ON SPARSELY TOTIENT NUMBERS

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1. Introduction. Following Masser and Shiu [6] we say that a positive integer n is *sparsely totient* if

$$m > n \Rightarrow \phi(m) > \phi(n). \tag{1}$$

Here ϕ is the familiar Euler totient function. We write $\mathcal{F}$ for the set of sparsely totient numbers. In [6] several results are proved about the multiplicative structure of $\mathcal{F}$. If we write $P(n)$ for the largest prime factor of n then it was shown (Theorem 2 of [6]) that

$$n \in \mathcal{F} \Rightarrow P(n) \leq (1 + o(1))\log^2 n, \tag{2}$$

and infinitely often

$$P(n) \geq (2 + o(1))\log n.$$

It was stated in [6] that the present author could improve (2) by reducing the exponent of the logarithm to $19/10 + \epsilon$. The proof of this result has never been published and here we shall establish a better result. We shall also discuss the relationship between results of this sort and the problem of the greatest prime factor of an integer in an interval [2]. For $j \geq 1$ we write $Q_j(n)$ for the jth smallest prime not dividing n, and $P_j(n)$ for the jth largest prime dividing n (so $P(n) = P_1(n)$). In [6] some results were obtained for these quantities. We shall also improve the bounds for these numbers. Our results are as follows.

THEOREM 1. *For $n \in \mathcal{F}$ and any $\epsilon > 0$ we have*

$$P(n) \ll (\log n)^{2-8/65+\epsilon}. \tag{3}$$

Here the implied constant depends only on ϵ.

THEOREM 2. *We have*

$$\limsup_{\substack{n\in\mathcal{F}\\ n\to\infty}} \frac{P_j(n)}{\log n} \leq \frac{j}{j-1} \quad \textit{for } j \geq 2. \tag{4}$$

Also

$$\liminf_{\substack{n\in\mathcal{F}\\ n\to\infty}} \frac{Q_j(n)}{\log n} \geq \frac{j}{j+1} \quad \textit{for } j \geq 1. \tag{5}$$

In the most interesting cases of Theorem 2 (P_2, Q_1) Masser and Shiu had $1 + \sqrt{2}$ and $\sqrt{2} - 1$ for the right hand sides of (4) and (5) respectively.

The basic idea in the proofs is to replace a factor m of n by another number t with $\phi(t) < \phi(m)$ and t not dividing n. If we write

$$r = \left[\frac{m}{t}\right] + 1$$

([ ] denotes integer part) then $nrt/m > n$, while if $r - m/t$ is sufficiently small we have $\phi(nrt/m) < \phi(n)$. This shows that $n \notin \mathcal{F}$. The difficult part is to show that $r - m/t$ is

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small. This reduces to a problem in Diophantine approximation; essentially we must show that

$$1-\epsilon<\left\{\frac{m}{t}\right\}<1$$

for some sufficiently small ϵ. This argument was also used in [**6**], but we are able to prove stronger results in Diophantine approximation and so obtain our results.

We note that if we write

$$R(n)=n\prod_{p|n}p^{-1}$$

then a slight alteration to the proof of Theorem 1 yields

$$R(n)\ll(\log n)^{2-8/65+\epsilon},$$

which improves [**6**, Corollary to Lemma 5].

2. Some preparatory lemmas. It will make the proof simpler if we make use of the following result from [**6**], rather than working *ab initio*.

LEMMA 1. *For $j\geqslant 3$ we have*

$$\limsup_{\substack{n\in\mathcal{F}\\ n\to\infty}}\frac{P_j(n)}{\log n}\leqslant\frac{j}{j-2},\tag{6}$$

and

$$\liminf_{\substack{n\in\mathcal{F}\\ n\to\infty}}\frac{Q_j(n)}{\log n}\geqslant\frac{j-1}{j+1}.\tag{7}$$

Proof. See [**6**, Lemma 7, Corollary].

LEMMA 2. *Let $r\geqslant 2$ be an integer and suppose that $\frac{1}{2}>\eta$, $\theta>0$ are given. Then for all sufficiently large x (in terms of η and θ), for each integer m with*

$$(x/3)^{r-1}\leqslant m\leqslant x^{r-1}\tag{8}$$

there are at least

$$\frac{\eta(\theta x)^r}{2(\log x)^r}\tag{9}$$

solutions to

$$1-\eta<\left\{\frac{q_1\cdots q_r}{m}\right\}<1,\tag{10}$$

where the q_j are distinct primes satisfying

$$1-2\theta\leqslant\frac{q_j}{x}\leqslant 1-\theta.\tag{11}$$

Proof. This will be established in Section 4.

LEMMA 3. *Let $r \geqslant 2$ be an integer, and suppose that $\frac{1}{2} > \theta$, $\eta > 0$ are given. Then, for all sufficiently large x, for each integer m with*

$$x^{r+1} \leqslant m \leqslant x^{r+2} \tag{12}$$

there are at least

$$\frac{\eta}{2}\left(\frac{\theta x}{\log 2x}\right)^r \tag{13}$$

solutions to

$$1-\eta < \left\{\frac{m}{q_1 \ldots q_r}\right\} < 1, \tag{14}$$

where the q_j are distinct primes satisfying

$$1+\theta \leqslant \frac{q_j}{x} \leqslant 1+2\theta. \tag{15}$$

Proof. See Section 6.

LEMMA 4. *Let $\epsilon > 0$ be given. Then there is a $K(\epsilon)$ such that for all x, v with $x > K(\epsilon)$ and*

$$v^{2-8/65+\epsilon} \leqslant x \leqslant 2v^2, \tag{16}$$

there are

$$\gg \frac{x}{v \log x} \tag{17}$$

solutions in primes p to

$$1-\frac{x}{16v^2} < \left\{\frac{x}{p}\right\} < 1, \quad \text{with } 2v \leqslant p < 3v. \tag{18}$$

Proof. See Section 6.

3. Proof of theorems. Suppose that $n \in \mathscr{F}$ and

$$(\log n)^\alpha \leqslant P(n) \leqslant 2(\log n)^2$$

with $\alpha = 2 - 8/65 + \epsilon$. By (2) we know that $P(n)$ cannot exceed $2(\log n)^2$ if n is sufficiently large. Let $p_1 = P(n)$ and write $m = n/p_1$. We apply Lemma 4 with $x = p_1$, $v = \log n$. It follows that there are $\gg p_1/(v^2 \log p_1)$ solutions to (18). Since there are at most three primes between $2v$ and $3v$ which divide n if n is sufficiently large (using (6)) we can deduce that (18) has a solution with $p \nmid n$. If we write $r = [p_1/p] + 1$ we then have

$$\begin{aligned}
\phi(mrp) &\leqslant r\phi(m)p(1-1/p) \leqslant \frac{rp(1-1/p)}{p_1(1-1/p_1)}\phi(n) \\
&= \phi(n)\left(1-\frac{1}{p}+\frac{1}{p_1}+O\left(\frac{1}{pp_1}\right)\right)\left(1+\frac{p}{p_1}\left(1-\left\{\frac{p_1}{p}\right\}\right)\right) \\
&\leqslant \phi(n)\left(1-\frac{7}{48v}+\frac{1}{p_1}+O\left(\frac{1}{v^2}\right)\right) < \phi(n) \quad \text{by (18) with } x = p_1
\end{aligned}$$

if n is sufficiently large. Of course $mrp > n$, and this contradicts our original hypothesis that $n \in \mathcal{F}$. We conclude that $P(n) < (\log n)^\alpha$ for all sufficiently large n, which establishes (3).

Proof of (4). Let $p_k = P_k(n)$ for $k = 1, 2, \ldots$. Suppose that

$$p_j > \frac{j \log n}{j - 1 - \epsilon}$$

for some small positive ϵ. Write

$$x = \log n, \qquad \theta = \eta = \epsilon^2, \qquad m = \prod_{k=1}^{j} p_k. \tag{19}$$

It then follows from (14) and (15) with $r = j - 1$ that there are distinct primes $q_1, \ldots, q_r$ with

$$1 - \eta < \left\{\frac{m}{q_1 \ldots q_r}\right\} < 1, \qquad 1 + \eta \leqslant \left\{\frac{q_t}{x}\right\} < 1 + 2\eta, \quad 1 \leqslant t \leqslant r. \tag{20}$$

By (6) n has $\ll \eta^{-1}$ prime divisors among the possible q_t, whereas the number of solutions to (20) exceeds

$$\frac{\eta^j (\log n)^r}{2(\log 2 \log n)^r}.$$

Hence, if n is sufficiently large, there is a solution to (20) in distinct primes none of which divide n.

Let

$$s = \frac{n}{m} q_1 \ldots q_r \left(1 + \left[\frac{m}{q_1 \ldots q_r}\right]\right).$$

Then $s > n$, while

$$\begin{aligned}\phi(s) &\leqslant \phi(n)\left(1 - \sum_{k=1}^{r} q_k^{-1} + \sum_{k=1}^{j} p_k^{-1} + O(x^{-2})\right)\left(1 + \frac{\eta q_1 \ldots q_r}{m}\right)\\ &\leqslant \phi(n)\left(1 - \frac{(j-1)}{x(1+2\eta)} + \frac{j - 1 - \epsilon}{x} + O(x^{-2})\right)\left(1 + \frac{\eta}{x}\right)\\ &< \phi(n) \quad \text{if } \epsilon \text{ is small enough and } n \text{ is large.}\end{aligned}$$

This contradiction gives

$$p_j \leqslant \frac{j \log n}{j - 1 - \epsilon},$$

which establishes (4).

The reader will notice that we can prove a slightly stronger result, namely that

$$\liminf_{\substack{n \in \mathcal{F} \\ n \to \infty}} \sum_{k=1}^{j} \frac{\log n}{P_k(n)} \geqslant j - 1.$$

Proof of (5). Let $p_k = Q_k(n)$ for $k = 1, 2, \ldots$. Suppose $p_j < j(j+1+\epsilon)^{-1}\log n$ for some small $\epsilon > 0$. We define m, θ, η, x as in (19). The proof may then be completed in a similar manner to the proof of (4). We apply Lemma 2 with $r = j+1$ in place of Lemma 3, and use (7) with (9) to deduce that there are solutions to (10) with $q_1 \ldots q_r$ *dividing* n. Again we could actually prove the stronger result that

$$\limsup_{\substack{n\in\mathcal{F}\\ n\to\infty}} \sum_{k=1}^{j} \frac{\log n}{Q_k(n)} \leqslant j+1.$$

4. Proof of Lemma 2. We first require two standard results, the first of which will be needed for Section 6 as well.

LEMMA 5. *Let $\mathcal{I}$ be a subinterval of* $[0, 1)$ *of length* γ, *and let* x_n, c_n *be two sequences of reals with* $c_n \geqslant 0$. *Suppose that* L, $N \geqslant 1$ *are given. Then*

$$\sum_{\substack{n\leqslant N\\ \{x_n\}\in\mathcal{I}}} c_n = \sum_{n\leqslant N} c_n(\gamma + O(L^{-1})) + O\Big(\sum_{h\leqslant L} \min(\gamma, h^{-1}) \Big|\sum_{n\leqslant N} c_n e(hx_n)\Big|\Big). \tag{21}$$

Proof. This may be deduced from Chapter 2 of [**1**]. Here the implied constants are absolute.

LEMMA 6. *Suppose that a_s, b_t are sequences of complex numbers bounded in modulus by some constant, and that M, N, L are given real quantities exceeding* 1. *Let m be an integer greater than* 1. *Then*

$$\sum_{h=1}^{L} \Big|\sum_{\substack{s\leqslant M\\ t\leqslant N}} a_s b_t e\Big(\frac{sth}{m}\Big)\Big|$$
$$\ll (\log mNL)^2 LMN(m^{-1/2} + (LMN/m)^{-1/2} + M^{-1/2} + (LN)^{-1/2}). \tag{22}$$

Proof. Results of this sort are well known. For example (22) may be obtained with a slight alteration to the argument in Chapter 25 of [**4**].

Proof of Lemma 2. Let $u = [(r+1)/2]$, $v = 1+u$, and write

$$a_s = \sum_{q_1\ldots q_u = s} 1, \qquad b_t = \sum_{q_v\ldots q_r = t} 1,$$

where the q_h are constrained by (11). We apply Lemma 5 with

$$N = (1-\theta)^r x^r, \qquad L = \log x, \qquad \mathcal{I} = (1-\eta, 1),$$

and

$$c_n = \sum_{st=n} a_s b_t.$$

For x sufficiently large we easily obtain from the prime number theorem that

$$2\Big(\frac{\theta x}{\log x}\Big)^r > \sum_{n\leqslant N} c_n > \frac{2}{3}\Big(\frac{\theta x}{\log x}\Big)^r.$$

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We also obtain from Lemma 6 that

$$\sum_{h\leq L}\left|\sum_{s,t} a_s b_t e\left(\frac{sth}{m}\right)\right| \leq (\log x)^3 x^{r-1/2}.$$

The lower bound (9) for the number of solutions to (10) subject to (11) then follows from (21) since the number of solutions where the q_h are not all distinct is $\ll (x/\log x)^{r-1}$. This completes the proof of Lemma 2.

5. Further exponential sum estimates. In this section we need to turn to deeper methods. We put $\eta = \epsilon^2$ throughout.

LEMMA 7. *Let c_h, a_s, b_t be sequences of complex numbers bounded in modulus by one. Suppose that*

$$v^{2-\alpha} < x \ll v^2, \quad Hx \ll v^{2+\eta}, \quad Hv^{\epsilon-\eta} \ll N \ll v^{1-5\alpha-\epsilon}, \quad M < M_1 \leq 2M, \quad N < N_1 \leq 2N, \quad H < H_1 \leq 2H. \tag{23}$$

Then

$$\sum_{M\leq s<M_1} a_s \sum_{\substack{N\leq t<N_1\\ 2v\leq st<3v}} b_t \sum_{H\leq h<H_1} c_h e\left(\frac{xh}{st}\right) \ll v^{1-\eta}. \tag{24}$$

Proof. This follows from Lemma 9 of **[2]** with the exponent pair $(\kappa, \lambda) = (\frac{1}{2}, \frac{1}{2})$ and $Q = Nv^{-\epsilon/2}H^{-1}$.

LEMMA 8. *Let $M < M_1 \leq 2M$, $N < N_1 \leq 2N$, $0 < \alpha < \frac{1}{7}$,*

$$v^{2-\alpha} \leq u \ll v^{2+\eta/2}, \max(v^{2\alpha+\epsilon}, v^{(46\alpha-1)/11+\epsilon}, v^{(62\alpha-5)/6+\epsilon}) \leq N. \tag{25}$$

Then

$$\sum_{M<m\leq M_1}\left|\sum_{\substack{N<n\leq N_1\\ 2v\leq mn<3v}} e\left(\frac{u}{mn}\right)\right| \ll v^{1-\eta-\alpha}. \tag{26}$$

Proof. For $N \geq v^{1/2+\alpha+\epsilon}$ this follows from the classical estimate of Van der Corput (Theorem 5.9 of **[7]**). Since $\alpha < \frac{1}{6}$ we may therefore suppose that $N < v^{1-2\alpha-\epsilon}$ in the following. We now need to adapt the proof of Lemma 5 in **[2]**. We put

$$T = \sum_{M<m\leq M_1}\left|\sum_{\substack{N<n\leq N_1\\ 2v\leq mn<3v}} e\left(\frac{u}{mn}\right)\right|.$$

Then (see (21) of **[2]**) for any positive integer Z we have

$$T^2 \leq M(1+NZ^{-1}) \sum_{M<m\leq M_1} \sum_{|k|\leq Z}\left(1-\frac{|k|}{Z}\right)\sum_{n^*} e\left(\frac{u}{m}((n+k)^{-1}-n^{-1})\right),$$

where n^* indicates the conditions

$$\max(N, 2vm^{-1}) \leq n, n+k < \min(N_1, 3vm^{-1}).$$

We choose $Z=v^{2\alpha+\epsilon/6}$, so that $1\leq Z \ll Nv^{-\epsilon/2}$ by (25). Put

$$T_k=\sum_{M<m\leq M_1}\sum_{n^*} e\left(\frac{u}{m}((n+k)^{-1}-n^{-1})\right).$$

Continuing to follow [2] we apply Lemma 3 of [2] to T_k. We require some further notation to do this. For the function of two variables $f(x,y)=f_k(x,y)=u((y+k)^{-1}-y^{-1})/x$ we write

$$H(f)=H(f;(x,y))=\frac{\partial^2 f(x,y)}{\partial x^2}\frac{\partial^2 f(x,y)}{\partial y^2}-\left(\frac{\partial^2 f(x,y)}{\partial x\,\partial y}\right)^2.$$

Let $\mathscr{D}$ be the domain of summation of (m,n) in T_k and write

$$\Delta=\left\{\left(\frac{\partial f(x,y)}{\partial x},\frac{\partial f(x,y)}{\partial y}\right):(x,y)\in\mathscr{D}\right\}.$$

For $\mathbf{v}\in\Delta\cap\mathbb{Z}^2$ let $\mathbf{x}(\mathbf{v})$ denote the solution (which in the present context can be shown to be unique) of

$$\left(\frac{\partial f(\mathbf{x})}{\partial x},\frac{\partial f(\mathbf{x})}{\partial y}\right)=\mathbf{v}$$

Now write

$$H_*(\mathbf{v})=H(f;\mathbf{x}(\mathbf{v})),$$

$$U_k=\sum_{\mathbf{v}\in\Delta} H_*(\mathbf{v})^{-1/2}e(f(\mathbf{x}(\mathbf{v}))-\mathbf{v}\,.\,\mathbf{x}(\mathbf{v})),$$

('.' denoting scalar product), and

$$B_k=uv^{-1}\,|k|\,N^{-1}.$$

Working as in [2] we obtain

$$\begin{aligned}T^2 &\ll v^2Z^{-1}+vZ^{-1}\sum_{1\leq|k|\leq Z}|T_k|\\ &\ll v^{2-2\alpha-\epsilon/6}+vZ^{-1}\sum_{1\leq|k|\leq Z}|U_k|+vZ^{-1}\sum_{1\leq|k|\leq Z}((N+M)\log v+B_k^{11/12}+vB_k^{-1})\\ &\ll v^{2-2\alpha-\epsilon/6}+vZ^{-1}\sum_{1\leq|k|\leq Z}|U_k|+vZ^{-1}\sum_{1\leq|k|\leq Z}(vB_k^{-1}+B_k^{11/12}).\end{aligned}\tag{27}$$

Now the last sum in (27) is

$$\begin{aligned}&\ll vZ^{-1}\sum_{1\leq|k|\leq Z}(v^2u^{-1}\,|k|^{-1}\,N+(u\,|k|/(vN))^{11/12})\\ &\ll v^2Z^{-1}(vu^{-1}N\log v+v^{11/6+\eta-1-11/12}Z^{1+11/12}N^{-11/12}).\end{aligned}\tag{28}$$

The first term in the brackets of (28) is much less than 1 since $N < v^{1-2\alpha-\epsilon}$. The second term in the brackets is much less than

$$v^{\eta-1/12}Z^{23/12}N^{-11/12},$$

which is much less than 1, using $N > v^{(46\alpha-1)/11+\epsilon}$. Thus

$$T^2 \ll v^{2-2\alpha-\epsilon/6} + vZ^{-1} \sum_{1\leq|k|\leq Z} |U_k|.$$

We estimate $|U_k|$ by the method of exponent pairs as in **[2]** and obtain, for any exponent pair (κ, λ), that

$$\begin{aligned} vZ^{-1} \sum_{1\leq|k|\leq Z} |U_k| &\ll v^2Z^{-1}(u^\lambda v^{-1-\lambda}Z^{1+\lambda}N^{1+\kappa-2\lambda} \\ &\quad + u^{\lambda-1}v^{1-\lambda}N^{\kappa+1-2\lambda}Z^\lambda + NZ/v + vNu^{-1}\log v) \\ &= v^2Z^{-1}(A_1 + A_2 + A_3 + A_4) \quad \text{say.} \end{aligned}$$

We have

$$A_1 \ll v^{\lambda-1+2(1+\lambda)(\alpha+\epsilon/6)}N^{1+\kappa-2\lambda}.$$

In the present context $(\kappa, \lambda) = (1/9, 13/18)$ is an efficient choice (see Section 5.20 of **[7]** for a brief discussion of exponent pairs). This gives $A_1 \ll 1$ if

$$N > v^{(62\alpha-5)/6+31\epsilon/36}.$$

The hypothesis (25) thus ensures that the contribution from A_1 is of a suitable size. Also

$$A_2 \ll v^{\lambda-1+2\lambda(\alpha+\epsilon/6)}N^{1+\kappa-2\lambda}$$

so the bound for A_1 holds *a fortiori* for A_2. The terms A_3 and A_4 may be bounded in a satisfactory manner also, using $N < v^{1-2\alpha-\epsilon}$. This completes the proof of this lemma.

LEMMA 9. *Let c_h, a_s, b_t be sequences of complex numbers, bounded in modulus by one. Suppose that*

$$x^r < MN < (3x)^r, \qquad N < N_1 \leq 2N, \qquad M < M_1 \leq 2M,$$
$$x^{r/4} < M < X^{3r/4}, \qquad x^{r+1} \leq m \leq x^{r+2}, \quad 1 \leq H \ll \log x.$$

Then there exists a $\delta(r) > 0$ such that

$$\sum_{N\leq s<N_1} a_s \sum_{M\leq t<M_1} b_t \sum_{1\leq h<H} c_h e\left(\frac{mh}{st}\right) \ll x^{r-\delta(r)}.$$

Proof. This may be obtained by using the Cauchy–Schwarz inequality in conjunction with Theorems 5.9, 5.11 and 5.13 of **[7]**.

6. Proof of Lemmas 3 and 4. Lemma 3 follows quickly from a combination of Lemmas 5 and 9. To prove Lemma 4 we need to employ Heath–Brown's generalized Vaughan identity **[5]**, whereby, for any function $f(x)$ with $|f(x)| \leq 1$ and any $\alpha \in (0, 1/7)$, the sum $\sum_{2v \le n < 3v} \Lambda(n) f(n)$ (here $\Lambda(n)$ is the von Mangoldt function)

may be decomposed into $\ll (\log v)^6$ sums of the form

$$\sum_{M<m<M_1} a_m \sum_{\substack{N<n<N_1 \\ 2v \le mn < 3v}} b_n f(mn), \qquad b_n \equiv 1 \text{ or } \log n, \qquad N > v^{(1-\alpha)/2-\epsilon/2}; \tag{I}$$

or

$$\sum_{M<m<M_1} a_m \sum_{\substack{N<n<N_1 \\ 2v \le mn < 3v}} b_n f(mn), \qquad v^{\alpha+\epsilon} \ll N \ll v^{1/3}. \tag{II}$$

Here a_m, b_n are real numbers with $|a_m|$, $|b_n| \ll v^{\eta}$ for any $\eta > 0$, for both types of sums. (See the proof of Lemma 2 in [3], where α here corresponds to 2γ there).

We complete the proof by appealing to Lemma 5 with $c_n = \Lambda(n)$ and $L = 16v^2x^{\eta/2-1}$. We use Lemma 8 to bound sums of type (I) (applying partial summation if $b_n = \log n$) and Lemma 7 to bound sums of type (II). We are able to do this since

$$\frac{1-\alpha-\epsilon}{2} > \frac{62\alpha-5}{6} + \epsilon, \tag{29}$$

$$\frac{1-\alpha-\epsilon}{2} > \frac{46\alpha-1}{11} + \epsilon,$$

and

$$1 - 5\alpha - \epsilon > \tfrac{1}{3}.$$

We also use the fact that the prime powers give a contribution

$$\ll v^{\frac{1}{2}} = o\left(\frac{x}{v \log x}\right).$$

We note that it is (29) which sets the limit of the method. The $(1-\alpha-\epsilon)/2$ term here arises naturally from Heath–Brown's identity, while the $(62\alpha-5)/6$ comes from the exponential sum estimates.

Finally we mention the connexion this problem has with the greatest prime factor of an integer in an interval. If

$$1 - \beta < \left\{\frac{y}{p}\right\} < 1$$

then there is a number in the interval $(y, y+\beta p)$ which is divisible by p. By altering the proof we have given one can deduce that for all large x there is an integer in the interval

$$[x, x + x^{1-1/(2-\alpha)})$$

with a prime factor of size approximately $x^{1/(2-\alpha)}$, for any $\alpha < 8/65$. By using similar arguments to [2] it can be shown that the interval contains an integer with a larger prime factor, but this is not helpful in the present context since the size of prime factor we need is linked directly to the interval length.

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