.

If the symbols of a tableau are all distinct, then the tableau may be associated with two substitutional operators—P, which is the sum of all permutations which rearrange the symbols within each row, without carrying any symbol from one row to another, and N, which is a linear combination of all the permutations which rearrange the symbols within each column of the tableau, the coefficient of each even permutation being $+1$ and that of each odd permutation -1. The product $E = PN$ is the Young operator of the tableau.

Let S_1 and S_2 be two tableaux of the same shape in the same set of symbols (not necessarily all distinct); if a tableau, S, also of the same shape, can be found such that each row of S contains the same set of symbols in some order as the corresponding row of S_1, and such that each column of S contains the same set of symbols in some order as the corresponding column of S_2, then S_1 will be said to be *convertible* to S_2. For example if S_1 is $\begin{array}{cccc} & 1 & 2 & 2 \\ & 3 & 5 & \\ 1 & 4 & & \end{array}$ and S_2 is $\begin{array}{cccc} & 1 & 1 & 2 \\ & 2 & 5 & \\ 4 & 3 & & \end{array}$ then S can be taken as $\begin{array}{cccc} & 2 & 1 & 2 \\ & 3 & 5 & \\ 4 & 1 & & \end{array}$. The situation may be represented diagrammatically by the gnomon $\begin{array}{|c|c|} \hline & S_2 \\ \hline S_1 & S \\ \hline \end{array}$ which in the example just cited becomes

$$\begin{array}{|c|c|} \hline & \begin{array}{cccc} & 1 & 1 & 2 \\ & 2 & 5 & \\ 4 & 3 & & \end{array} \\ \hline \begin{array}{cccc} & 1 & 2 & 2 \\ & 3 & 5 & \\ 1 & 4 & & \end{array} & \begin{array}{cccc} & 2 & 1 & 2 \\ & 3 & 5 & \\ 4 & 1 & & \end{array} \\ \hline \end{array} .$$

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It can be seen from the diagram at a glance that the rows of S are obtained by permuting the symbols within each row of S_1 separately, while the same may be said of the relation between the columns of S and those of S_2.

Now in the special case where S_1 and S_2 are quadrantal tableaux in n distinct symbols, the P-, N-, and E-operators corresponding to them being P_1, N_1, E_1 and P_2, N_2, E_2, respectively, S_1 is convertible to S_2 if and only if $N_2 P_1$ is non-zero, and this happens if and only if any pair of symbols appearing in the same row of S_1 appear in different columns of S_2. On the basis of these statements the quasi-idempotency of the Young operator, namely the relation $E^2 = \theta E$, where θ is an integer, and the theory of the irreducible representations of the symmetric group may be deduced. (Rutherford [1]).

But if S_1 and S_2 are non-quadrantal tableaux in n distinct symbols the relations are not so simple. Certainly if S_1 is convertible to S_2 then $N_2 P_1$ is non-zero, as in the quadrantal case, but the converse does not necessarily hold, as may be seen by taking S_1 as $\begin{smallmatrix} & 1 & 3 \\ 2 & 4 & \end{smallmatrix}$ and S_2 as $\begin{smallmatrix} & 1 & 2 \\ 3 & 4 & \end{smallmatrix}$. In other words S_1 may not be convertible to S_2 but the inconvertibility is now not necessarily due to the occurrence of a pair of symbols in the same row of S_1 and in the same column of S_2. The proof of the quasi-idempotency of the Young operator of a non-quadrantal tableau thus breaks down and indeed the relation $E^2 = \theta E$, with θ a numerical constant, does not necessarily hold for such operators.

The conditions under which two echelon tableaux may be subject to the relation of convertibility must therefore be studied afresh; the investigation will be confined to standard tableaux.

§ 2. *The Convertibility Theorem.*

Let the standard echelon tableaux of some fixed shape in any set of n symbols, say a set of n numbers, repetitions being permitted, selected from the numbers 1, 2, 3, ..., m, be ordered and labelled S_1, S_2, ..., S_f in such a way that S_i precedes S_j (*i.e.* $i < j$) if the first symbol of S_i, reading along each row in turn from left to right, which is not equal to the symbol in the corresponding position of S_j, is less than that symbol in numerical value.

Let S_i and S_j be any two tableaux of the ordered set S_1, S_2, ..., S_f and consider the convertibility of S_i to S_j; that is, consider the construction of a tableau S which may be carried into S_i by row operations and into S_j by column operations. Suppose that the symbol 1 occurs a_i times in the first row of S_i and a_j times in the first row of S_j. Any further occurrences of the symbol 1 in later rows of S_i or S_j must be to the left of the first position

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of the first row. Convertibility of S_i to S_j implies that the a_i 1's from the first row of S_i are carried into the first row of S and thence into the first row of S_j, since the 1's in this row of S_j are the only ones vertically above the first row of S. Moreover, further 1's from later rows of S_i may be carried into positions of S to the right of a vertical line through the first symbol of the first row, and these must also be carried into the first row of S_j. Hence $a_i \leqslant a_j$. If $a_i < a_j$ then S_j precedes S_i. Suppose that $a_i = a_j = a$ and let the symbol 2 occur b_i times in the first row of S_i, b_j times in the first row of S_j. Any further occurrences of 2 in either tableau must be to the left of a vertical line through the $(a+1)$-th position of the first row. The b_i 2's from the first row of S_i will be carried into the first row of S, the first a positions of which are, of course, occupied by 1's, and will then be carried into the first row of S_j, while some 2's from later rows of S_i may be carried into positions of S to the right of a vertical line through the $(a+1)$-th element of the first row and thence into the first row of S_j. And so $b_i \leqslant b_j$. If $b_i < b_j$ then S_j precedes S_i; otherwise let $b_i = b_j = b$, and continue the argument by considering the occurrences of 3, 4, ... in the first rows of S_i and S_j. The result is that either S_j precedes S_i or the first rows of S_i, S, S_j are all identical. In the latter case the first rows of S_i and S_j may be removed and the argument may be continued for the truncated tableaux. The final result is that if S_i is convertible to S_j then S_i cannot precede S_j.

This is a direct generalisation of the result already known for quadrantal tableaux in distinct symbols.

§3. *Double Forms and Young's Standard Theorem.*

Let $u^{(1)}, u^{(2)}, \ldots$ be a number of m-ary row vectors, and $x^{(1)}, x^{(2)}, \ldots$ a number of m-ary column vectors, all the elements in each case being independent variables. The compound inner product of the set of vectors $u^{(1)}, u^{(2)}, \ldots, u^{(r)}$ into the set $x^{(1)}, x^{(2)}, \ldots, x^{(r)}$ is written $(u^{(1)} u^{(2)} \ldots u^{(r)} \mid x^{(1)} x^{(2)} \ldots x^{(r)})$ and is equal to the determinant

$$\begin{vmatrix} u^{(1)}_{x^{(1)}} & u^{(1)}_{x^{(2)}} & \cdots & u^{(1)}_{x^{(r)}} \\ u^{(2)}_{x^{(1)}} & u^{(2)}_{x^{(2)}} & \cdots & u^{(2)}_{x^{(r)}} \\ \vdots & & & \\ u^{(r)}_{x^{(1)}} & u^{(r)}_{x^{(2)}} & \cdots & u^{(r)}_{x^{(r)}} \end{vmatrix}$$

where the symbols of the type u_x stand for inner products: $u_x \equiv \sum_{i=1}^{m} u_i x_i$.

Let U and X be echelon tableaux of the same shape whose symbols are the $u^{(i)}$ and $x^{(j)}$, respectively, the symbols in each tableau not necessarily being all distinct. The double form $\{U \mid X\}$ is defined as the product of

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all the determinants obtained by forming the compound inner product of the set of vectors in each column of U into the set of vectors in the corresponding column of X. And the polarized double form $\{\underline{U}|X\}$ is the sum of all double forms obtained by permuting the symbols of each row of U separately among themselves in all possible ways.

Young's Standard Theorem (Turnbull, [1], p. 357) states that, if U and X are quadrantal tableaux, and X is non-standard, then $\{U|X\}$ is a linear combination of forms $\{U|X_i\}_j$, where the X_i are standard tableaux in the symbols of X and of the same shape as X, added to double forms whose tableaux are "deeper" than X. The "deepening" process consists in increasing the lengths of certain columns at the expense of others. The presence of these "deepened" forms would be troublesome in the case of non-quadrantal tableaux: but a useful form of Young's Standard Theorem for general echelon tableaux may be obtained by considering $\{\underline{U}|X\}$. The same type of determinantal reduction as was used to prove Young's theorem in the quadrantal case may now be applied, and all forms in which a lengthening of some column of X, and of the corresponding column of U, takes place are annulled at each step by the symmetrisation with respect to the rows of U. The result is that

$$\{\underline{U}|X\} = \sum_i c_i \{\underline{U}|X_i\}$$

where the X_i are standard echelon tableaux in the symbols of X and of the same shape as X, while the values of the constants c_i are independent of U.

Returning again to quadrantal tableaux, we may state the further result that if U_i and X_i run through all the standard tableaux in a given set of u- and x-symbols, respectively, then the forms $\{\underline{U_i}|X_j\}$ are linearly independent. As will be seen later this result does not hold in the case of non-quadrantal tableaux, even if the tableaux are restricted to one fixed shape.

§ 4. *The T_0-matrix and Invariant Matrix corresponding to an Echelon Tableau.*

Let $A = [a_{ij}]$ be an m-rowed square matrix whose elements are independent variables, and let

$$a_{ij} = \alpha_i a_j = \beta_i b_j = \gamma_i c_j = \ldots$$

be equivalent Clebsch-Aronhold symbolisations of these elements. Let the set of numbers in which 1 appears ρ_1 times, 2 appears ρ_2 times, and so on, where $\rho_1 + \rho_2 + \ldots + \rho_r = n$, be selected from the numbers 1, 2, ..., m, and called the set (ρ). Form all the standard echelon tableaux $S_{(\rho)1}$, $S_{(\rho)2}$, ...

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of a given shape, and having the set (ρ) as symbols. Also let $\alpha_{(\rho)i}$ and $a_{(\rho)i}$ be defined, respectively, as the product of Greek letters $\alpha\beta\gamma\ldots$ and that of Latin letters $abc\ldots$ with the symbols of $S_{(\rho)i}$ in order, reading along each row in turn from left to right, as subscripts. Then if E is the Young operator of the tableau $S(a)$ of the given shape with the letters $a, b, c, \ldots$ in alphabetical order, reading along each row in turn, as symbols, the polynomials

$$\alpha_{(\rho)i} E a_{(\sigma)j}$$

for all $(\rho)i$, $(\sigma)j$ may be constructed, and, regarding a_{ij} as the symbolic inner product i_j, may be written as

$$\{\underline{S_{(\rho)i}|}\, S_{(\rho)j}\},$$

the left-hand tableau being formed from row suffixes of A, and the right-hand one from column suffixes. Define the matrix $T_0(A)$ as that whose $(\rho)i$-$(\sigma)j$-th element is $\{\underline{S_{(\rho)i}|}\, S_{(\rho)j}\}$, $(\rho)i$ being the row-label and $(\sigma)j$ the column-label.

In particular, replace A by the unit matrix. Then $\{\underline{S_{(\rho)i}|}\, S_{(\sigma)j}\}$ is certainly zero for $(\rho) \neq (\sigma)$, and also $\{\underline{S_{(\rho)i}|}\, S_{(\rho)j}\}$ is certainly zero if $S_{(\rho)i}$ is not convertible to $S_{(\rho)j}$; thus $\{\underline{S_{(\rho)i}|}\, S_{(\rho)j}\}$ is zero for $i < j$. And $\{\underline{S_{(\rho)i}|}\, S_{(\rho)i}\}$ is non-zero. The matrix $T_0(I)$ is therefore a quasi-diagonal matrix, with a submatrix on the diagonal for each value of (ρ); and these submatrices are triangular with zeros above the diagonal and non-zero numbers on the diagonal. The form of $T_0(I)$ implies that it is non-singular.

Let B be a second square matrix of order $m \times m$ whose elements b_{ij} are independent variables. Each column of AB is a linear combination of columns of A. Differently expressed, this implies that each Latin letter a_i, b_i, ... in the Clebsch-Aronhold symbolisation of A is replaced by a linear combination of symbols of the same letter name (*i.e.*, a_i is replaced by a linear combination of a's, b_i by a linear combination of b's). Substituting these linear combinations for the symbols in the polynomials $\alpha_{(\rho)i} E a_{(\sigma)j}$ and expanding, we see that each column of $T_0(AB)$ is a linear combination of column vectors in each of which the elements are symbolic polarised double forms having a common right-hand tableau, while in each vector the left-hand tableaux are the same $S_{(\rho)i}$ as mark the rows of $T_0(A)$. By Young's Standard Theorem (§ 3) it follows that each column of $T_0(AB)$ is a linear combination of columns of $T_0(A)$. In matrix notation

$$T_0(AB) = T_0(A)\, S(B), \tag{1}$$

where $S(B)$ depends on the elements of B, but not on those of A. Put $A = I$ in (1):

$$T_0(B) = T_0(I)\, S(B).$$

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Hence

$$T_0(I)\,S(AB) = T_0(AB) = T_0(A)\,S(B) = T_0(I)\,S(A)\,S(B).$$

Divide by the non-singular $T_0(I)$:

$$S(AB) = S(A)\,S(B).$$

And so $S(A)$ and the equivalent $T(A) = T_0(I)\,S(A)\,T_0^{-1}(I) = T_0(A)\,T_0^{-1}(I)$ are both invariant matrices of A (*cf.* the construction in Littlewood [1], p. 184). It is also worth noting that the elements of the matrix $T_0^{-1}(I)$ correspond to the correction factors M_i introduced by Young in order to obtain orthogonal idempotents from the tableaux operators (*cf.* Rutherford [1] for references).

The results of this paragraph were derived for the case of quadrantal tableaux in a previous paper (Wallace [1], p. 106) in a slightly different way. The non-singularity of $T_0(I)$ is proved here in a rather more direct way, revealing that this matrix is triangular.

By a method similar to that used for quadrantal tableaux, a further result may be obtained for general echelon tableaux, namely the value of the trace of the matrix $T(A)$ constructed above. Write the $(\rho)i$-$(\rho)j$-th element of $T_0(I)$ as $\xi_{(\rho)ij}$ and that of $T_0^{-1}(I)$ as $\zeta_{(\rho)ij}$. The trace of $T(A)$ is a symmetric function of the latent roots of A (Wallace [1], p. 110) which may therefore be taken to be a diagonal matrix with $a_{ij} = \omega_i \delta_{ij}$, where δ_{ij} is the Kronecker δ. The $(\rho)i$-$(\rho)i$-th element of $T(A)$ is

$$\sum_h \alpha_{(\rho)i}\, E a_{(\rho)h}\, \zeta_{(\rho)hi} = \sum_h \alpha_{(\rho)i}\, a_{(\rho)i}\, \xi_{(\rho)ih}\, \zeta_{(\rho)hi} + \text{certain vanishing terms}$$

$$= \alpha_{(\rho)i}\, a_{(\rho)i}.$$

Summing with respect to $(\rho)i$ and using Aitken's theorem on the monomial expansion of the bialterant associated with the given echelon tableau (Aitken [1]), we find that the trace of $T(A)$ is the isobaric determinant in $\omega_1, \omega_2, \ldots, \omega_m$ corresponding to the echelon tableau in question. For example, if the tableau is

```
          ×  ×  ×  ×
       ×  ×  ×
 ×  ×  ×
 ×  ×
```

in which the numbers of symbols in the successive rows are 4, 3, 3, 2 and the displacements of the left-hand end of the successive rows, beyond the row above in each case, are 1, 2, 0, then the corresponding isobaric determinant is

$$\begin{vmatrix} h_2 & h_5 & h_8 & h_{10} \\ 1 & h_3 & h_6 & h_8 \\ \cdot & 1 & h_3 & h_5 \\ \cdot & \cdot & h_2 & h_4 \end{vmatrix}$$

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where the differences between the suffixes of successive pairs of rows, first and second, second and third, third and fourth are $1+1$, $2+1$, $0+1$, and the suffixes of the diagonal elements read in reverse are 4, 3, 3, 2; the function h_r is the sum of all r-th degree products of the ω_i.

§ 5. *Linear Dependence Properties of Polarised Double Forms.*

Since $T(BA) = T(B)\,T(A)$ it follows that

$$T_0(BA) = T(B)\,T_0(A).$$

And so any row of $T_0(BA)$ is a linear combination of rows of $T_0(A)$. Taking B as a permutation matrix, this may be interpreted in terms of double forms as the theorem that $\{U\,|\,X\}$, where U is non-standard, is a linear combination of the forms $\{\underline{U_i}|\,X\}$ where the U_i are standard tableaux in the symbols of U, and of the same shape, and the values of the coefficients of the linear combination are independent of X (*cf.* Turnbull [1], p. 362; Wallace [1], p. 110).

$T_0(A)$ is non-singular in the sense that $|\,T_0(A)|$ is not zero identically in the a_{ij}: for in particular $|\,T_0(I)\,| \neq 0$. In terms of double forms this implies that the forms $\{\underline{U}|\,X_i\}$, where the X_i run through all the standard tableaux of given shape in the x-symbols and where U is any fixed tableau of the same shape in the u-symbols, are linearly independent. For if a relation

$$\sum_i c_i\{\underline{U}|\,X_i\} = 0 \tag{2}$$

could be found, it would continue to hold if U were replaced by a tableau U_0 in which all the symbols within any row are the same, different rows, however, containing different symbols, since the forms involving U are derived by polarisations from forms involving U_0 instead; and then U_0 could be replaced by any tableau, by means of further polarisations. Thus a relation of the type (2) would hold with the values of the c_i independent of U, and this would imply the identical vanishing of $|\,T_0(A)\,|$.

There is also the weaker linear independence condition that, if the U_i are the standard tableaux of some given shape in the u-symbols, then no relation

$$\sum_i c_i\,\{\underline{U_i}|\,X\} = 0$$

can exist with c_i whose values are independent of X, for such a relation would imply the identical vanishing of $|\,T_0(A)\,|$. In particular this implies that, if the symbols of X are all distinct, the forms $\{\underline{U_i}|\,X\}$ are linearly independent. But the linear independence of the forms $\{\underline{U_i}|\,X\}$ need not hold for every X. For example, if X is the tableau $\begin{smallmatrix} x & y \\ y & y \end{smallmatrix}$ and if the U_i

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are standard tableaux of the same shape in $u, v, \ldots$, then the $\{\underline{U_i}|X\}$ cannot be linearly independent, because, in particular

$$\left\{\begin{matrix} u & v \\ \underline{u} & \underline{v} \end{matrix}\,\middle|\,\begin{matrix} & x & y \\ y & y & \end{matrix}\right\} = \left\{\begin{matrix} u & v \\ u & v \end{matrix}\,\middle|\,\begin{matrix} & x & y \\ y & y & \end{matrix}\right\} + \left\{\begin{matrix} v & u \\ v & u \end{matrix}\,\middle|\,\begin{matrix} & x & y \\ y & y & \end{matrix}\right\} = 0.$$

The fact that the invariant matrix $T(A)$ corresponding to a non-quadrantal tableau is reducible shows that the elements of $T_0(A)$ are not linearly independent, and so the polarised double standard forms $\{\underline{U_i}|X_j\}$ are *not* linearly independent unless the tableaux are quadrantal.

§ 6. *Composite Tableaux.*

The exclusion in § 1 of tableaux of the type $\begin{matrix} & & \times & \times & \times \\ & & \times & \times & \\ \times & \times & & & \end{matrix}$ was purely a matter of convenience, and not in any way essential for the subsequent discussion. Such a tableau may be called *composite.* It is clear that a composite tableau $S = \begin{matrix} & S_1 \\ S_2 & \end{matrix}$ is standard if and only if S_1 and S_2 are standard; and if $S = \begin{matrix} & S_1 \\ S_2 & \end{matrix}$, $S' = \begin{matrix} & S_1' \\ S_2' & \end{matrix}$ are similarly decomposed, then S is convertible to S' if and only if S_1 is convertible to S_1' and S_2 to S_2'. Any double form or polarised double form corresponding to $S = \begin{matrix} & S_1 \\ S_2 & \end{matrix}$ is equal to the product of two such forms, one corresponding to S_1, the other to S_2. In particular this may be applied to the symbolic double forms which give the traces of invariant matrices. Thus the trace of the invariant matrix associated with S is the product of the traces of those associated with S_1 and S_2; or, in other words, the invariant matrix associated with S is the direct product of those associated with S_1 and S_2.

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