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PRIMITIVE GENERATORS FOR ALGEBRAS

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1. Introduction. Let H be a graded commutative algebra with a nice set of algebra generators. Let H also be a comodule over a Hopf algebra A. In Section 2 we give conditions under which certain of these generators of H can be rechosen to be primitive. In addition we give explicit formulas expressing these primitive generators in terms of the original set of generators.

In Section 3 we apply the theory of Section 2 to the mod p homology of the Thom spectra MO, MU and MSp. In particular we give two explicit descriptions of the image of the Hurewicz homomorphism for MO. One of these makes explicit the recursive computation of E. Brown and F. Peterson [**1**].

In Section 4 we give a variation of the theory of Section 2 which computes primitive generators of certain Hopf algebras. This theory is applied to study the primitive elements of $H^*(BU)$ and $H_*(SO; Z_2)$.

Other applications of the theory of Section 2 will appear in [**3**] and [**4**].

2. A change of generators theorem. Let $H = R[Y_1, \ldots, Y_n, \ldots]$ be a graded polynomial algebra which is a comodule over a Hopf algebra A. It may happen that $G = R\{1 = Y_0, \ldots, Y_n, \ldots\}$ is a subcomodule of H. This may occur because there is $\alpha: X \to W$ with $A = E_*E$ and $\alpha_*: E_*X \to E_*W$ the inclusion of G into H. Or, we may have made a clever choice of polynomial generators. (See [**4**].) Write

$$\psi(Y_n) = \sum_{k=0}^{n} \theta_{n,k} \otimes Y_k.$$

We then ask whether it is possible to choose new polynomial generators of H so that some are primitive and the non-primitive ones span a subcomodule of H. This will split $H = H_1 \otimes H_2$ as algebras and comodules with H_1 primitive. The following theorem accomplishes this change of generators under two assumptions. We require that $\{Y_n | n \in S\}$, which we wish to replace by a set of primitive generators, spans a subcomodule $R\{Y_n | n \in S\}$ of H. We also require analogues $\phi_{n,k}$ in (H, ψ) of the $\theta_{n,k}$

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in (A, Δ). That is,

$$\Delta(\theta_{n,k}) = \sum_{i=k}^{n} \theta_{n,i} \otimes \theta_{i,k}$$

by coassociativity and we need to be able to choose the $\phi_{n,k}$ with

$$\psi(\phi_{n,k}) = \sum_{i=k}^{n} \theta_{n,i} \otimes \phi_{i,k}.$$

Note that it is possible for the subset S of Z^+ to be finite or all of $\mathbf{Z}^+$.

THEOREM 2.1. *Let R be a commutative ring, and let A be a Hopf algebra over R. Let $H = R[Y_1, \ldots, Y_n, \ldots]$ or $H = E(Y_1, \ldots, Y_n, \ldots)$, $R = Z_2$. Let $Y_0 = 1$ and $\deg Y_n = \alpha_n$ where $0 = \alpha_0 < \alpha_1 \leqq \ldots \leqq \alpha_n \leqq \ldots$. Let H be an A-comodule whose coproduct ψ is an algebra homomorphism. Assume that:*

(a) *The R-submodule of H spanned by $\{Y_0, \ldots, Y_n, \ldots\}$ is a subcomodule of H. Write*

$$\psi(Y_n) = \sum_{k=0}^{n} \theta_{n,k} \otimes Y_k \quad \textit{with } \theta_{n,k} \in A_{\alpha_n - \alpha_k}.$$

(b) *There are $\phi_{n,k} \in H_{\alpha_n - \alpha_k}$ for $0 \leqq k \leqq n$ such that*

$$\psi(\phi_{n,k}) = \sum_{i=k}^{n} \theta_{n,i} \otimes \phi_{i,k}.$$

(c) *There is a set of positive integers S such that $\theta_{n,k} = 0$ for $n \in S$ and $k \notin S$, $k > 0$. In addition $\theta_{n,0} = 0$ and $\phi_{n,0} = Y_n$.*
Then there is a Hopf algebra structure with coproduct Δ on H, and there are $p_n \in H_{\alpha_n}$ for $n \in S$, $Q_m \in H_{\alpha_m}$ for $m \notin S$ such that:

(i) $\Delta(Y_n) = \sum_{k=0}^{n} \phi_{n,k} \otimes Y_k.$

(ii) *The P_n are primitive under both ψ and Δ.*

(iii) $Q_0 = 1$ *and for* $m > 0$:

$$\psi(Q_m) = \sum_{i \leqq m,\, i \notin S} \theta_{m,i} \otimes Q_i;$$

$$\Delta(Q_m) = \sum_{0 < i \leqq m,\, i \notin S} \phi_{m,i} \otimes Q_i + Q_m \otimes 1.$$

(iv) *$\{P_n | n \in S\} \cup \{Q_m | m \notin S\}$ is a set of algebra generators of H.*

(v) $Y_n = P_n + \sum_{k<n} \phi_{n,k} P_k$ *for* $n \in S$;

$$Y_m = Q_m + \sum_{k<m,\, k \in S} \phi_{m,k} P_k \quad \textit{for } m \notin S.$$

(vi) $P_n = Y_n + \sum_{k<n} \chi(\phi_{n,k}) Y_k$ *for* $n \in S$;

$$Q_m = Y_m + \sum_{k<m,\, k \in S} \chi(\phi_{m,k}) Y_k \quad \textit{for } m \notin S;$$

where χ is the conjugation of H.

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Assume now that $H = R[Y_n|n \geqq 1]$.

(vii) *If* A *is a free* R*-module then* $P_\psi H = R[P_n|n \in S]$.

(viii) *Assume that* char $R = \gamma$ *is either* 0 *or a prime. Let*

$$T = \{m \notin S|Q_m \text{ is } \Delta\text{-primitive}\}.$$

Then $P_\Delta H$ *is* R*-free with basis*

$$\begin{cases} \{Q_m, P_n|n \in S, m \in T\} & \text{if } \gamma = 0 \\ \{Q_m^{\gamma^r}, p_n^{\gamma^r}|n \in S, m \in T, r \geqq 0\} & \text{if } \gamma > 0. \end{cases}$$

Proof. Define the Hopf algebra structure on H by the formula in (i). Then define the P_n and Q_m inductively on their degree so that (v) holds. Clearly $P_n \equiv Y_n$ and $Q_m \equiv Y_m$ modulo decomposables, so (iii) is valid. We check (ii) and (iii):

$$\psi(P_n) = \psi(Y_n) - \sum_{k<n} \psi((_{n,k})\psi(P_k)$$

$$= \sum_{i<n} \theta_{n,i} \otimes Y_i - \sum_{k<n} \sum_{i=k}^{n} \theta_{n,i} \otimes \phi_{i,k}P_k = 1 \otimes Y_n - \sum_{k<n} 1 \otimes \phi_{n,k}P_k$$

$$+ \sum_{i=1}^{n-1} \theta_{n,i} \otimes \left[Y_i - P_i - \sum_{k=1}^{i-1} \phi_{i,k}P_k \right] = 1 \otimes P_n.$$

Similarly, $\Delta(P_n) = P_n \otimes 1 + 1 \otimes P_n$.

$$\psi(Q_m) = \psi(Y_m) - \sum_{\substack{k<m \\ k \in S}} \psi(\phi_{m,k})\psi(P_k)$$

$$= \sum_{i \leqq m} \theta_{m,i} \otimes Y_i - \sum_{\substack{k<m \\ k \in S}} \sum_{i=k}^{m} \theta_{m,i} \otimes \phi_{i,k}P_k$$

$$= 1 \otimes Y_m - \sum_{\substack{k<m \\ k \in S}} 1 \otimes \phi_{m,k}P_k$$

$$+ \sum_{\substack{i<m \\ i \in S}} \theta_{m,i} \otimes \left[Y_i - P_i - \sum_{k<i} \phi_{i,k}P_k \right]$$

$$+ \sum_{\substack{i<m \\ i \notin S}} \theta_{m,i} \otimes \left[Y_i - \sum_{\substack{k<i \\ k \in S}} \phi_{i,k}P_k \right] = 1 \otimes Q_m + \sum_{\substack{i<m \\ i \notin S}} \theta_{m,i} \otimes Q_i.$$

Similarly,

$$\Delta(Q_n) = 1 \otimes Q_m + \sum_{0<i<m,\, i \notin S} \phi_{m,i} \otimes Q_i + Q_m \otimes 1.$$

We now prove (vi). Let λ be either a fixed Q_m or P_m. Let $N - 1$ be the cardinality of the set $\{k \in S|\alpha_k < \alpha_m\}$. Then the equations in (v) include N linear equations with coefficients in H and with unknowns $\{\lambda\} \cup \{P_k|\alpha_k < \alpha_m\}$. Observe that the matrix of coefficients is lower tri-

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angular with ones on the diagonal. We solve such a system of equations in a general setting in Lemma 2.2. This solution gives the formulas in (vi).

Clearly $P_\psi H$ and $P_\Delta H$ contain the R-modules which we assert they equal. If $Z \in H$ then we can write Z as a polynomial in the P_n and Q_m. If $Z \notin R[P_n | n \in S]$ then choose a monomial summand of Z,

$$\alpha P_{a_1} \ldots P_{a_s} Q_{b_1}{}^{e_1} \ldots Q_{b_t}{}^{e_t}$$

with $\alpha \in R$, $t > 0$ and $0 < b_1 < \ldots < b_t$. Assume that we have chosen the monomial with t least and among such monomials $(e_t, \ldots, e_1)$ is least in the lexicographical order. Then $\psi(Z)$ contains

$$\alpha e_t \theta_{b_t,0} \otimes P_{a_1} \ldots P_{a_s} Q_{b_1}{}^{e_1} \ldots Q_{b_{t-1}}^{e_{t-1}} Q_{b_t}{}^{e_t - 1}$$

as a nonzero summand, and hence $\psi(Z)$ is nonzero. This proves (vi). If Z has a summand

$$\alpha P_{a_1} \ldots P_{a_s} Q_{c_1} \ldots Q_{c_u} Q_{b_1}{}^{e_1} \ldots Q_{b_t}{}^{e_t}$$

with $\alpha \in R$, $t > 0$, $0 < b_1 < \ldots < b_t$, $\{b_1, \ldots, b_t\} \not\subset T$ then choose the monomial with t least and among such monomials let $(e_t, \ldots, e_1)$ be least in the lexicographical order. Then $\bar{\Delta}(Z)$ contains

$$\alpha \phi_{b_t,b} \otimes P_{a_1} \ldots Q_{b_{t-1}}^{e_{t-1}} Q_b{}^{e_t}$$

as a nonzero summand for some $0 < b < b_t$. Thus

$$P_\Delta H = P_\Delta R[Q_m, P_n | m \in T, n \in S]$$
$$= R\{Q_m{}^{\gamma^r}, P_n{}^{\gamma} | n \in S, m \in T, r \geqq 0\}.$$

LEMMA 2.2. *Let H be a commutative Hopf algebra. Let b_{ij}, $1 \leqq i, j \leqq n$ and c_i, $1 \leqq i \leqq n$, be elements of H. Let $B = (b_{ij})$, $C = (c_1, \ldots, c_n)^\top$ and $Z = (z_1, \ldots, z_n)^\top$. Consider the n equations in n unknowns $BZ = C$. Assume that:*

(i) *B is lower triangular.*

(ii) *The diagonal entries of B are all ones.*

(iii) $\Delta(b_{i,j}) = \sum_{s=i}^{j} b_{i,s} \otimes b_{s,j}$.

Then

$$z_k = c_k + \sum_{i=1}^{k-1} \chi(b_{k,i}) c_i \quad \text{for } 1 \leqq k \leqq n.$$

Proof. By Cramer's rule, $z_k = \det B_k / \det B = \det B_k$ where $B_k = (b'_{ij})$ is the $k \times k$ matrix with $b'_{ij} = b_{ij}$ for $j < k$ and $b'_{ik} = c_i$. We expand $\det B_k$ by minors of the last column to obtain:

$$z_k = c_k + \sum_{i=1}^{k-1} (-1)^{i+k} c_i \det B_{k,i}.$$

Observe that $B_{k,i}$ has the shape:

$$\begin{array}{c} \\ i-1 \\ \\ k-i \end{array}\left(\begin{array}{ccc|c} & i-1 & & k-i \\ 1 & & 0 & \\ & \ddots & & 0 \\ * & & 1 & \\ \hline & * & & B'_{k,i} \end{array}\right).$$

Thus

$$z_k = c_k + \sum_{i=1}^{k-1} (-1)^{i+k} c_i \det B'_{k,i}.$$

The matrices $B'_{k,i}$ have the form:

$$B = \begin{pmatrix} \beta_{11} & 1 & & & & 0 & \\ \beta_{21} & \beta_{22} & 1 & \cdot & & & \\ & \cdot & & & \cdot & & \\ & \cdot & & & & \cdot & \\ & \cdot & & & & 1 & \\ \beta_{u-1,1} & & \cdots & & & \beta_{u-1,u-1} & 1 \\ \beta_{u1} & & \cdots & & & \beta_{u,u-1} & \beta_{uu} \end{pmatrix}.$$

Observe that

$$\det B = \sum_{e=0}^{u-2} \sum_{1<q_1<\ldots<q_e<u} (-1)^{u+e+1} \beta_{u,q_e} \beta_{q_e,q_{e-1}} \ldots \beta_{q_2,q_1} \beta_{q_1,1}.$$

Thus

$$z_k = c_k + \sum_{i=1}^{k-1} \sum_{e=0}^{k-i-2} \sum_{i<q_1<\ldots<q_e<k} (-1)^{e+1} c_i b_{k,q_e} \ldots b_{q_1,i}.$$

The following lemma applies to show that

$$z_k = c_k + \sum_{i=1}^{k-1} c_i \chi(b_{k,i}).$$

LEMMA 2.3. *Let H be a connected Hopf algebra with $\bar{\psi}^{(0)} = 1$ and $\bar{\psi}^{(k)}$ the kth iterated reduced coproduct. Define*

$$\Psi(Y) = \sum_{k=0}^{\infty} \sum_{i} (-1)^{k+1} Y_{i1}^{(k)} \ldots Y_{i,k+1}^{(k)}$$

where

$$\bar{\psi}^{(k)}(Y) = \sum_{i} Y_{i1}^{(k)} \otimes \ldots \otimes Y_{i,k+1}^{(k)}.$$

Then $\Psi = \chi$.

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Proof. We prove $\Psi(Y) = \chi(Y)$ by induction on the degree of Y. By coassociativity,

$$\Psi(Y) = -Y - \sum_i Y_{i1}^{(1)}\Psi(Y_{i2}^{(1)}) = -Y - \sum_i Y_{i1}^{(1)}\chi(Y_{i2}^{(1)})$$
$$= \chi(Y).$$

3. Applications to Thom spectra. Recall [**5**] that

$$H_*(MO; Z_2) = Z_2[b_1, \dots, b_n, \dots]$$

where

$$H_*(MO(1); Z_2) = H_*(RP^\infty; Z_2) = Z_2\{1, b_0, b_1, \dots, b_n, \dots\}$$

and

$$b_n \in H_n(MO; Z_2).$$

$H_*(MO; Z_2)$ is a comodule over the dual of the mod 2 Steenrod algebra $\mathfrak{A}_*$ and $H_*(MO(1); Z_2)$ is a subcomodule. By [**8**] the component of $\psi(b_n)$ in $\mathfrak{A}_* \otimes b_{2^s-1}$ is

$$\xi_{t-s}^{2s} \otimes b_{2^s-1} \text{ if } n = 2^t - 1$$

and is zero otherwise. Thus Theorem 2.1 applies to $H_*(MO; Z_2)$ with $S = \{n > 0| \quad n \neq 2^t - 1\}$. The coproduct Δ which Theorem 2.1 imposes upon $H_*(MO; Z_2)$ is not of geometric origin. It is not induced from

$$HZ_2 \wedge MO \xrightarrow{\simeq} HZ_2 \wedge S \wedge MO \xrightarrow{1 \wedge \eta \wedge 1} HZ_2 \wedge MO \wedge MO.$$

Also neither of the canonical H-space structures on BO induces Δ via the Thom isomorphism. Nevertheless, Δ is the key to unravelling problems of geometric relevance about $H_*(MO; Z_2)$. Note that conclusions (i)-(vi) of the following theorem were derived by E. Brown and F. Peterson [**1**].

THEOREM 3.1. *There is a Hopf algebra structure on $H_*(MO; Z_2)$ and there are $V_n \in H_n(MO; Z_2)$ for $n \neq 2^t - 1$ and $\zeta_m \in H_{2^m-1}(MO; Z_2)$ such that:*

(i) *The V_n are primitive under both ψ and Δ.*

(ii) $\psi(\zeta_m) = \sum_{i=0}^{m} \xi_{m-1}^{2^i} \otimes \zeta_i$ *and* $\Delta(\zeta_m) = \sum_{i=0}^{m} \zeta_{m-i}^{2^i} \otimes \zeta_i$.

(iii) $H_*(MO; Z_2) = Z_2[V_n| \quad n \neq 2^t - 1] \otimes Z_2[\zeta_1, \dots, \zeta_m, \dots]$.

(iv) Image $[h: \mathfrak{N}_* \to H_*(MO; Z_2)] = Z_2[V_n| \, n \neq 2^t - 1]$.

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(v) $b_n = V_n + \sum_{k<n} \phi_{n,k} V_k$ *for* $n \neq 2^t - 1$;

$$b_{2^m-1} = \zeta_m + \sum_{\substack{k<2^m-1 \\ k\neq 2^s-1}} \phi_{2^m-1,k} V_k;$$

$$\phi_{n,k} = (1 + \zeta_1 + \ldots + \zeta_r + \ldots)_{n-k}^{k+1} \quad \text{for } n \geqq k \geqq 0.$$

(vi) $F(V_n) = U_n$ *and* $F(\zeta_m) = \xi_m$ *where* F *is the Liulevicius isomorphism* [**5**]:

$$F: H_*(MO; Z_2) \rightarrow \mathfrak{A}_* \otimes Z_2[U_n| \quad n \neq 2^t - 1].$$

(vii) $V_n = b_n + \sum_{k<n} \chi(\phi_{n,k}) b_k$ *for* $n \neq 2^t - 1$;

$$\zeta_m = b_{2^m-1} + \sum_{\substack{k<2^m-1 \\ k\neq 2^s-1}} \chi(\phi_{2^m-1,k}) b_k;$$

$$\chi(\phi_{n,k}) = (1 + \chi(\zeta_1) + \ldots + \chi(\zeta_r) + \ldots)_{n-k}^{k+1}$$
$$= (1 + m_1 + \ldots + m_{2^r-1} + \ldots)_{n-k}^{k+1}$$

in the notation of Theorem 3.2.

(viii) $\Delta(b_n) = \sum_{k=0}^{n} \phi_{n,k} \otimes b_k$.

(ix) $H^*(MO; Z_2) \cong \mathfrak{A} \otimes \Gamma[V_n^*| \quad n \neq 2^t - 1]$ *as Hopf algebras.*

There is an analogue of Theorem 3.1 where b_n, V_n, ξ_n is replaced by m_n, β_n, $\chi(\xi_n)$, respectively. Here m_n is the coefficient of X^{n+1} in the inverse power series of $X + b_1X^2 + \ldots + b_kX^{k+1} + \ldots$. Then

$$H_*(MO; Z_2) = Z_2[m_1, \ldots, m_n, \ldots].$$

To define the β_n we consider the Liulevicius map [**7**]

$$G: H_*(MO; Z_2) \xrightarrow{\psi} \mathfrak{A}_* \otimes H_*(MO; Z_2) \xrightarrow{1 \otimes g} \mathfrak{A}_* \otimes Z_2[U_n| \quad n \neq 2^s - 1]$$

where g is the algebra homomorphism induced by $g(m_n) = U_n$ if $n \neq 2^s - 1$ and $g(m_{2^s-1}) = 0$. Then G is an isomorphism of Z_2-algebras and $\mathfrak{A}_*$-comodules. Define β_n as the primitive element $G^{-1}(U_n)$. A. Liulevicius [**7**] computes

$$\psi(m_n) = \sum_{s2^k+2^k-1=n} \chi(\xi_k) \otimes m_s^{2^k}.$$

Thus $\{1, m_1, m_2, \ldots\}$ is not a subcomodule of $H_*(MO; Z_2)$. Thus Theorem 2.1 does not seem to apply to relate the m_n and β_n. However, this first impression is erroneous.

THEOREM 3.2.
(i) $H_*(MO; Z_2) = Z_2[\beta_n| \quad n \neq 2^s - 1] \otimes Z_2[\zeta_1, \ldots, \zeta_s, \ldots]$.
(ii) Image $[h: \mathfrak{N}_* \rightarrow H_*(MO; Z_2)] = Z_2[\beta_n| \quad n \neq 2^s - 1]$.

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(iii) $m_{2n} = \beta_{2n}$.
(iv) $m_{2^s-1} = \chi(\zeta_s)$.

$$\text{(v)}\ m_{2n-1} = \beta_{2n-1} + \sum_{k=1}^{e} m_{2^k-1}\beta^{2^k}_{2^{e-k}(2p+1)-1}$$

where $n \neq 2^s$ *and* $2n = 2^e(2p + 1)$.

$$\text{(vi)}\ \beta_{2n-1} = m_{2n-1} + \sum_{k=1}^{e} \zeta_k m^{2^k}_{2^{e-k}(2p+1)-1}$$

where $n \neq 2^s$ *and* $2n = 2^e(2p + 1)$.

Proof. (i)-(iv) are known; see [**7**]. To prove (v) it suffices to show by induction on n that

$$\beta'_{2n-1} = m_{2n-1} + \sum_{k=1}^{e} \chi(\zeta_k)\beta^{2^k}_{2^{e-k}(2p+1)-1}$$

is primitive and that $G(\beta'_{2n-1}) = U_{2n-1}$:

$$\psi(\beta'_{2n-1}) = \psi(m_{2n-1}) + \sum_{k=1}^{e} \psi\chi(\zeta_k)(1 \otimes \beta^{2^k}_{2^{e-k}(2p+1)-1})$$

$$= \sum_{k=0}^{e} \chi(\xi_k) \otimes m^{2^k}_{2^{e-k}(2p+1)-1}$$

$$+ \sum_{k=1}^{e}\sum_{i=0}^{k} \chi(\xi_i) \otimes \chi(\zeta_{k-i})^{2^i}(1 \otimes \beta^{2^k}_{2^{e-k}(2p+1)-1}) = 1 \otimes \beta'_{2n-1}$$

$$+ \sum_{k=1}^{e} \chi(\xi_k) \otimes \left[m^{2^k}_{2^{e-k}(2p+1)-1} + \sum_{j=0}^{e-i} \chi(\zeta_j)^{2^k}\beta^{2^{j+k}}_{2^{e-j-k}(2p+1)-1}\right] = 1 \otimes \beta'_{2n-1}$$

$$+ \sum_{k=1}^{e} \chi(\xi_k) \otimes \left[m_{2^{e-k}(2p+1)-1} + \sum_{j=0}^{e-i} \chi(\zeta_j)\beta^{2^j}_{2^{e-j-k}(2p+1)-1}\right]^{2^k} = 1 \otimes \beta'_{2n-1}$$

$$+ \sum_{k=1}^{e} \chi(\xi_k) \otimes \left[m_{2^{e-k}(2p+1)-1} + \beta_{2^{e-k}(2p+1)-1} + \sum_{j=1}^{e-i} \chi(\zeta_j)\beta^{2^j}_{2^{e-j-k}(2p+1)-1}\right]^{2^k}$$

$$= 1 \otimes \beta'_{2n-1}.$$

$$G(\beta'_{2n-1}) = (1 \otimes g) \circ \psi(\beta'_{2n-1}) = 1 \otimes g(\beta'_{2n-1})$$

$$= 1 \otimes g(m_{2n-1}) + \sum_{k=1}^{e} 1 \otimes g\chi(\zeta_k)g(\beta^{2^k}_{2^{e-k}(2p+1)-1}) = 1 \otimes g(m_{2n-1}).$$

To prove (vi) observe that we have the following $e + 1$ equations in the $e + 1$ unknowns $\beta^{2^s}_{2^{e-s}(2p+1)-1}$, $0 \leqq s \leqq e$:

$$(*) \quad m^{2^k}_{2^{e-k}(2p+1)-1} = \beta^{2^k}_{2^{e-k}(2p+1)} + \sum_{h=1}^{e-k} \chi(\zeta_h)^{2^k}\beta^{2^{h+k}}_{2^{e-h-k}(2p+1)-1} \quad (0 \leqq k \leqq e).$$

Observe that the coefficient matrix is $A = (a_{ij})$ with $a_{ij} = \chi(\zeta_{i-j})^{2^{e-i}}$

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for $0 \leqq j \leqq i \leqq e$. Hence

$$\Delta(a_{ij}) = \Delta\chi(\zeta_{i-j})^{2^{e-i}} = (\chi \otimes \chi) \circ T \circ \Delta(\zeta_{i-j})^{2^{e-i}}$$
$$= (\chi \otimes \chi) \circ T\left(\sum_{s=0}^{i-j} \zeta_{i-j-s}^{2^s} \otimes \zeta_s\right)^{2^{e-i}} = \sum_{s=0}^{i-j} \chi(\zeta_s)^{2^{e-i}} \otimes (\zeta_{i-j-s})^{2^{s+e-i}}$$
$$= \sum_{s=0}^{i-j} a_{i,i-s} \otimes a_{i-s,j} = \sum_{t=i}^{j} a_{i,t} \otimes a_{t,j} \quad \text{where } t = i - s.$$

Thus Lemma 2.2 applies to the system of equations (*) to give:

$$\beta_{2^{e-k}(2p+1)-1}^{2^k} = m_{2^{e-k}(2p+1)}^{2^k} + \sum_{h=1}^{e-k} \zeta_h^{2^k} m_{2^{e-h-k}(2p+1)-1}^{2^{h+k}} \quad (0 \leqq k \leqq e).$$

We now let $k = 0$ to obtain (vi).

Theorem 3.1 has an analogue where we replace MO, 2, b_n, RP^∞ by MU, p prime, a_n, CP^∞, respectively. Then

$$a_n,\ V_n \in H_{2n}(MU; Z_p) \quad \text{and} \quad \zeta_m \in H_{2(p^m-1)}(MU; Z_p).$$

Conclusions (i), (iii), (iv), (v) are identical. For p odd we get conclusion (ii) and for $p = 2$ we now have

$$\psi(\zeta_m) = \sum_{i=0}^{m} \xi_{m-1}^{2^{i+1}} \otimes \zeta_i.$$

Conclusion (vi) becomes $F(V_n) = U_n$, $F(\zeta_m) = \xi_m$ for p odd and $F(\zeta_m) = \xi_m^2$ for $p = 2$ where F is the Liulevicius isomorphism [**6**]:

$$F\colon H_*(MU; Z_p) \to Z_p[\xi_1, \ldots, \xi_m, \ldots] \otimes Z_p[U_n|\quad n \neq p^t - 1] \quad \text{for } p \text{ odd},$$

$$F\colon H_*(MU; Z_2) \to Z_2[\xi_1^2, \ldots, \xi_m^2, \ldots] \otimes Z_2[U_n|\ n \neq 2^t - 1] \quad \text{for } p = 2.$$

Theorem 3.1 has another analogue where we replace MO, b_n, RP^∞ by MSp, d_n, HP^∞, respectively. Then

$$d_n,\ V_n \in H_{4n}(MSp; Z_2) \quad \text{and} \quad \zeta_m \in H_{4(2^m-1)}(MSp; Z_2).$$

Conclusions (i), (iii), (iv), (v) are identical. Conclusion (ii) becomes

$$\psi(\zeta_n) = \sum_{i=0}^{n} \xi_{n-i}^{2^{i+2}} \otimes \zeta_i.$$

Conclusion (vi) becomes

$$F(V_n) = U_n \quad \text{and} \quad F(\zeta_m) = \xi_m^4$$

where F is the Liulevicius isomorphism [**6**]:

$$F\colon H_*(MSp; Z_2) \to Z_2[\xi_1^4, \ldots, \xi_m^4, \ldots] \otimes Z_2[U_n|\quad n \neq 2^t - 1].$$

4. Applications to Hopf algebras. A commutative Hopf algebra H with coproduct Δ becomes a comodule over H with coproduct $\psi(Y) =$

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$\Delta(Y) - Y \otimes 1$. Then ψ-primitives are the same as Δ-primitives. However, Theorem 2.1 does not apply to this situation because ψ is not an algebra homomorphism. In Theorem 4.1 we consider a special case when an analogue of Theorem 2.1 holds. We apply Theorem 4.1 to study $PH^*(BU)$ and $PH_*(SO; Z_2)$.

THEOREM 4.1. *Let H be a graded connected commutative cocommutative Hopf algebra over a commutative ring R. Let H have a set of algebra generators $Y_n \in H_{\alpha n}$, $n \geqq 1$, such that*

$$Y_0 = 1 \quad \text{and} \quad \Delta(Y_n) = \sum_{i=0}^{n} Y_i \otimes Y_{n-i}.$$

Then there are $P_n \in H_{\alpha n}$, $n \geqq 1$, such that:

(i) *The P_n are primitive.*

(ii) *If R is a field of characteristic zero, then $\{P_1, P_2, \ldots\}$ generates H.*

(iii) $nY_n = P_n + \sum_{0<k<n} Y_{n-k}P_k$.

(iv) $P_n = nY_n + \sum_{0<k<n} \chi(Y_{n-k})kY_n$.

(v) $\chi(Y_n) = \sum_{e_1+2e_2+\ldots+ne_n=n} (-1)^{e_1+\ldots+e_n}(e_1, \ldots, e_n)Y_1^{e_1} \ldots Y_n^{e_n}$

where $(e_1, \ldots, e_n)$ denotes the multinomial coefficient.

(vi) $P_n = \sum_{e_1+2e_2+\ldots+ne_n=n} (-1)^{e_1+\ldots+e_n+1} \dfrac{n(e_1, \ldots, e_n)}{e_1 + \ldots + e_n} Y_1^{e_1} \ldots Y_n^{e_n}$.

Proof. We make H into a comodule over H by $\psi(Y) = \Delta(Y) - Y \otimes 1$. Then Theorem 2.1 almost applies to this situation with $\theta_{n,k} = Y_{n-k}$. The only problem is that ψ is not an algebra homomorphism. In the proof of Theorem 2.1 we only used this property of ψ in verifying (iv). Thus in our case we have

$$\psi(\theta_{n,k}P_k) = \Delta(\theta_{n,k})(1 \otimes P_k) + \psi(\theta_{n,k})(P_k \otimes 1)$$

instead of

$$\psi(\theta_{n,k}P_k) = \psi(\theta_{n,k})(1 \otimes P_k)$$

which we had in Theorem 2.1. To take advantage of this new situation we replace Y_n by nY_n in the formula which defines P_n. Then

$$\psi(nY_n) = \sum_{k=1}^{n} (kY_k) \otimes Y_{n-k} + \sum_{k=1}^{n} Y_k \otimes (n-k)Y_{n-k}.$$

Thus the P_n defined inductively by (iii) are primitive. Now the proof of Theorem 2.1 applies to prove (i)-(iv). Observe that P_n can replace Y_n as an algebra generator of H if and only if n is a unit in R. (v) follows from Lemma 2.3. We combine (iv) and (v) to obtain (vi).

Recall that $H^*(BU; R) = R[C_1, \ldots, C_n, \ldots]$ with

$$\Delta(C_n) = \sum_{k=0}^{n} C_k \otimes C_{n-k}.$$

Thus Theorem 4.1 applies to $H = H^*(BU; R)$ and derives Newton's formula, Corollary 4.2 (iii), and Girard's formula, Corollary 4.2 (vi). See [**9**, p. 195]. In addition we obtain a formula for P_n in terms of the C_k and $\chi(C_k)$ as well as a formula for $\chi(C_n)$.

COROLLARY 4.2. *There are* $P_n \in H^{2n}(BU; R) = R[C_1, \ldots, C_n, \ldots]$ *such that:*

(i) *The* P_n *are primitive.*

(ii) *If* R *is a field of characteristic zero then*

$$H^*(BU; R) = R[P_1, \ldots, P_n, \ldots].$$

(iii) $nC_n = P_n + \sum_{0<k<n} C_{n-k}P_k.$

(iv) $P_n = nC_n + \sum_{0<k<n} \chi(C_{n-k})kC_k.$

(v) $\chi(C_n) = \sum_{e_1+2e_2+\ldots+ne_n=n} (-1)^{e_1+\ldots+e_n}(e_1, \ldots, e_n)C_1^{e_1} \ldots C_n^{e_n}.$

(vi) $P_n = \sum_{e_1+2e_2+\ldots+ne_n=n} (-1)^{e_1+\ldots+e_n+1} \dfrac{n(e_1, \ldots, e_n)}{e_1,+ \ldots + e_n} C_1^{e_1} \ldots C_n^{e_n}.$

Note that there are analogues of Theorem 4.2 for $H^*(BO; Z_p)$, $H^*(BSp; R)$, $H_*(BU; R)$, $H_*(BO; Z_p)$ and $H_*(BSp; R)$. Next we apply Theorem 4.1 to $H_*(SO; Z_2)$. This is the only example in this paper which is not a polynomial algebra. Recall from [**2**, pp. 17-10] that

$$H_*(SO; Z_2) = E(U_1, \ldots, U_n, \ldots)$$

where

$$\deg U_n = n \quad \text{and} \quad \psi(U_n) = \sum_{i=0}^{n} U_i \otimes U_{n-i}.$$

COROLLARY 4.3. *There are* $P_n \in H_{2n-1}(SO; Z_2) = E(U_1, \ldots, U_n, \ldots)$, $n \geqq 1$, *such that:*

(i) *The* P_n *are primitive.*

(ii) $U_{2n-1} = P_n + \sum_{0<k<n} U_{2n-2k}P_k.$

(iii) $U_{2n} = \sum_{0<k<n} U_{2n-2k+1}P_k.$

(iv) $P_n = U_{2n-1} + \sum_{0<k<n} \chi(U_{2n-2k})U_{2k-1}.$

(v) $\chi(U_m) = U_m.$

(vi) $P_n = U_{2n-1} + \sum_{0<k<n} U_{2n-2k}U_{2k-1}.$

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Proof. Theorem 4.2 defines a primitive in every degree. However, the primitive P_n' in an even degree $2n$ is zero by induction on n because

$$P_n' = 2nU_{2n} + \sum_{0<k<n} U_{2n-2k+1}P_k = 0$$

by (vi). This explains (iii). Note that (v) follows immediately from the formula for $\psi(U_n)$. Then (iv) combined with (v) gives (vi).

This corollary is not a deep result because the formula in (vi) is known [**2**, pp. 17-11] and (i)-(iv) follow easily from (vi).

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