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NOTES ON DIFFERENTIAL CALCULUS IN TOPOLOGICAL LINEAR SPACES, II

S. YAMAMURO

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Throughout this note, let E and F be locally convex Hausdorff spaces over the real number field R. We denote real numbers by Greek letters. The sets of all continuous semi-norms on E and F will be denoted by $P(E)$ and $P(F)$ respectively, and A will always stand for an open subset of E.

The purpose of this note is to continue our study on the properties of differentiable maps; in particular, the differentiability of the inverse maps and the weak injectivity of maps with invertible derivatives.

1. Definitions of differentiability

A map $f: A \to F$ is said to be *differentiable at* $a \in A$ if there exists a continuous linear map u of E into F such that

$$\lim_{\varepsilon \to 0} \sup_{x \in B} q(\varepsilon^{-1} r(f, a, \varepsilon x)) = 0$$

for any $q \in P(F)$ and any bounded subset B of E, where

$$r(f, a, x) = f(a + x) - f(x) - u(x).$$

If f is differentiable at a, the map u is determined uniquely, and it is called the *derivative of* f *at* a and is denoted by $f'(a)$.

A map $f: A \to F$ is said to be *strongly differentiable at* $a \in A$ *with respect to* $p_0 \in P(E)$ or p_0*-differentiable at* a if there exists a continuous linear map u of E into F such that

$$\lim_{\varepsilon \to 0} \sup_{p_0(x) \leqq 1} q(\varepsilon^{-1} r(f, a, \varepsilon x)) = 0$$

for any $q \in P(F)$. Every strongly differentiable map is differentiable. We use the same symbol $f'(a)$ to denote the strong derivative of f at a.

Some fundamental properties of these two differentiabilities have been collected by Yamamuro (1974). We only note here that these are respectively the weakest and the strongest among all the definitions proposed so far which coincide with the usual Fréchet differentiability when the spaces involved are normed spaces.

The strong differentiability has been introduced by Keller (1963–64), who observed the following fact.

(1.1) *A map* $f: A \to F$ *is* p_0*-differentiable at* $a \in A$ *if and only if there exists a continuous linear map* $u: E \to F$ *such that the following condition is satisfied: for any* $\varepsilon > 0$ *and* $q \in P(F)$ *there exists* $\delta > 0$ *such that* $p_0(x) \leqq \delta$ *implies* $q(r(f, a, x)) \leqq \varepsilon p_0(x)$.

We add two remarks which will be used later. First, the following is obvious.

(1.2) *If* f *is* p_0*-differentiable at a point and* $p \geqq p_0$ *(i.e.,* $p(x) \geqq p_0(x)$ *for every* $x \in E$*), then* f *is* p*-differentiable at the point.*

Secondly, let us denote the set $\{x: p(x) \leqq 1\}$ by U_p. Then, if $f: A \to F$ is p_0-differentiable at $a \in A$, there exists $p \in P(E)$ such that $p \geqq p_0$ and $a + U_p \subset A \cap (a + U_{p_0})$. Hence,

(1.3) *If* $f: A \to F$ *is* p_0*-differentiable at* $a \in A$*, we can suppose that* $a + U_{p_0} \subset A$.

2. Strong differentiability of the inverse maps

Throughout this section, we assume that $f: A \to F$ is an injective map of A onto an open set $f(A)$, strongly differentiable at $a \in A$ and $f'(a)$ is an isomorphism. We shall give a condition for the inverse map g of f be strongly differentiable at $f(a)$.

Suhinin (1969) has proved that g is strongly differentiable if g is continuous at every point in $f(A)$. This sufficient condition is somewhat strange; in the case of normed spaces, only the continuity of g at $f(a)$ is needed. We shall fill this gap by giving an exact condition for g to be strongly differentiable at $f(a)$.

By considering the transformation

$$x \mapsto f'(a)^{-1}[f(a+x) - f(a)],$$

we can, throughout this section, suppose that $E = F$, $a = 0$, $f(0) = 0$ and $f'(a) = 1$ (the identity map).

The following fact is an immediate consequence of (1.1) with $q = p_0$.

(2.1) *There is a positive constant* $\beta(f)$ *such that* $p_0(x) \leqq \beta(f)$ *implies* $\frac{1}{2}p_0(x) \leqq p_0(f(x)) \leqq \frac{3}{2}p_0(x)$.

In particular,

(2.2) *The map* f *is* p_0*-continuous at zero, i.e.,* $p_0(f(x_\lambda)) \to 0$ *for any net* (x_λ) *such that* $p_0(x_\lambda) \to 0$.

As a matter of fact, f is continuous in a stronger sense; it follows from (1.1) that $p_0(x_\lambda) \to 0$ implies that $f(x_\lambda) \to 0$. However, we shall only need the p_0-continuity, as the following result shows.

(2.3) *Let $p \geqq p_0$. Then, the inverse map g is p-differentiable at zero if and only if g is sequentially p-continuous at zero.*

PROOF. Assume that $p \geqq p_0$ and g is p-differentiable at zero. Since, by (1.2), f is also p-differentiable at zero, the chain rule implies that $g'(0) = f'(0)^{-1} = 1$. Hence, (2.2) holds with g and p instead of f and p_0. Conversely, let us assume that g is sequentially p-continuous at zero for some $p \in P(E)$ such that $p \geqq p_0$. If g is not p-differentiable at zero, there exist a null sequence $(\varepsilon_n) \subset R$, a sequence $(y_n) \supset U_p$, $q \in P(E)$ and a positive number δ such that

$$q(\varepsilon_n^{-1} g(\varepsilon_n y_n) - y_n) > \delta \text{ for all } n.$$

Put $x_n = \varepsilon_n^{-1} g(\varepsilon_n y_n)$. Then, since $p(\varepsilon_n y_n) \to 0$ and g is p-continuous at zero, we have

$$p_0(\varepsilon_n x_n) \leqq p(\varepsilon_n x_n) = p(g(\varepsilon_n y_n)) \to 0.$$

Hence, $p_0(\varepsilon_n x_n) \leqq \frac{1}{2}\beta(f)$ for large n, and, by (2.1), we have

$$p_0(\varepsilon_n x_n) \leqq p_0(f(\varepsilon_n x_n)) = p_0(\varepsilon_n y_n),$$

from which it follows that $p_0(x_n) \leqq 1$ for large n. Hence, for large n,

$$\begin{aligned} q(\varepsilon_n^{-1} g(\varepsilon_n y_n) - y_n) &= q(x_n - \varepsilon_n^{-1} f(\varepsilon_n x_n)) \\ &\leqq \sup_{p_0(x) \leqq 1} q(\varepsilon_n^{-1} f(\varepsilon_n x) - x) \to 0, \end{aligned}$$

a contradiction.

Thus, if there is $p \in P(E)$ such that $p \geqq p_0$ and g is sequentially p-continuous at zero, then g is strongly differentiable at zero. This corresponds exactly to the normed space version of this theorem. As to the Suhinin's result mentioned above, we can improve it slightly as follows.

(2.4) *Assume that the function $\tau \mapsto g(\tau y)$ is continuous when $\tau y \in f(A)$. Then, there exists $p \in P(E)$ such that $p \geqq p_0$ and g is sequentially p-continuous at zero.*

PROOF. We choose $p \in P(E)$ such that $U_p \subset f(A) \cap U_{p_0}$. Then, it follows that $p \geqq p_0$. To prove that g is sequentially p-continuous at zero, assume that $p(f(x_n)) \to 0$ and $f(x_n) \in U_p$. Then, $p(\varepsilon_n^{-1} f(x_n)) \to 0$ for some positive null sequence $(\varepsilon_n) \subset R$. Put $z_n = \varepsilon_n^{-1} x_n$. Then, we can suppose that $p(z_n) \geqq \varepsilon_n^{-\frac{1}{2}}$, because, if $p(z_n) < \varepsilon_n^{-\frac{1}{2}}$ for infinite n, we have $p(x_n) = p(\varepsilon_n z_n) < \varepsilon_n^{\frac{1}{2}} \to 0$, which completes the proof. Now, put $y_n = \varepsilon_n^{-1} f(x_n)$. Then, since $\varepsilon_n y_n \in U_p$, the function

$\tau \mapsto \varepsilon_n^{-1} p(g(\varepsilon_n \tau y_n))$ is continuous, and, for $z_n(\tau) = \varepsilon_n^{-1} g(\varepsilon_n \tau y_n)$, we have $z_n(0) = 0$ and $z_n(1) = z_n$. Hence, there exist τ_n in $(0, 1]$ such that

$$p(z_n(\tau_n)) = \varepsilon_n^{-\frac{1}{2}} \qquad \text{for all } n.$$

Since $p(\varepsilon_n z_n(\tau_n)) \to 0$, it follows from (1.2) and (2.1) that, for large n,

$$p(\varepsilon_n z_n(\tau_n)) \leqq p(f(\varepsilon_n z_n(\tau_n))) = p(\varepsilon_n \tau_n y_n),$$

which implies $p(z_n(\tau_n)) \leqq p(\tau_n y_n)$. This is impossible because $p(z_n(\tau_n)) \to \infty$ and $p(\tau_n y_n) \leqq 1$.

3. Differentiability of the inverse maps

Throughout this section, we assume that $f\colon A \to F$ is an injective map onto an open set $f(A)$, differentiable at $a \in A$ and $f'(a)$ is an isomorphism. By the same reason as in the previous section, we may suppose that $E = F$, $a = 0$, $f(0) = 0$ and $f'(0) = 1$.

To define the continuity which we need here, we shall use a notion stronger than the *brakedness* considered in Yamamuro (to appear). A sequence $(x_n) \subset E$ will be said to be *(α_n)-braked* if $\alpha_n \to \infty$ and $(\alpha_n x_n)$ is bounded. Obviously, such sequences converge to zero in the sense of Mackey, but for our purpose, we must specify the sequence (α_n). We call a map $f\colon A \to F$ *brakedly continuous at $a \in A$* if, for any (α_n)-braked sequence (x_n), the sequence $(f(a + x_n) - f(a))$ is also an (α_n)-braked sequence.

(3.1) *The map f is brakedly continuous at zero.*

PROOF. Let (x_n) be an (α_n)-braked sequence and B be a bounded set which contains $(\alpha_n x_n)$. Then, by the definition of differentiability,

$$\begin{aligned} \alpha_n f(x_n) &= \alpha_n f(\alpha_n^{-1} \alpha_n x_n) \\ &= \alpha_n r(f, 0, \alpha_n^{-1} \alpha_n x_n) + \alpha_n x_n, \end{aligned}$$

which implies that the sequence $(\alpha_n f(x_n))$ is bounded.

Now, we have the corresponding statement to (2.3).

(3.2) *The inverse map g is differentiable at zero if and only if it is brakedly continuous at zero.*

PROOF. If g is differentiable at zero, then we can apply (3.1) to g. Conversely, let g be brakedly continuous at zero. If g is not differentiable at zero, there exist a null sequence $(\varepsilon_n) \subset R$, a bounded sequence $(y_n) \subset E$, $q \in P(E)$ and a positive number δ such that

$$q(\varepsilon_n^{-1} g(\varepsilon_n y_n) - y_n) > \delta \text{ for all } n.$$

Put $x_n = \varepsilon_n^{-1} g(\varepsilon_n y_n)$. Then, since $(\varepsilon_n y_n)$ is an (ε_n^{-1})-braked sequence, $(g(\varepsilon_n y_n))$ is also an (ε_n^{-1})-braked sequence, which means that (x_n) is a bounded sequence. But this is impossible, because, for a bounded set B containing (x_n),

$$\begin{aligned} q(\varepsilon_n^{-1} g(\varepsilon_n y_n) - y_n) &= q(\varepsilon_n^{-1} f(\varepsilon_n x_n) - x_n) \\ &\leqq \sup_{x \in B} q(\varepsilon_n^{-1} f(\varepsilon_n x) - x) \to 0 \quad \text{when} \quad n \to \infty . \end{aligned}$$

As a notion of continuity, the braked continuity can not be said to be in a natural form. We add a remark to explain how distant is the differentiability of g at zero from the continuity of g at zero.

Let us consider the following property: for any map f which satisfies all conditions posed in the beginning of this section, the inverse map g is differentiable at zero whenever it is continuous at zero. In Yamamuro (to appear) we have proved that E has this property if and only if E is boundedly levered. A space E is said to be *boundedly levered* if, for any null sequence $(x_n) \subset E$ such that $x_n \neq 0$ for all n, there exists a sequence $(\alpha_n) \subset R$ such that $(\alpha_n x_n)$ is bounded but not convergent to zero. Every normed space and every strict inductive limit of an increasing sequence of Banach spaces is boundedly levered. However,

(3.3) *Every boundedly levered metrizable locally convex Hausdorff space E is normable.*

PROOF. If E is not normable, there exists a fundamental increasing sequence (p_i) of semi-norms which are mutually non-equivalent. Since E can be assumed to be complete, we can apply a lemma in Bessaga, Pełczyński and Rolewicz (1961; page 680) to obtain a sequence (x_n) such that

$$p_1(x_n) \geqq 1, \; n p_i(x_n) < p_{i+1}(x_n) \;\text{ if }\; 1 \leqq i, \; n < +\infty .$$

Then, for $\alpha_n = 1/n p_n(x_n)$, $(\alpha_n x_n)$ is a null sequence and $\alpha_n x_n \neq 0$ for all n. Suppose that $(\xi_n \alpha_n x_n)$ is a bounded sequence. Then,

$$\begin{aligned} p_i(\xi_n \alpha_n x_n) &< n^{-1} p_{i+1}(\xi_n \alpha_n x_n) \\ &\leqq n^{-1} \sup_{n \geqq 1} p_{i+1}(\xi_n \alpha_n x_n) \to 0, \end{aligned}$$

which means that $(\xi_n \alpha_n x_n)$ converges to zero. This shows that E is not boundedly levered.

4. Weak injectivity

A map $f: A \to F$ is said to *have an invertible* (*strong*) *derivative at* $a \in A$ f f is (strongly) differentiable at a and $f'(a)$ has a continuous inverse. (It is not assumed that $f'(a)$ is surjective.)

When E and F are normed spaces and f has an invertible derivative at a, then f is *weakly injective ar at* by which we mean that there exists a neighbourhood U of zero such that $f(a) \neq f(a+x)$ for all non-zero $x \in U$. (See Dieudonné (1960; page 269, Problem 1)).

On the other hand, Keller (1963–64) has constructed a map f of a countable product of R with the product topology into itself such that $f(0) = 0, f(x_n) = 0$ for some null sequence (x_n) and f has an invertible derivative at zero.

The conclusions of this section are that the fact stated above for normed spaces can be generalized by using the strong differentiability whereas the differenbtiability again behaves very baddly.

(4.1) *If $f: A \to F$ has an invertible strong derivative at $a \in A$, then f is weakly injective at a.*

Proof. By assumption, there exists $p \in P(E)$ such that

$$\sup_{p(x) \leqq 1} q(\varepsilon^{-1}[f(a+\varepsilon x) - f(a)] - f'(a)(x)) \to 0 \text{ when } \varepsilon \to 0$$

for any $q \in P(F)$. Assume that f is not weakly injective at a. Then, there exists a net (x_λ) in E which converges to zero and $f(a + x_\lambda) = f(a)$ for all λ. Since we can assume that $p(x_\lambda) \leqq 1$ for all λ, we have

$$\varepsilon^{-1}[f(a+\varepsilon x_\lambda) - f(a)] - f'(a)(x_\lambda) \to 0 \quad \text{if} \quad \varepsilon \to 0$$

uniformly with respect to λ. Assume that $p(x_\lambda) = 0$ for all λ. Then, since $p(\varepsilon^{-1} x_\lambda) = 0$ for any non-zero ε, we have

$$-f'(a)(\varepsilon^{-1}x_\lambda) = \varepsilon^{-1}[f(a+\varepsilon\varepsilon^{-1}x_\lambda) - f(a)] - f'(a)(\varepsilon^{-1}x_\lambda) \to 0$$

when $\varepsilon \to 0$, which means that $x_\lambda = 0$ for all λ, because $f'(a)$ has an inverse. Hence, taking a subsequence if necessary, we may suppose that $p(x_\lambda) \neq 0$ for all λ. Then, since $p(p(x_\lambda)^{-1}x_\lambda) = 1$ for all λ,

$$\begin{aligned} &-f'(a)(p(x_\lambda)^{-1}x_\lambda) \\ &= p(x_\lambda)^{-1}[f(a + p(x_\lambda)p(x_\lambda)^{-1}x_\lambda) - f(a)] - f'(a)(p(x_\lambda)^{-1}x_\lambda) \to 0, \end{aligned}$$

which means that $p(x_\lambda)^{-1}x_\lambda \to 0$, because $f'(a)$ has a continuous inverse. However, this is impossible.

A topological space E is said to be *sequential* if, for any subset M of E and a point $a \in \bar{M} \backslash M$ (the upper bar denotes the closure), there is a sequence (x_n) in M such that $x_n \to a$. Metrizable spaces are sequential, but the converse is not true. For a detailed account of this notion, we refer to Yamamuro (1974; Appendix 1).

(4.2) *Every map with an invertible derivative at a point is weakly injective there if and only if E is sequential and boundedly levered.*

PROOF. We suppose that E has the property that every map with an invertible derivative at a point is weakly injective there. First, we prove that E is sequential.

If E is not sequential, then there exists a subset M of E and a point $a \in \tilde{M} \backslash M$ such that no sequence in M converges to a. By considering the set $M - a$ instead of M, we may suppose that $a = 0$. Then, let $k: E \to R$ be the characteristic function of the set M and define a map $f: E \to E$ by

$$f(x) = x - k(x)x.$$

This map is not weakly injective at zero, because, if we take a net (x_λ) in M converging to zero, we have $f(x_\lambda) = f(0) = 0$ for all λ. On the other hand, f is differentiable at zero and $f'(0) = 1$. To see this, let (ε_n) be a null sequence and (x_n) be a bounded sequence; then

$$\varepsilon_n^{-1}[f(\varepsilon_n x_n) - f(0)] - x_n = -k(\varepsilon_n x_n)x_n.$$

Since $\varepsilon_n x_n \to 0$ and M does not contain null sequences, we see that $k(\varepsilon_n x_n)x_n = 0$ except for finite n's. This shows that the identity map is the derivative of f at zero, which is a contradiction.

Next, we prove that E is boundedly levered. If E is not boundedly levered, there exists a null sequence (e_n) such that $e_n \neq 0$ for all n and $\alpha_n e_n \to 0$ whenever $(\alpha_n e_n)$ is bounded. Let $k: E \to R$ be the characteristic function of $M = (e_n)$ and define a map $f: E \to E$ as above. Then, f is not weakly injective at zero. However, f has the identity map as its derivative at zero. To see this, let (ε_n) be a null sequence and (x_n) be a bounded sequence; we need to show that

$$\varepsilon_n^{-1} r(f, 0, \varepsilon_n x_n) = \varepsilon_n^{-1}[f(\varepsilon_n x_n) - f(0)] - x_n \to 0 \quad \text{when } n \to \infty.$$

We only need to consider those $\varepsilon_n x_n$ which belong to M, because, if $\varepsilon_n x_n$ is not in M, we have $\varepsilon_n^{-1} r(f, 0, \varepsilon_n x_n) = 0$. Hence, taking a subsequence if necessary, we can suppose that $\varepsilon_n x_n = e_n$ for all n. Then,

$$\varepsilon_n^{-1} r(f, 0, \varepsilon_n x_n) = -x_n = -\varepsilon_n^{-1} e_n \to 0 \quad \text{when} \quad n \to \infty,$$

because $(\varepsilon_n^{-1} e_n)$ is a bounded sequence. Thus, we have proved that E is sequential and boundedly levered.

To prove the converse, let us assume that E is sequential, boundedly levered and $f: A \to F$ has an invertible derivative at $a \in A$. If f is not weakly injective at a, since E is sequential, there exists a sequence (x_n) such that $x_n \neq 0$, $x_n \to 0$ and $f(a) = f(a + x_n)$ for all n. Since E is boundedly levered, there exists a sequence (α_n) in R such that $(\alpha_n x_n)$ is bounded but not convergent to zero. Taking a subsequence if necessary, we can suppose that $\alpha_n \to \infty$. Then, since f is differentiable at a, we have

$$\alpha_n[f(a + \alpha_n^{-1}(\alpha_n x_n)) - f(a)] - f'(a)(\alpha_n x_n) \to 0 \quad \text{when} \quad n \to \infty,$$

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which implies $f'(a)(\alpha_n x_n) \to 0$ when $n \to \infty$. This is a contradiction, because $f'(a)$ has a continuous inverse and $(\alpha_n x_n)$ is not a null sequence.

References

C. Bessaga, A. Pełczyński and S. Rolewicz (1961), 'On diametral approximative dimension and linear homogeneity of F-spaces', *Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys.* **9**, 677–683.

J. Dieudonné (1960), *Foundations of Modern Analysis* (Academic Press, New York, 1960).

H. H. Keller (1963–64), 'Differenzierbarkeit in topologischen Vektorräumen', *Comment. Math. Helv.* **38**, 308–320.

M. F. Suhinin (1969), 'Two versions of a theorem on the differentiation of the inverse function in certain linear topological space (Russian)', *Vestnik Moskow Univ. Ser.* 1, **24**, 34–38.

S. Yamamuro (to appear), 'Notes on differential calculus in topological linear spaces', *J. Reine Angew. Math.*

S. Yamamuro (1974), *Differential Calculus in Topological Linear Spaces*, Lecture Notes in Math., Vol. 374 (Springer-Verlag, 1974).

Department of Mathematics
Institute of Advanced Studies
Australian National University
Canberra.