MODULAR FITTING FUNCTORS IN FINITE GROUPS

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We consider Fitting classes for which the injectors in any finite
solvable group are modular subgroups. It is shown that only normal
Fitting classes have this property. In fact, we prove two more
general results demonstrating that modular Fitting functors and
submodular Fitting classes are normal.

1. Introduction

All groups considered in this paper are assumed to be finite. A class
X of groups (closed under isomorphisms) is called a *Fitting class* if it
satisfies the following two properties:

(1)   If G is a group in X, N a normal subgroup of G, then N is in
X .

(2)   If G is the product of two normal subgroups belonging to X, then
G is in X .

The relevance of Fitting classes lies in the theory of finite solvable
groups and stems from the fact that each Fitting class X gives rise to
a distinguished conjugacy class of subgroups in any finite solvable group,

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475

Peter Hauck and Reinhold Kienzle

the so-called *X-injectors*. A subgroup U of G is called an X-injector
if $U \cap N$ is maximal among all subgroups of N contained in X
(X-maximality) for all normal subgroups N of G . The well-known result
of Fischer, Gaschütz and Hartley ([3], Satz 1) states that every finite
solvable group G possesses X-injectors precisely when X is a
Fitting class, and in this case all X-injectors of G are conjugate.

It follows from the definition that, for a given Fitting class X ,
every group G has a unique largest normal subgroup contained in X ,
the X-*radical* $\mathrm{Rad}_X(G)$. Clearly, any X-injector of G contains $\mathrm{Rad}_X(G)$.

The concepts of injector and radical have been generalized by
Beidleman, Brewster and Hauck [1], [2] in the following way. Let U be
either the class F of all finite groups or the class S of all finite
solvable groups.

A *Fitting functor* f (in U) assigns to each group G in U a
nonempty set $f(G)$ of subgroups of G such that the following conditions
are satisfied:

(1)   If α is an isomorphism of G onto $\alpha(G)$, then $f(\alpha(G)) =$
$\{\alpha(X) \mid X \in f(G)\}$.

(2)   If N is a normal subgroup of G , then $f(N) = \{X \cap N \mid X \in f(G)\}$.

If X is a Fitting class, then, denoting by $\mathrm{Inj}_X(G)$ the set of
X-injectors of a given group G , it is not hard to see that $f = \mathrm{Rad}_X$ is
a Fitting functor in F and in S and that $f = \mathrm{Inj}_X$ is a Fitting
functor in S . In fact, these are examples of so-called *conjugate*
Fitting functors, that is $f(G)$ consists of a single conjugacy class of
subgroups for any group G in the respective universe U .

Our main interest in this paper is in *normal* Fitting functors, that
is those Fitting functors f such that, for all groups G in U , $f(G)$
consists only of normal subgroups of G .

It is an easy exercise to prove that a normal conjugate Fitting
functor f in $U = S$ or F (that is $|f(G)| = 1$ for all groups G in

U) is of the form $f = \text{Rad}_X$ for some Fitting class X ([1], 3.10.a).
Considerably deeper is the statement that, in general, a normal Fitting
functor is a union $f = \cup \{\text{Rad}_{X_i} \mid i \in I\}$ of radical-functors for certain
Fitting classes X_i ([1], 7.5).

The first purpose of the present paper is to prove that *modular*
Fitting functors are normal (and hence of the form described above), where,
of course, a Fitting functor f in U is called modular if, for any
group G in U , all members of $f(G)$ are modular subgroups of G . We
recall that a subgroup U of G is modular if the following two properties
are satisfied:

$$<U,M> \cap N = <M, U \cap N> \quad \text{whenever} \quad M \leq N \leq G \quad \text{and}$$

$$<U,M> \cap N = <U, M \cap N> \quad \text{whenever} \quad U \leq N \leq G, M \leq G .$$

Since modularity of subgroups is not a transitive relation, it is natural
to consider the concept of a *submodular* subgroup. (The concept of
"submodularity" has been introduced and investigated by I. Zimmermann
(Freiburg) in a yet unpublished thesis.) Clearly, a subgroup U of G is
termed submodular if there exists a chain $U = U_O \leq U_1 \leq \ldots \leq U_n = G$ such
that U_i is modular in U_{i+1} for $i = 0, \ldots, n-1$.

If, for a given group G, $f_{sm}(G)$ is defined to be the set of all
submodular subgroups of G , then it is obvious that f_{sm} is a non-normal
Fitting functor. Therefore, a result like the one mentioned above for
modular Fitting functors does not hold for submodular Fitting functors. The
same phenomenon occurs for subnormal Fitting functors; but in this case,
it is rather easy to show that conjugate subnormal Fitting functors (or
more generally, subnormal Fitting functors satisfying the Frattini
argument; see [1], 3.5) are normal.

Whether an analogous result holds for conjugate submodular Fitting
functors, remains an open question. We have, however, a positive answer
for the important case of injectors for Fitting classes in finite solvable
groups.

Peter Hauck and Reinhold Kienzle

2. Notation

The group theoretical notation used in this paper is mainly standard.

C_n stands for the cyclic group of order n and $\mathbb{F}_q$ for the field with q elements.

The proofs of the theorems depend on certain wreath product constructions. For groups G and H, $G \, wr \, H$ denotes the regular wreath product. If K is a subgroup of G, then K^* stands for the direct product of $|H|$ copies of K in the base group of $G \, wr \, H$, and K_i denotes the image of K under the natural embedding into the i-th component of K^*. The projection from G^* onto the i-th component is always denoted by π_i.

Finally, for a group G with subgroup H, $\mathrm{Core}_G(H)$ stands for the largest normal subgroup of G contained in H and H^G for the smallest normal subgroup of G containing H.

3. The results

THEOREM 1. *Let f be a modular Fitting functor in $U = F$ or S. Then f is normal, that is there exists a family of Fitting classes $\{X_i \mid i \in I\}$ such that $f(G) = \cup \{Rad_{X_i}(G) \mid i \in I\}$ for every group G in U.*

THEOREM 2. *Let X be a Fitting class such that the X-injectors of any finite solvable group G are submodular in G. Then X is a normal Fitting class, that is $Rad_X(G)$ is X-maximal in G for all finite solvable groups G.*

4. The proofs

Proof of Theorem 1. Suppose the theorem is false and let $G \in U$ be a group of minimal order such that there exists a non-normal $V \in f(G)$.

Let K be a finite solvable group of nilpotent length at least 4

and of order coprime to the order of G and set $H = G \ wr \ K \in U$. Choose $T \in f(H)$ such that $T \cap G_1 = V_1$. We prove first that T is contained in G^* .

Let π be the set of prime divisors of the order of K . By the Schur-Zassenhaus theorem, T has a Hall π-subgroup Q . By the same theorem, there exists some $x \in G^*$ such that $Q^x \leq K$. Note that $T^x \in f(H)$. If $Q \neq 1$, we may assume without loss of generality that $G_2^{\ q} = G_1$ for some $q \in Q$. It follows that $G_1 \times G_2 = \langle G_2, q \rangle \cap (G_1 \times G_2)$ $= \langle G_2, T^x \rangle \cap (G_1 \times G_2) = \langle G_2, T^x \cap (G_1 \times G_2) \rangle$ by the modularity of T^x . Consequently, $\pi_1(T^x \cap (G_1 \times G_2)) = G_1 = \pi_1(T \cap (G_1 \times G_2))$. Using the fact that $T \cap G_1$ is normal in $\pi_1(T \cap G^*)$, we conclude that V is normal in G , a contradiction. Hence $Q = 1$.

This shows that $T \leq G^*$, whence $T^H \leq G^*$. We set $C = C_H(T^H/\mathrm{Core}_H(T))$ and claim that C is contained in G^* .

Assuming the contrary, the fact that C is normal in H and the Schur-Zassenhaus theorem imply the existence of some non-trivial element $k \in C \cap K$. Without loss of generality, $G_1^{\ k} = G_2$. Let $g \in V$. Then $(g,1,\ldots,1) \in T$ and $(1,g,1,\ldots,1) = (g,1,\ldots,1)^k \in T^H$. Thus $(g^{-1},g,1,\ldots,1) \in \mathrm{Core}_H(T)$ as $k \in C$. Then $(g^{-1}g^y,1,\ldots,1) =$ $(g^{-1},g,1,\ldots,1)(g,g^{-1},1,\ldots,1)^{(y,1,\ldots,1)} \in \mathrm{Core}_H(T) \leq T$ for any $y \in G$. Consequently, $[V,G] \leq \pi_1(T \cap G_1) = V$ that is V is normal in G , a contradiction.

Therefore C is contained in G^* and, by the choice of K, H/C has nilpotent length at least 4 . Since T is modular in H , this contradicts Satz 5 of [8].

Thus f is a normal Fitting functor. The structure of f now follows from Korollar 7.5 of [1].

The following lemma is essential for the proof of Theorem 2. In its statement we use the well-known fact that a group with a unique minimal normal subgroup possesses a faithful irreducible module over $\mathbb{F}_p$, p a prime, if this minimal normal subgroup is not a p-group.

LEMMA. *Let* p *and* q *be different primes,* $A = C_p$, B *a faithful irreducible* A-*module over* $\mathbb{F}_q$, C *a faithful irreducible* AB-*module over* $\mathbb{F}_p$, D *a faithful irreducible* ABC-*module over* $\mathbb{F}_q$, *and set* $H = ABCD$. *(All products are semidirect products with respect to the obvious action.) Then none of the groups* ABC, BC, *and* AC *is submodular in* H.

Proof. ABC is a maximal subgroup of H, $\text{Core}_H(ABC) = 1$. Since ABC is not nilpotent, it follows from Lemma 1 of [8] that ABC is not modular and hence not submodular in H.

Now let X stand for BC or AC. Assume that X is submodular in H. The only maximal subgroups of H containing X are XD and all complements of D in H that contain X. Since H is primitive, the latter ones are conjugate to ABC ([6], II.3.2) and thus not modular in H by what we have proved above. Since in solvable groups maximal modular subgroups are maximal subgroups ([8], Lemma 1), there exists a chain $1 = D_0 < D_1 < \ldots < D_r = D$ of X-invariant subgroups in D such that D_{i+1}/D_i is an irreducible X-module and XD_i is a (maximal) modular subgroup of XD_{i+1} for $i = 0, \ldots, r-1$. We claim that at least one of the modules D_{i+1}/D_i is not 1-dimensional. For otherwise, X' centralizes all modules D_{i+1}/D_i. Since X is not abelian (C is a faithful AB-module), X' is a non-trivial subgroup of C. Being a p-group centralizing a series of a q-group, $p \neq q$, X' centralizes D. This contradicts the fact that D is a faithful ABC-module.

So there exists some j, $0 \leq j \leq r-1$, such that D_{j+1}/D_j is at least 2-dimensional. Thus q^2 divides the order of $XD_{j+1}/\text{Core}_{XD_{j+1}}(XD_j)$.

Since XD_j is maximal modular in XD_{j+1} , this contradicts Lemma 1 of [8].

Proof of Theorem 2. We may assume that $X \subseteq S$ (by replacing X by $X \cap S$ if necessary) and $X \neq (1)$. Then X contains C_r for any prime r .

For otherwise, there exist primes p and q such that $C_p \in X$ and $C_q \notin X$. Taking H to be the group of the Lemma, it follows that the X-injectors of H are just the Sylow p-subgroups of H (see [4], section 3.3). Since AC is not submodular in H , this is a contradiction.

We now show that $G^2 \, wr \, C_r \in X$ for all primes r (where G^2 stands for $G \times G$).

Assuming the contrary, there is a group G in X of minimal order such that there exists a prime q with $G^2 \, wr \, C_q \notin X$. By the remark made above, $G \neq 1$. Let N be a maximal normal subgroup of G , of index p say (p a prime). Then $N \in X$ and thus $N^2 \, wr \, C_q \in X$ and $N^2 \, wr \, C_p \in X$ by the minimal choice of G . It follows that $G^2 \, wr \, C_p$, being a product of the two subnormal subgroups $N^2 \, wr \, C_p \in X$ and $(G^2)^* \in X$, is contained in X . In particular, $p \neq q$.

Let H be the group of the Lemma (with the same meaning of p and q as here) and set $K = G^2 \, wr \, H$.

Let V be an X-injector of K . Clearly, $(G^2)^* \leq V$.

Suppose there exists a non-trivial subgroup D_1 of D such that $(G^2)^* D_1 \in X$. We may assume D_1 to be of order q . Since $(G^2)^* D_1 \cong G^{2m} \, wr \, C_q$, where m is the index of D_1 in H , it follows from 5.11.a of [5] that $G^2 \, wr \, C_q \in X$, a contradiction.

We conclude that $V \cap (G^2)^* D = (G^2)^*$.

Since $G^2 \ wr \ C_p \in X$, 5.1 and 5.9a of [5] imply that

$(G^2) * C \cong G^{2n} \ wr \ C$ (n is the index of C in H) is contained in X .

Consequently, by passing to a suitable conjugate of V , we may assume

that $V \cap (G^2)*CD = (G^2)*C$.

If $V/(G^2)*$ is a p-group, then V is conjugate to $(G^2)* \ AC$

since the latter group is isomorphic to $G^{2l} \ wr \ AC$ (l is the index of

AC in H) and thus contained in X by 5.1 and 5.9a of [5] again.

Otherwise, it follows easily from the fact that injectors cover or avoid

chief-factors ([4], Lemma 1) that V is conjugate to $(G^2)*BC$ or to

$(G^2)* \ ABC$. Since V is submodular in K, $V/(G^2)*$ is submodular in

$K/(G^2)*$. We conclude that one of the groups AC , BC , or ABC is

submodular in H . This contradicts the Lemma. We have proved that

$G^2 \ wr \ C_r \in X$ for all $G \in X$ and all primes r . This implies that X

is a normal Fitting class by 5.11b of [5] and the theorem of Makan [7].

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