Proceedings of the Edinburgh Mathematical Society (1999) 42, 113–125 ©

SOME UNDECIDABLE EMBEDDING PROBLEMS FOR FINITE SEMIGROUPS

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(Received 17th February 1997)

Let S be a finite semigroup, A be a given subset of S and $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$, and $\mathcal{J}$ be Green's equivalence relations. The problem of determining whether there exists a supersemigroup T of S from the class of all semigroups or from the class of finite semigroups, such that A lies in an $\mathcal{L}$ or $\mathcal{R}$-class of T has a simple and well known solution (see for example [7], [8] or [3]). The analogous problem for $\mathcal{J}$ or $\mathcal{D}$ relations is trivial if T is of arbitrary size, but undecidable if T is required to be finite [4] (even if we restrict ourselves to the case $|A| = 2$ [6]). We show that for the relation $\mathcal{H}$, the corresponding problem is undecidable in both the class of finite semigroups (answering Problem 1 of [9]) and in the class of all semigroups, extending related results obtained by M. V. Sapir in [9]. An infinite semigroup with a subset never lying in a $\mathcal{H}$-class of any embedding semigroup is known and, in [9], the existence of a finite semigroup with this property is established. We present two eight element examples of such semigroups as well as other examples satisfying related properties.

1991 *Mathematics subject classification:* primary 20M05; secondary 20F10.

1. Introduction

Throughout this paper we will adopt the notation that if S is a semigroup, then $\mathbf{S}^1$ denotes simply S if S has an identity, or otherwise the semigroup $\mathbf{S}^1$ is obtained from S by adjoining an identity element. S is the universe or underlying set of S.

On any semigroup S we can define the following equivalence relations

$$\mathcal{L}^{\mathbf{S}} = \{(a, b) : \exists\, x, y \in S^1 \textit{ such that } xa = b, yb = a\},$$

$$\mathcal{R}^{\mathbf{S}} = \{(a, b) : \exists\, x, y \in S^1 \textit{ such that } ax = b, by = a\},$$

$$\mathcal{J}^{\mathbf{S}} = \{(a, b) : \exists\, w, x, y, z \in S^1 \textit{ such that } wax = b, ybz = a\},$$

$$\mathcal{H}^{\mathbf{S}} = \mathcal{L} \wedge \mathcal{R},$$

$$\mathcal{D}^{\mathbf{S}} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}.$$

When there is no confusion as to what semigroup a particular relation is being defined on, the superscripts of these relations will be dropped. These five equivalence relations are known as Green's relations, and are fundamental concepts in the study of semigroups.

Denote by L_a (resp. R_a, H_a, J_a, D_a) the equivalence class of $\mathcal{L}$ (resp. $\mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D}$) containing a. The following two lemmas will play an important role in due course. Proofs will be omitted since they are well known and can be found in almost any semigroup textbook (see [1] for example).

Lemma 1.1 (Green). *Let a and b be two $\mathcal{R}$ equivalent elements of a semigroup* $\mathbf{S}$ *and let $s, t \in S^1$ be such that $as = b$ and $bt = a$ (s, t exist by the definition of $\mathcal{R}$). Then the mappings given by $x \mapsto xs$ and $y \mapsto yt$ for $x \in L_a$, $y \in L_b$ are $\mathcal{R}$-class preserving, mutually inverse, injective mappings from L_a to L_b and from L_b to L_a respectively. The dual statement for $\mathcal{L}$ equivalent elements also holds.*

Recall that an element $a \in \mathbf{S}$ is regular if there is an x such that $axa = a$.

Lemma 1.2. *(i) If a $\mathcal{D}$-class D of a semigroup* $\mathbf{S}$ *contains a regular element then every element of D is regular and D is called a regular $\mathcal{D}$-class of* $\mathbf{S}$.

(ii) If a $\mathcal{D}$-class D of a semigroup $\mathbf{S}$ *is regular, then every $\mathcal{L}$-class and every $\mathcal{R}$-class in D contains an $\mathcal{H}$-class that is a subgroup of* $\mathbf{S}$.

2. Preliminaries

Let $\mathcal{U}$ represent one of Green's relations on a semigroup $\mathbf{S}$. While the property of being within a $\mathcal{U}$-class of a semigroup $\mathbf{S}$ is retained under all embeddings of $\mathbf{S}$ into larger semigroups, the restriction of a $\mathcal{U}^{\mathbf{T}}$-class of a semigroup $\mathbf{T}$ to some subsemigroup $\mathbf{S}$ need not be a $\mathcal{U}^{\mathbf{S}}$-class.

Definition 2.1. If $\mathbf{S}$ is a finite semigroup and $A \subseteq S \times S$ then we say A is *eventually $\mathcal{U}$-related* if $A \subseteq \mathcal{U}^{\mathbf{T}}$ for some supersemigroup $\mathbf{T}$ containing $\mathbf{S}$. If $\mathbf{T}$ can be chosen from a particular class $\mathcal{K}$ of semigroups (the class of finite semigroups for example) then we say A is *eventually $\mathcal{U}$-related in $\mathcal{K}$*. If $A \subseteq S$ then call *A eventually $\mathcal{U}$-embeddable in a class $\mathcal{K}$* if $A \times A$ is eventually $\mathcal{U}$-related in $\mathcal{K}$.

If there is an algorithm determining whether a given finite subset of $S \times S$ is eventually $\mathcal{U}$-related then there certainly exists an algorithm determining if a given finite subset of S is eventually $\mathcal{U}$-embeddable.

Define the following relations on a semigroup $\mathbf{S}$:

$$\mathcal{L}^* = \{(a, b) : ax = ay \Leftrightarrow bx = by \ \forall x, y \in S^1\},$$

$$\mathcal{R}^* = \{(a, b) : xa = ya \Leftrightarrow xb = yb \ \forall x, y \in S^1\},$$

$$\mathcal{H}^* = \mathcal{L}^* \wedge \mathcal{R}^*.$$

We have the following well known result (for example, see [7], [8] or [3]):

Lemma 2.2. *If* **S** *is a semigroup, then a subset* $A \subseteq S \times S$ *is eventually* $\mathcal{L}$*-related (resp. eventually* $\mathcal{R}$*-related) if and only if* $A \subseteq \mathcal{L}^*$ *(resp.* $A \subseteq \mathcal{R}^*$*). Furthermore if* **S** *is finite, then a subset* $A \subseteq S \times S$ *is eventually* $\mathcal{L}$*-related (resp. eventually* $\mathcal{R}$*-related) if and only if it is eventually* $\mathcal{L}$*-related within the class of finite semigroups (resp. eventually* $\mathcal{R}$*-related within the class of finite semigroups).*

This lemma works for $\mathcal{L}^*$ (resp. $\mathcal{R}^*$) because of the left (resp. right) regular representation of **S** by inner left (resp. right) translations on the set S^1. There is no natural analogue of this for the $\mathcal{H}$-relation.

Lemma 2.2 provides a simple algorithm for testing whether a given subset of a finite semigroup is eventually $\mathcal{L}$-embeddable (or eventually $\mathcal{R}$-embeddable). In [**9**] however, M. V. Sapir has shown that the problem of determining, for two disjoint subsets A, B of a finite semigroup **S**, whether or not $(A \times A) \cup (B \times B)$ is eventually $\mathcal{H}$-related is undecidable. This, along with Lemma 2.2, implies the existence of a finite semigroup **S** and a subset $(A \times A) \cup (B \times B)$ of $S \times S$ for which $(A \times A) \cup (B \times B) \subseteq \mathcal{H}^*$ but are not eventually $\mathcal{H}$-related (Corollary 1 of [**9**]). The main aim of this paper is to present small examples of such semigroups and the following undecidability results, the first of which is an extension of results in [**9**].

Theorem 2.3. *The problem of determining whether or not a subset A of a finite semigroup* **S** *is eventually* $\mathcal{H}$*-embeddable in the class of finite semigroups or in the class of all semigroups is undecidable.*

Theorem 2.4. *The problem of determining for two disjoint subsets A and B of a finite semigroup* **S** *whether there is a supersemigroup* **T** *of* **S** *such that A lies in an* $\mathcal{R}^{\mathbf{T}}$*-class of* **T** *and B lies in an* $\mathcal{L}^{\mathbf{T}}$*-class of* **T** *is undecidable in the class of finite semigroups and in the class of all semigroups.*

Problem 1 of [**9**] asks if there is an algorithm for determining whether a subset A of a finite semigroup **S** is eventually $\mathcal{H}$-embedded in the class of finite semigroups. Theorem 2.3 answers this in the negative. It is also remarked in [**9**] that there is an algorithm for determining whether or not a subset A of a finite semigroup **S** is eventually $\mathcal{H}$-embedded in the class of all semigroups. This statement is not proved in [**9**] and in fact Theorem 2.3 shows that it is not true.

3. Proof of Theorem 2.3

The proof of Theorem 2.3 is a modification of that used by Sapir in [**9**] in which he established the previous undecidability result. For the sake of completeness, definitions of important concepts used in that paper will be given here. However, while Sapir's *split systems* play a central role in the arguments used in [**9**], for the purposes of this paper it will be more convenient to introduce the notion of a *split pair*, a very similar but slightly simpler concept.

Definition 3.1. A split pair is a pair of sets (A, B) with an associated operation $A \times A \to B$. An embedding of a split pair into a semigroup $\mathbf{S}$ is a pair of maps (j, k) such that the maps $j : A \to S$ and $k : B \to S$ are injective and $j(a)j(b) = k(ab)$, for each $a, b \in A$.

By a partial group $\mathbf{G}$ we will mean a set with an element 1 and a partially defined binary operation such that for every $x \in G$, $1x = x1 = x$ and if both $(xy)z$ and $x(yz)$ are defined then they are equal. The following definition appears in [**4**]:

Definition 3.2. Let $\mathbf{G_0}$ and $\mathbf{G}$ be partial groups such that $\mathbf{G_0}$ is embedded in $\mathbf{G}$. For each $i = 0, 1, 2, \ldots,$ let G_0^i be the subset of the universe of $\mathbf{G}$ defined as follows: $G_0^0 = \{1\}$ (the identity element), $G_0^1 = G_0$, $G_0^{i+1} = G_0^i G_0$. Then for $k \geq 2$, the partial group $\mathbf{G}$ is *an extension of rank* k of $\mathbf{G_0}$ if and only if

1. $G = \bigcup_{i=0}^{k} G_0^i$,
2. for every pair of positive integers i, j with $i + j \leq k$ and every pair of elements $x \in G_0^i$, $y \in G_0^j$, the product xy exists and is contained in G_0^{i+j},
3. if $i + j > k$ and $x \in G_0^i \backslash G_0^{i-1}$, $y \in G_0^j \backslash G_0^{j-1}$ then the product xy is not defined,
4. if $i + j + l \leq k$ and $x \in G_0^i$, $y \in G_0^j$, $z \in G_0^l$, then $(xy)z$ and $x(yz)$ are defined and equal,
5. for $f, g, h \in G$, if $fg = fh$ or $gf = hf$, then $g = h$.

For the arguments to follow, let $\mathbf{G}$ always denote an extension of rank 2 of a partial group $\mathbf{G_0}$ with the elements of G_0 labelled $\{g_1, g_2, \ldots, g_n\}$ so that g_1 is the identity element. Let the remaining elements of G be labelled $\{g_{n+1}, \ldots, g_m\}$.

From Connection 2.2 in [**5**] we have that the unsolvability of the uniform word problem in the pseudovariety of groups and in the pseudovariety of finite groups imply that the problem of determining whether a finite partial group is embeddable in a group or in a finite group is undecidable. A group $\mathbf{H}$ can be viewed trivially as an extension of arbitrary rank of itself. So for every k, if a partial group $\mathbf{G}$ is embeddable in a group (or a finite group), $\mathbf{H}$, then there is an extension of rank k of $\mathbf{G}$ that is embeddable in $\mathbf{H}$ (just take an appropriate "partial subgroup" of $\mathbf{H}$). If the problem of determining whether or not an extension of rank k of a partial group is embeddable in a group (or a finite group) is decidable then we would obtain the following algorithm for determining when an arbitrary finite partial group $\mathbf{G}$ is embeddable in a group (or a finite group), contradicting the fact that this second problem is undecidable:

1. Construct all extensions of rank k of $\mathbf{G}$ (there are only finitely many and they can be effectively listed);
2. If one of the extensions of rank k is embeddable in a group (or a finite group), $\mathbf{H}$, then $\mathbf{G}$ is embeddable in $\mathbf{H}$. Otherwise $\mathbf{G}$ is not embeddable in a group (or a finite group).

We therefore have the following lemma:

Lemma 3.3 ([4]). *The problem of determining whether or not an extension of rank k of a partial group is embeddable in a group or in a finite group is undecidable.*

The following argument and proof is simply that of Lemma 3 of [9], using split pairs instead of split systems. For a given **G**, an extension of rank 2 of a finite partial group $\mathbf{G}_0$, we can construct an associated split pair (A, B) where $A = \{a_1, \ldots, a_n\}$ is a copy of G_0, $B = \{b_1, \ldots, b_m\}$ a copy of G, and with operation $a_i a_j = b_k$ whenever $g_i g_j = g_k$ in **G**.

Lemma 3.4. *Let (A, B) be the split pair associated with* **G**, *an extension of rank 2 of a finite partial group $\mathbf{G}_0$. Then (A, B) is embeddable in a group if and only if* **G** *is embeddable in the same group.*

Proof. Let (A, B) and **G** be as in the statement of the lemma.

An embedding, θ, of **G** in a group induces a natural embedding of the split pair into that group (that is with $j(a_i) = k(b_i) = \theta(g_i)$).

So assume that (j, k) constitutes an embedding of (A, B) into a group, **H**, and let g_1 be the identity element of **G**. Then we have

$$\begin{aligned} k(a_i a_j) = j(a_i)j(a_j) &= j(a_i)j(a_1)j(a_1)^{-1}j(a_1)^{-1}j(a_1)j(a_j) \\ &= k(b_i)j(a_1)^{-1}j(a_1)^{-1}k(b_j). \end{aligned}$$

So the map $\theta : \mathbf{G} \to \mathbf{H}$ given by $\theta(g_i) = k(b_i)j(a_1)^{-1}j(a_1)^{-1}$ is an embedding of **G** into the group, since it is injective, and

$$\begin{aligned} \theta(g_i g_j) &= k(a_i a_j)j(a_1)^{-1}j(a_1)^{-1} \\ &= k(b_i)j(a_1)^{-1}j(a_1)^{-1}k(b_j)j(a_1)^{-1}j(a_1)^{-1} \\ &= \theta(g_i)\theta(g_j). \end{aligned}$$

□

Definition 3.5. For the split pair (A, B) associated with **G**, an extension of rank 2 of a partial group $\mathbf{G}_0$, define $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0)}$ to be the semigroup whose universe, $S_{(G,G_0)}$, is the set $\{0\} \cup A \cup B$ and with multiplication $a_i \cdot a_j = b_k$ if $a_i a_j = b_k$ in (A, B) and 0 otherwise.

$\mathbf{S}_{(\mathbf{G},\mathbf{G}_0)}$ is a semigroup, since the product of any three elements in $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0)}$ is zero (that is, $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0)}$ is 3-nilpotent).

Definition 3.6. If **C** is a group then define $\overline{\mathbf{C}}$ as the semigroup whose universe is $C \cup A_c \cup B_c \cup \{0\}$, where A_c, B_c are disjoint copies of the set C, and with multiplication

(for $a_i \in A_c$, $b_i \in B_c$, $c_i \in C$, and where x_i is one of a_i, b_i, or c_i)

$$a_i \cdot a_j = b_k, \textit{ if } c_i c_j = c_k \textit{ in } \mathbf{C},$$

$$x_i \cdot c_j = c_i \cdot x_j = x_k, \textit{ if } c_i c_j = c_k \textit{ in } \mathbf{C},$$

and all other products take the value 0.

$\overline{\mathbf{C}}$ is a semigroup since the subscripts of the elements behave as in the group $\mathbf{C}$ and the letter names of the elements behave according to the following table:

$\cdot$	0	A_c	B_c	C
0	0	0	0	0
A_c	0	B_c	0	A_c
B_c	0	0	0	B_c
C	0	A_c	B_c	C

which is a commutative, 3-nilpotent semigroup with adjoined identity element, C (indeed, $\overline{\mathbf{C}}$ is an extension of this semigroup). Note that since $\mathbf{C}$ is a group, the $\mathcal{H}^{\overline{\mathbf{C}}}$-classes of $\overline{\mathbf{C}}$ are $\{0\}$, A_c, B_c, and C.

Theorem 2.3 follows from the following lemma and Lemma 3.3.

Lemma 3.7. *Let (A, B) be the split pair associated with* $\mathbf{G}$, *an extension of rank 2 of a partial group* $\mathbf{G_0}$. *The subset A of* $\mathbf{S}_{(\mathbf{G},\mathbf{G_0})}$ *is eventually $\mathcal{H}$–embeddable if and only if* $\mathbf{G}$ *is embeddable in a group.*

Proof. Suppose θ is an embedding of $\mathbf{G}$ into a group $\mathbf{C}$, with the elements of $\mathbf{C}$ labelled so that $\theta(g_i) = c_i$. Then $\theta' : \mathbf{S}_{(\mathbf{G},\mathbf{G_0})} \to \overline{\mathbf{C}}$ defined by

$$\theta'(a_i) = a_i \in A_c, \theta'(b_i) = b_i \in B_c, \theta'(0) = 0$$

is an embedding of $\mathbf{S}_{(\mathbf{G},\mathbf{G_0})}$ in $\overline{\mathbf{C}}$ which sends A to the $\mathcal{H}^{\overline{\mathbf{C}}}$-class A_c.

So now assume that $\mathbf{S}_{(\mathbf{G},\mathbf{G_0})}$ is the subsemigroup of a bigger semigroup $\mathbf{T}$, in which A lies in an $\mathcal{H}^{\mathbf{T}}$-class, H_A. We may assume that $\mathbf{T}$ is regular, since every (finite) semigroup can be embedded into a (finite) regular semigroup, and its $\mathcal{H}$-classes will still be within $\mathcal{H}$-classes of the regular semigroup. Now for every $g_i, g_j \in \mathbf{G_0}$, whenever $xa_i = a_j$ and $ya_j = a_i$, for some $x, y \in \mathbf{T}$, we have $xa_ia_1 = a_ja_1$ and $ya_ja_1 = a_ia_1$, or $xb_i = b_j$ and $yb_j = b_i$, so therefore $b_i\mathcal{L}^{\mathbf{T}}b_j$. Similarly, $b_i\mathcal{R}^{\mathbf{T}}b_j$ and thus $b_i\mathcal{H}^{\mathbf{T}}b_j$. For $b_k \in B$, with $g_k \notin \mathbf{G_0}$, we can find (by the definition of A) $a_i, a_j \in A$ with $a_ia_j = b_k$. Since $A \subseteq H_A$, there exist $x, y \in \mathbf{T}^1$ with $xa_i = a_1$, $ya_1 = a_i$. So

$$xb_k = xa_ia_j = a_1a_j = b_j \text{ and } yb_j = ya_1a_j = a_ia_j = b_k$$

and hence it follows that $b_k \mathcal{L}^{\mathbf{T}} b_j$. Similarly $b_k \mathcal{R}^{\mathbf{T}} b_i$ and, since $b_i \mathcal{H}^{\mathbf{T}} b_j$, we have shown that B is contained in an $\mathcal{H}^{\mathbf{T}}$-class, H_B.

Since $\mathbf{T}$ is regular, by Lemma 1.2 there are $p, q \in \mathbf{T}^1$ such that pH_A is a group $\mathcal{H}^{\mathbf{T}}$-class (call it H_{pA}) and $qpa = a$, for all $a \in H_A$. So $qpb_k = qpa_ia_j = a_ia_j = b_k$ (for each $b_k \in B$ and some i, j). So pB is contained in an $\mathcal{H}^{\mathbf{T}}$-class which lies in the same $\mathcal{D}^{\mathbf{T}}$-class as H_B. Consider then $(pa_1)^{-1}pa_1$, the identity element of H_{pA}. By Green's Lemma, the map $\alpha : B \to pa_1^{-1}pB$ given by $\alpha(b) = (pa_1)^{-1}pb$ is injective since, for $b_k = a_ia_j$, $qpa_1(pa_1)^{-1}pb_k = qpa_1(pa_1)^{-1}pa_ia_j = qpa_ia_j = qpb_k = b_k$. Furthermore, by Lemma 1.2, $pa_1^{-1}pB$ is contained in an $\mathcal{H}$-class, H_{pB}, which lies in the same $\mathcal{D}$-class as H_B.

Now there exist $r, s \in \mathbf{T}^1$ such that $H'_{pB}r$ is a group $\mathcal{H}^{\mathbf{T}}$-class (call it H_{pBr}) and $xrs = x$, for all $x \in H_{pB}$. Now $((pa_1)^{-1}pa_1a_1r)((pa_1)^{-1}pa_1a_1r)^{-1}$ is the identity of H_{pBr}. Putting $e = (pa_1)^{-1}pa_1a_1r$ we have (with $b_k = a_ia_j$ as above),

$$\begin{aligned}
(pa_1)^{-1}pb_kre^{-1} &= (pa_1)^{-1}pa_ia_jre^{-1} \\
&= (pa_1)^{-1}pa_i((pa_1)^{-1}pa_1)a_jre^{-1} \\
&= (pa_1)^{-1}pa_iee^{-1}((pa_1)^{-1}pa_1)a_jre^{-1} \\
&= (pa_1)^{-1}pa_i(pa_1)^{-1}pa_1a_1re^{-1}((pa_1)^{-1}pa_1a_jre^{-1}) \\
&= ((pa_1)^{-1}pa_ia_1re^{-1})((pa_1)^{-1}pa_1a_jre^{-1}) \\
&= ((pa_1)^{-1}pa_1a_ire^{-1})((pa_1)^{-1}pa_1a_jre^{-1}) \\
&= ((pa_1)^{-1}pb_ire^{-1})((pa_1)^{-1}pb_jre^{-1}).
\end{aligned}$$

By Green's Lemma, the maps $j : A \to H_{pBr}$ and $k : B \to H_{pBr}$ defined by

$$j(a_i) = (pa_1)^{-1}pb_ire^{-1}, \text{ and}$$

$$k(b_i) = (pa_1)^{-1}pb_ire^{-1}$$

are injective and satisfy $j(a_i)j(a_j) = k(a_ia_j)\,(= k(b_k))$. Thus we have an embedding of the split pair (A, B) into the subgroup H_{pBr}. Therefore, by Lemma 3.4, $\mathbf{G}$ can be embedded in the subgroup H_{pBr}. □

Theorem 2.3 is proved.

Note 3.8. Notice that this result need not be restricted to finite semigroups. We can make infinite semigroups with similar behaviour by considering the 0-direct join of $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0)}$ with an infinite null semigroup (recall a null semigroup is one in which multiplication always gives 0, and that the 0-direct join of semigroups $\mathbf{N}$ and $\mathbf{M}$ is the semigroup whose universe is $\{0\} \cup (N \backslash \{0\}) \cup (M \backslash \{0\})$ with multiplication as within the subsemigroups $\mathbf{N}$ and $\mathbf{M}$, and 0 otherwise). We can follow the same arguments as above (replacing $\overline{\mathbf{C}}$ with the 0-direct join of $\overline{\mathbf{C}}$ and the infinite null semigroup), and the finite subset A is still eventually $\mathcal{H}$-embeddable if and only if $\mathbf{G}_0$ is embeddable in a group.

4. Proof of Theorem 2.4

Definition 4.1. If A and B are disjoint subsets of a finite semigroup $\mathbf{S}$, then the pair $[A, B]$ is *eventually* $\mathcal{R}-\mathcal{L}$*-embeddable* if there is a supersemigroup $\mathbf{T}$ containing $\mathbf{S}$ in which A is contained in a $\mathcal{R}^{\mathbf{T}}$-class and B is contained in an $\mathcal{L}^{\mathbf{T}}$-class. $[A, B]$ is *eventually* $\mathcal{R}-\mathcal{L}$*-embeddable in* $\mathcal{K}$ if $\mathbf{T}$ can be chosen from a particular class $\mathcal{K}$ of semigroups.

For all arguments to follow in this section, let $\mathbf{G}$ be an extension of rank 2 of a partial group $\mathbf{G}_0$.

Definition 4.2. Let $\mathbf{G}_2$ be an extension of rank 3 of $\mathbf{G}$, and let G_1 be the set $G^2 \cup G$. Let A, B, C, D be disjoint copies of the sets G_0, G, G_1, G_2 respectively. Then define $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0,\mathbf{G}_1,\mathbf{G}_2)}$ to be the semigroup whose universe is $A \cup B \cup C \cup D \cup \{0\}$ and which has the following operation:

$$\begin{aligned} a_i a_j &= b_k, \text{ whenever } g_i, g_j \in G_0, \text{ and } g_i g_j = g_k \in G, \\ a_i b_j = b_i a_j &= c_k, \text{ whenever } g_i g_j = g_k \in G_1 \text{ and } g_i \in G_0, g_j \in G \text{ or reverse}, \\ a_i c_j = c_i a_j &= d_k, \text{ whenever } g_i g_j = g_k \in G_2 \text{ and } g_i \in G_0, g_j \in G_1 \text{ or reverse}, \\ b_i b_j &= d_k, \text{ whenever } g_i, g_j \in G \text{ and } g_i g_j = g_k \in G_2, \\ & 0, \text{ otherwise.} \end{aligned}$$

Note that $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0,\mathbf{G}_1,\mathbf{G}_2)}$ is a semigroup, since the subscripts of elements behave according to the extension of rank 3 of $\mathbf{G}$, which is associative, and the letter names behave according to the 5-nilpotent semigroup

$\cdot$	0	A	B	C	D
0	0	0	0	0	0
A	0	B	C	D	0
B	0	C	D	0	0
C	0	D	0	0	0
D	0	0	0	0	0

for which associativity can be routinely verified.

Theorem 2.4 now follows from Lemma 3.3 and the following lemma.

Lemma 4.3. *Let* $\mathbf{G}$ *be an extension of rank 2 of a partial group* $\mathbf{G}_0$*. Then* $\mathbf{G}$ *is embeddable in a group if and only if there exists an extension* $\mathbf{G}_2$ *of rank 3 of* $\mathbf{G}$ *such that,* *for the subsets A and B of the semigroup $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0,\mathbf{G}_1,\mathbf{G}_2)}$ (with G_1 appropriately defined), $[A, B]$ is eventually $\mathcal{R}-\mathcal{L}$-embeddable.*

Proof. Firstly assume **G** is embeddable in a group **H** and $\mathbf{G}_2$ is an extension of rank 3 of **G** that is compatible with the multiplication of **H** (that is, $\mathbf{G}_2$ is embeddable in **H**). Then, by adjoining an identity element, 1, to the table above and then constructing a new semigroup **T** by replacing the letters $A, B, C, D, 1$ with disjoint copies of the group **H** as in Definition 3.6, it is quickly seen that $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0,\mathbf{G}_1,\mathbf{G}_2)}$ is embedded in **T** such that all of the sets $A, B, C, D, \{0\}$ lie in $\mathcal{H}^{\mathbf{T}}$-classes. So certainly $[A, B]$ is eventually $\mathcal{R}-\mathcal{L}$-embeddable. Notice also that **T** is finite if and only if **H** is finite.

So now assume there is an extension $\mathbf{G}_2$ of rank 3 of **G** such that the semigroup $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0,\mathbf{G}_1,\mathbf{G}_2)}$ (with G_1 defined as before) is embedded in a semigroup **T** in which $[A, B]$ is $\mathcal{R}-\mathcal{L}$-embedded. Proceeding as in the proof of Lemma 3.7 from the last section, we have that A being $\mathcal{R}^{\mathbf{T}}$-related implies that B is $\mathcal{R}^{\mathbf{T}}$-related. But B is $\mathcal{L}^{\mathbf{T}}$-related by our assumption, so therefore B is eventually $\mathcal{H}$-embeddable. We now show that there is an extension $\mathbf{G}_3$ of rank 2 of **G** (itself an extension of rank 2 of $\mathbf{G}_0$) for which the semigroup $\mathbf{S}_{(\mathbf{G}_3,\mathbf{G})}$ is the subsemigroup of $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0,\mathbf{G}_1,\mathbf{G}_2)}$ generated by the set B and therefore, by Lemma 3.7, $\mathbf{G}_3$, hence **G**, is embeddable in a group (and if **T** is finite, then **G** is embeddable in a finite group).

Let $D' = \{d_k \in D : b_i b_j = d_k\}$. Consider the extension $\mathbf{G}_3$ of rank 2 of **G** whose universe is the set G_1 and whose multiplication is $g_i g_j = g_k$, if both $g_i, g_j \in G$ and $g_i g_j = g_k$ in the extension of rank 3 $\mathbf{G}_2$; $g_i g_1 = g_1 g_i = g_i$, if $g_i \in G_1$; and undefined otherwise. This is a "sub partial group" of $\mathbf{G}_2$ and therefore the semigroup $\mathbf{S}_{(\mathbf{G}_3,\mathbf{G})}$ is isomorphic to the subsemigroup of $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0,\mathbf{G}_1,\mathbf{G}_2)}$ on the set $\{0\} \cup B \cup D'$. Since B is $\mathcal{H}$-related in **T**, Lemma 3.7 applies and so **G** is embeddable in a group. □

Note 4.4. As in Note 3.8, Theorem 2.4 can be modified to the class of infinite semigroups.

5. Examples

Example 1. An eight element semigroup with a three element $\mathcal{H}^*$-class that is not eventually $\mathcal{H}$-embeddable.

In [9] it is proved that there exists a finite semigroup **S** with a subset A of $\mathbf{S} \times \mathbf{S}$ that satisfies $A \subseteq \mathcal{H}^*$ but which is not eventually $\mathcal{H}$-related. Similarly Theorem 2.3 implies the existence of a finite semigroup for which there is an $\mathcal{H}^*$-class that is not eventually $\mathcal{H}$-embeddable. Such an example is not presented in [9] nor seems to have been published elsewhere. By constructing $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0)}$ for the partial group $\mathbf{G}_0$:

$\cdot$	g_1	g_2	g_3
g_1	g_1	g_2	g_3
g_2	g_2	g_3	
g_3	g_3		g_2

and its extension, **G**, of rank 2:

$\cdot$	g_1	g_2	g_3	g_4
g_1	g_1	g_2	g_3	g_4
g_2	g_2	g_3	g_4	
g_3	g_3	g_4	g_2	
g_4	g_4			

we can show that the three element subset A of $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0)}$ (which has 8 elements) is not eventually $\mathcal{H}$-embeddable: in **G** we have $(g_2g_2)(g_2g_2) = g_3g_3 = g_2 = g_2g_1$ and $g_2(g_2(g_2g_2)) = g_2g_4$ so therefore $g_2g_1 = g_2g_4$, a property not satisfied by any group. Thus **G** is not embeddable in a group and the claim follows immediately from Lemma 3.7. It is easily verified that A is an $\mathcal{H}^*$-class of $\mathbf{S}_{(\mathbf{G},\mathbf{G}_0)}$.

Example 2. An eight element semigroup with an $\mathcal{H}^*$-related pair that is not eventually $\mathcal{H}$-embeddable (alternative technique).

While Lemma 3.7 shows that any extension of rank 2 of a partial group not embeddable in a group will give rise to a semigroup with a subset that is not $\mathcal{H}$-embeddable, it is a very simple and routine exercise to show that any 3 element extension of rank 2 of a partial group is always embeddable in a group and so no smaller examples can be obtained by exactly the methods used above. This fact also makes it impossible to use the above method to construct semigroups with an $\mathcal{H}^*$-related pair that are not eventually $\mathcal{H}$-related. The following 3-nilpotent semigroup **S** (with Theorem 5.1) shows that such examples nevertheless exist:

$\cdot$	0	a_1	a_2	b_1	b_2	c_1	c_2	c_3
0	0	0	0	0	0	0	0	0
a_1	0	0	0	c_1	c_3	0	0	0
a_2	0	0	0	c_2	c_3	0	0	0
b_1	0	c_1	c_3	0	0	0	0	0
b_2	0	c_2	c_2	0	0	0	0	0
c_1	0	0	0	0	0	0	0	0
c_2	0	0	0	0	0	0	0	0
c_3	0	0	0	0	0	0	0	0

We have

Theorem 5.1. *The set $A = \{a_1, a_2\}$ in* **S** *is an $\mathcal{H}^*$-class of* **S** *but is not eventually $\mathcal{H}$-embeddable.*

Proof. A is an $\mathcal{L}^*$-class of **S**, since for $i \in \{1, 2\}$, $a_i x = a_i y$, for $x, y \in S^1$, $x \neq y$, if and only if both x and y are contained in $\{0, a_1, a_2, c_1, c_2, c_3\}$. Likewise, A is an $\mathcal{R}^*$-class and therefore an $\mathcal{H}^*$-class.

Now let **T** be any semigroup in which **S** can be embedded so that A is $\mathcal{L}^{\mathbf{T}}$-related. So there is an $x \in T^1$ such that $xa_1 = a_2$. Therefore,

$$(xb_1)a_1 = x(b_1 a_1) = xc_1 = xa_1 b_1 = a_2 b_1 = c_2 = b_2 a_1.$$

However

$$(xb_1)a_2 = x(b_1 a_2) = xc_3 = xa_1 b_2 = a_2 b_2 = c_3 \neq c_2 = b_2 a_2.$$

So therefore A is not $\mathcal{R}^*$-related. That is, whenever A is $\mathcal{L}$-related in some embedding semigroup, it is neither $\mathcal{R}$-related nor eventually $\mathcal{R}$-related in that semigroup. □

Example 3. Infinite examples.

In view of Note 3.8, the two previous examples can be modified to provide infinite semigroups with 3 element and 2 element $\mathcal{H}^*$-classes respectively that are not eventually $\mathcal{H}$-embeddable. Infinite examples consisting of single $\mathcal{H}^*$-classes that are not eventually $\mathcal{H}$-related are also known. For example, Fountain has noted (see comment in **[9]**) that any cancellative semigroup not embeddable in a group is $\mathcal{H}^*$-related but not eventually $\mathcal{H}$-embeddable (see **[1]** for such an example by Malcev). On the other hand, it is a simple task to prove that a finite semigroup for which $\mathcal{H}^*$ is the universal relation is a group.

Example 4. A ten element semigroup with two subsets that are not eventually $\mathcal{R} - \mathcal{L}$-embeddable.

To the multiplication table for **S** in Example 2 above, add two elements d_1, d_2 with the multiplication $d_i x = y$ whenever $a_i x = y$, $xd_i = y$ whenever $xa_i = y$ and all other products not already defined take the value 0. Let the resulting 3-nilpotent semigroup be denoted by **U**.

Theorem 5.2. *The subsets $\{d_1, d_2\}$ and $\{a_1, a_2\}$ of U are $\mathcal{R}^*$ and $\mathcal{L}^*$ classes of* **U** *respectively but $[\{d_1, d_2\}, \{a_1, a_2\}]$ is not eventually $\mathcal{R} - \mathcal{L}$-embeddable.*

Proof. The proof is similar to that of Theorem 5.1.

Since $\{a_1, a_2\}$ is an $\mathcal{H}^*$-class of **S**, then $\{a_1, a_2\}$ and $\{d_1, d_2\}$ lie within $\mathcal{H}^*$-classes of **U** (in fact they lie within the same $\mathcal{H}^*$-class).

Now let **T** be any semigroup in which **U** can be embedded so that a_1 and a_2 are

$\mathcal{L}^{\mathbf{T}}$-related. So there is an $x \in T$ such that $xa_1 = a_2$. Therefore,

$$(xb_1)d_1 = x(b_1d_1) = xc_1 = xa_1b_1 = a_2b_1 = c_2 = b_2d_1.$$

However

$$(xb_1)d_2 = x(b_1d_2) = xc_3 = xa_1b_2 = a_2b_2 = c_3 \neq c_2 = b_2d_2.$$

So therefore d_1 and d_2 are not $\mathcal{R}^*$-related in $\mathbf{T}$. So $[\{d_1, d_2\}, \{a_1, a_2\}]$ is not eventually $\mathcal{R} - \mathcal{L}$-embeddable. □

Example 5. A four element semigroup with a 3 element $\mathcal{D}^*$-class that is not eventually $\mathcal{D}$-embeddable in the class of finite semigroups.

We will show that with $\mathcal{D}^*$ defined as $\mathcal{L}^* \vee \mathcal{R}^*$ it is easy to construct examples of $\mathcal{D}^*$-classes of finite semigroups that are not eventually $\mathcal{D}$-embeddable (or $\mathcal{J}$-embeddable) within the class of finite semigroups (recall that every semigroup is eventually $\mathcal{D}$ and $\mathcal{J}$-embeddable in a (possibly infinite) semigroup and that on a finite semigroup, the relations $\mathcal{D}$ and $\mathcal{J}$ coincide; see **[1]**).

Define $\mathbf{D}$ to be the following 3-nilpotent semigroup:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	c	0	0
b	0	c	0	0
c	0	0	0	0

Theorem 5.3. *The set $\{a, b, c\}$ is a $\mathcal{D}^*$-class of* $\mathbf{D}$ *but it is not eventually $\mathcal{D}$-embeddable (or eventually $\mathcal{J}$-embeddable) in a finite semigroup.*

Proof. The $\mathcal{L}^*$ classes of $\mathbf{D}$ are $\{a, b\}, \{c\}, \{0\}$ and the $\mathcal{R}^*$ classes of $\mathbf{D}$ are $\{a\}, \{b, c\}$ and $\{0\}$. Hence $\{a, b, c\}$ is a $\mathcal{D}^*$-class. However if $\{a, b, c\}$ is $\mathcal{D}$-embeddable in a finite semigroup, then it is $\mathcal{D}$-embeddable in a finite 0-simple semigroup. In a finite 0-simple semigroup we have $xyz = 0 \Leftrightarrow xy = 0$ or $yz = 0$ (this property is called *categorical at* 0); however in $\mathbf{D}$ we have $aaa = 0$ with $aa \neq 0$. (This is a direct application of Theorem 2.5 of **[4]** which states that a 3-nilpotent semigroup is embeddable in a completely 0-simple semigroup if and only if it is categorical at 0.) Hence $\mathbf{D}$ is not embeddable in a finite 0-simple semigroup, and therefore $\{a, b, c\}$ is not eventually $\mathcal{D}$- or $\mathcal{J}$-embeddable within the class of finite semigroups. □

Note that Fountain (Example 2.2 in **[2]**) has found an 8 element example with $\mathcal{D}^*$-related idempotents e and f satisfying $e > f$ (recall that for idempotents e, f, we define $e < f$ to mean $ef = fe = e$). Since $\mathcal{D}$-classes containing idempotents e, f with $e > f$ are infinite (see **[1]**) these two elements are not eventually $\mathcal{D}$-embeddable in a finite semigroup.

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