HOMOTOPICAL NILPOTENCY OF LOOP-SPACES

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1. Introduction. In this paper we shall work in the category of countable CW-complexes with base point and base point preserving maps. All homotopies shall also respect base points. For simplicity, we shall frequently use the same symbol for a map and its homotopy class. Given spaces X, Y, we denote the set of homotopy classes of maps from X to Y by $[X, Y]$. We have an isomorphism τ: $[\Sigma X, Y] \rightarrow [X, \Omega Y]$ taking each map to its adjoint, where Σ is the suspension functor and Ω is the loop functor. We shall denote $\tau(1_{\Sigma X})$ by e' and $\tau^{-1}(1_{\Omega X})$ by e.

Suppose that $X_1, \ldots, X_n$ are spaces. For $0 \leqq i \leqq n - 1$, let $T_i(X_1, \ldots, X_n)$ be the subset of the cartesian product $X_1 \times X_2 \times \ldots \times X_n$ consisting of those n-tuples with at least i coordinates at the base points. If the spaces $X_1, \ldots, X_n$ are understood, we shall frequently abbreviate $T_i(X_1, \ldots, X_n)$ by T_i. Since maps preserve base points, given maps f_k: $X_k \rightarrow Y_k$ for $k = 1, \ldots, n$, we see that $f_1 \times \ldots \times f_n$ maps $T_i(X_1, \ldots, X_n)$ to $T_i(Y_1, \ldots, Y_n)$. We denote the restriction of $f_1 \times \ldots \times f_n$ to $T_i(X_1, \ldots, X_n)$ by $T_i(f_1, \ldots, f_n)$. Thus, $T_i(f_1, \ldots, f_n)$ is a map from $T_i(X_1, \ldots, X_n)$ to $T_i(Y_1, \ldots, Y_n)$. We see that T_0 is the usual cartesian product functor, T_1 is the so-called "fat wedge", and T_{n-1} is the one-point union functor. We have natural transformations of functors $T_i \rightarrow T_{i-1}$ for $i = 1, \ldots, n$, induced by the obvious inclusions. Let us denote the composition $T_i \rightarrow T_0$ by j_i, where we shall drop the suffix i if it is understood from the context. The quotient T_0/T_1 is the smash product functor $\wedge$. We may consider j_i as a natural fibration. Let F_i be the fibre and u_i: $F_i \rightarrow T_i$ the inclusion.

2. Given spaces $X_1, \ldots, X_n$, let us consider the fibration

$$F_i(X_1, \ldots, X_n) \xrightarrow{u_i} T_i(X_1, \ldots, X_n) \xrightarrow{j_i} T_0(X_1, \ldots, X_n).$$

Then it can be checked that there is map θ_i: $\Omega T_0 \rightarrow \Omega T_i$ such that $(\Omega j_i)\theta_i \simeq 1_{\Omega T_0}$. In fact, if p_k: $T_0(X_1, \ldots, X_n) \rightarrow X_k$ is the projection and ι_k: $X_k \rightarrow T_i(X_1, \ldots, X_n)$ the obvious inclusion, then we can and shall take $\theta_i = \Omega(\iota_1 p_1) + \ldots + \Omega(\iota_n p_n)$. Further, θ_i is an H-map if $i \leqq n - 2$; see (**7**, Lemma 1 or **3**, Theorem 2.14). Thus we, have a split short exact sequence of H-spaces:

$$* \rightarrow \Omega F_i \xrightarrow{\Omega u_i} \Omega T_i \xrightarrow{\Omega j_i} \Omega T_0 \rightarrow *.$$

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Consider $\theta_i(\Omega j_i)\colon \Omega T_i \to \Omega T_i$. From the exact sequence we see that there exists a unique element d_i of $[\Omega T_i, \Omega F_i]$ such that

$$1_{\Omega T_i} = (\Omega u_i)d_i + \theta_i(\Omega j_i) = (\Omega u_i)d_i + \sum_{k=1}^{n} \Omega(\iota_k p_k j_i).$$

Given a map $f\colon X \to T_i(X_1, \ldots, X_n)$ we define a map

$$H_i^n(f) = d_i(\Omega f)\colon \Omega X \to \Omega F_i(X_1, \ldots, X_n).$$

Then we have that

$$\Omega f = (\Omega u_i)H_i^n(f) + \sum_{k=1}^{n} \Omega(\iota_k p_k j_i f).$$

Note that if $n = 2$ and $i = 1$, then $F_1(X_1, X_2) = X_1 \flat X_2$, the "flat product", and if $f\colon X \to X_1 \vee X_2$ is a map, then $H_1^2(f)\colon \Omega X \to \Omega(X_1 \flat X_2)$ is the dual of the Hopf construction used in (**4**). In the general case, we note that if $g\colon Y \to X$ is another map, then $H_i^n(fg) = H_i^n(f)(\Omega g)$. If $f_k\colon X_k \to Y_k$ are maps for $k = 1, \ldots, n$, then we have a map

$$T_i(f_1, \ldots, f_n)\colon T_i(X_1, \ldots, X_n) \to T_i(Y_1, \ldots, Y_n).$$

This map induces a map

$$F_i(f_1, \ldots, f_n)\colon F_i(X_1, \ldots, X_n) \to F_i(Y_1, \ldots, Y_n)$$

such that $T_i(f_1, \ldots, f_n)u_i \simeq u_i F_i(f_1, \ldots, f_n)$. Then, if $g\colon Z \to T_i(Y_1, \ldots, Y_n)$ is a map, we easily check that

$$\Omega\{T_i(f_1, \ldots, f_n)g\} = (\Omega u_i)\Omega\{F_i(f_1, \ldots, f_n)\}H_i^n(g) + \sum_{k=1}^{n} \Omega\{\iota_k p_k j_i T_i(f_1, \ldots, f_n)g\}$$

and $H_i^n\{T_i(f_1, \ldots, f_n)g\} = \Omega\{F_i(f_1, \ldots, f_n)\}H_i^n(g)$.

We now apply our results to the nilpotency of loop-spaces; see (**1**) for definitions. Let $X_1, \ldots, X_n$ be spaces. For each $k = 1, \ldots, n$, let $e\colon \Sigma\Omega X_k \to X_k$ be a map such that $\tau(e) = 1_{\Omega X_k}$. Let $e_k = \iota_k e\colon \Sigma\Omega X_k \to T_i(X_1, \ldots, X_n)$. Then $\tau(e_k)\colon \Omega X_k \to \Omega T_i$. Note, of course, that $\tau(e_k)$ is $\Omega\iota_k$. Let $c_n\colon T_0(\Omega T_i, \ldots, \Omega T_i) \to \Omega T_i$ be the commutator map of weight n; see (**1**). Let $\bar{c}_n = \tau^{-1}\{c_n(\tau(e_1) \times \ldots \times \tau(e_n))\}\colon \Sigma T_0\Omega(X_1, \ldots, X_n) \to T_i(X_1, \ldots, X_n)$. We have, of course, a map $\bar{c}_n$ for each i. However, the i shall be understood from the context, and we shall suppress it from the notation for $\bar{c}_n$ unless it is absolutely necessary in order to avoid confusion, in which case we shall write $\bar{c}_{n,i}$ for the obvious $\bar{c}_n$. Applying the above method, we now have maps $H_i^n(\bar{c}_n)\colon \Omega\Sigma T_0\Omega(X_1, \ldots, X_n) \to \Omega F_i(X_1, \ldots, X_n)$ satisfying the relation

$$\Omega\bar{c}_n = (\Omega u_i)H_i^n(\bar{c}_n) + \sum_{k=1}^{n} \Omega(\iota_k p_k j_i \bar{c}_n).$$

LEMMA 1. $\Omega\bar{c}_n = (\Omega u_i)H_i^n(\bar{c}_n)$.

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Proof. Consider $j_i\bar{c}_n$: $\Sigma T_0\Omega(X_1, \ldots, X_n) \to T_0(X_1, \ldots, X_n)$. We have that $\tau(j_i\bar{c}_n)$: $T_0\Omega \to \Omega T_0$. We note that $T_0\Omega(X_1, \ldots, X_n) \cong \Omega T_0(X_1, \ldots, X_n)$. Now, using the fact that Ωj_i and Ωp_k are H-maps, we have that

$$\tau(j_i\bar{c}_n) = (\Omega j_i)\tau(\bar{c}_n) = (\Omega j_i)c_n(\tau(e_1) \times \ldots \times \tau(e_n)) = c_n(\tau(j_ie_1) \times \ldots \times \tau(j_ie_n)).$$

Hence, for each k, we have that

$$\begin{aligned}(\Omega p_k)\tau(j_i\bar{c}_n) &= (\Omega p_k)c_n(\tau(j_ie_1) \times \ldots \times \tau(j_ie_n)) \\ &= c_n(\tau(p_kj_ie_1) \times \ldots \times \tau(p_kj_ie_n)).\end{aligned}$$

We have used c_n to denote the commutator maps of weight n for various different spaces. Since $\tau(p_kj_ie_r) = 0$ for $r \neq k$, it is clear that the last map on the right-hand side is homotopically trivial. Hence, $\tau(p_kj_i\bar{c}_n) = 0$ for all k, and hence $p_kj_i\bar{c}_n = 0$ for all k since τ is an isomorphism. It follows then that $j_i\bar{c}_n = 0$ by the property of direct products. Thus, we have that $\Omega\bar{c}_n = (\Omega u_i)H_i^n(\bar{c}_n)$.

LEMMA 2. *There exists a map* b_i: $\Sigma T_0\Omega(X_1, \ldots, X_n) \to F_i(X_1, \ldots, X_n)$ *such that* $u_ib_i = \bar{c}_n$ *and* $H_i^n(\bar{c}_n) = \Omega b_i$.

Proof. According to the proof above, we have that $j_i\bar{c}_n = 0$. From the exact sequence of the fibration

$$F_i \xrightarrow{u_i} T_i \xrightarrow{j_i} T_0,$$

we have a map b_i: $\Sigma T_0\Omega \to F_i$ such that $u_ib_i = \bar{c}_n$. Hence, $(\Omega u_i)(\Omega b_i) = (\Omega\bar{c}_n)$. Since $\Omega c_n = (\Omega u_i)H_i^n(\bar{c}_n)$ and $(\Omega u_i)_\#$ is a monomorphism, it follows that $H_i^n(\bar{c}_n) = \Omega b_i$.

Now suppose that $X_1 = X_2 = \cdots = X_n = X$ and $i = n - 1$. Then we have a generalized folding map ∇: $T_{n-1}(X, \ldots, X) \to X$. It is easily checked that $\nabla\bar{c}_n = \tau^{-1}(c_n)$, where c_n: $T_0\Omega(X, \ldots, X) \to \Omega X$ is the commutator map of weight n for ΩX.

THEOREM 1. $c_n = \Omega(\nabla u_{n-1})H_{n-1}^n(\bar{c}_n)e'$, *where* c_n *is the commutator map of weight* n *for* ΩX. *Hence,* nil $X < n$ *if and only if* $\nabla u_{n-1}b_{n-1} \simeq *$.

Proof. By Lemma 1, we have that $\Omega\bar{c}_n = (\Omega u_{n-1})H_{n-1}^n(\bar{c}_n)$. Hence, $\Omega(\nabla\bar{c}_n) = \Omega(\nabla u_{n-1})H_{n-1}^n(\bar{c}_n)$. Since $\nabla\bar{c}_n = \tau^{-1}(c_n)$, we have that $\tau(\nabla\bar{c}_n) = c_n$. Hence, $c_n = \Omega(\nabla\bar{c}_n)e' = \Omega(\nabla u_{n-1})H_{n-1}^n(\bar{c}_n)e'$. Since $H_{n-1}^n(\bar{c}_n) = \Omega b_{n-1}$, we have that $c_n = \tau(\nabla u_{n-1}b_{n-1})$ and the result follows.

Remark. We observe that if $\Omega(\nabla u_{n-1}) \simeq *$, then $\Omega(\nabla i) \simeq *$: $\Omega(X \flat X) \to \Omega X$. This is easily seen by embedding $X \times X$ in $T_0(X, \ldots, X)$ as the first two coordinates. This induces maps which yield a diagram

$$\begin{array}{ccccc}
X \flat X & \xrightarrow{i} & X \vee X & \xrightarrow{j} & X \times X \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\
F_{n-1}(X, \ldots, X) & \xrightarrow[u_{n-1}]{} & T_{n-1}(X, \ldots, X) & \xrightarrow[j_{n-1}]{} & T_0(X, \ldots, X)
\end{array}$$

Since $\nabla f_2\colon X \vee X \to X$ is actually the folding map, the result follows. We observe that $\Omega(\nabla i) \simeq *$ is a condition for $\operatorname{nil} X \leqq 1$; see (**2**; **4**). Thus, if $\Omega(\nabla u_{n-1}) \simeq *$, then $\operatorname{nil} X \leqq 1$.

Now, suppose that X is an H′-space with comultiplication $\phi\colon X \to X \vee X$ such that $j\phi \simeq \Delta\colon X \times X$, where $j\colon X \vee X \to X \times X$ is the inclusion and Δ is the diagonal map. Define $\phi_3 = (\phi \vee 1)\phi\colon X \to X \vee X \vee X$, $\phi_3' = (1 \vee \phi)\phi\colon X \to X \vee X \vee X$. Let Y be a space and let $f_1, f_2, f_3\colon X \to Y$ be maps. Then $T_2(f_1, f_2, f_3)\colon T_2(X, X, X) \to T_2(Y, Y, Y)$ and $\nabla T_2(f_1, f_2, f_3)\phi_3$, $\nabla T_2(f_1, f_2, f_3)\phi_3'\colon X \to Y$. It is easily checked that in $[X, Y]$, $\nabla T_2(f_1, f_2, f_3)\phi_3 = (f_1 + f_2) + f_3$ and $\nabla T_2(f_1, f_2, f_3)\phi_3' = f_1 + (f_2 + f_3)$. By the above methods, we see that

$$\Omega\{T_2(f_1, f_2, f_3)\phi_3\} = (\Omega u_2)\Omega\{F_2(f_1, f_2, f_3)\}H_2^3(\phi_3) + \sum_{k=1}^{3} \Omega\{\iota_k p_k j_2 T_2(f_1, f_2, f_3)\phi_3\}.$$

Hence,

$$\Omega\{(f_1 + f_2) + f_3\} = \Omega\{\nabla T_2(f_1, f_2, f_3)\phi_3\} = \Omega(\nabla u_2)\Omega\{F_2(f_1, f_2, f_3)\}H_2^3(\phi_3) + \sum_{k=1}^{3} \Omega\{\nabla \iota_k p_k j_2 T_2(f_1, f_2, f_3)\phi_3\}.$$

Now

$$\begin{aligned}\nabla \iota_k p_k j_2 T_2(f_1, f_2, f_3)\phi_3 &= p_k j_2 T_2(f_1, f_2, f_3)\phi_3\\ &= p_k T_0(f_1, f_2, f_3) j_2 \phi_3.\end{aligned}$$

It is easily checked that $j_2\phi_3 \simeq \Delta\colon X \to T_0(X, X, X)$, the generalized diagonal map. Hence, $\nabla \iota_k p_k j_2 T_2(f_1, f_2, f_3)\phi_3 = f_k p_k j_2 \phi_3 \simeq f_k p_k \Delta = f_k$. Thus,

$$\Omega\{(f_1 + f_2) + f_3\} = \Omega(\nabla u_2)\Omega\{F_2(f_1, f_2, f_3)\}H_2^3(\phi_3) + \sum_{k=1}^{3} \Omega f_k.$$

Similarly,

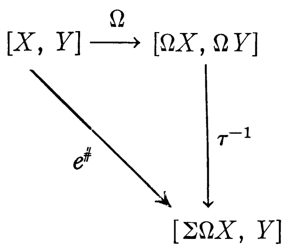
$$\Omega\{f_1 + (f_2 + f_3)\} = \Omega(\nabla u_2)\Omega\{F_2(f_1, f_2, f_3)\}H_2^3(\phi_3') + \sum_{k=1}^{3} \Omega f_k.$$

Thus, if $\Omega(\nabla u_2) \simeq *\colon \Omega F_2(Y, Y, Y) \to \Omega Y$, then

$$\Omega\{(f_1 + f_2) + f_3\} = \Omega\{f_1 + (f_2 + f_3)\}$$

in $[\Omega X, \Omega Y]$.

We now consider the function $\Omega\colon [X, Y] \to [\Omega X, \Omega Y]$ induced by the loop functor. We observe that since X is an H′-space, this function is a one-to-one into function. In fact, we have the commutative triangle



where $e: \Sigma\Omega X \to X$ is such that $\tau(e) = 1_{\Omega X}$. Now τ^{-1} is an isomorphism, Since X is an H′-space, there is a map $s: X \to \Sigma\Omega X$ such that $es \simeq 1_X$. Hence $e^{\#}$ is an injection, and hence Ω is an injection.

It follows then that if $\Omega(\nabla u_2) \simeq *: \Omega F_2(Y, Y, Y) \to \Omega Y$, then $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$, that is, the induced operation in $[X, Y]$ is associative. However, we have observed above that this condition also implies that $[X, Y]$ is abelian. For $\Omega(\nabla u_2) \simeq *$ implies that $[\Omega X, \Omega X]$ is an abelian group, and our observations above now show that $\Omega: [X, Y] \to [\Omega X, \Omega Y]$ is a monomorphism. We now see that $[X, Y]$ is a commutative, associative loop, and hence is an abelian group. Thus, we have the following theorem.

THEOREM 2. *Let (X, ϕ) be an* H′*-space and let Y be a space such that $\Omega(\nabla u_2) \simeq *: \Omega F_2(Y, Y, Y) \to \Omega Y$. Then $[X, Y]$ is an abelian group and $\Omega: [X, Y] \to [\Omega X, \Omega Y]$ is a monomorphism of abelian groups.*

The above results can be generalized to n functions. Suppose that (X, ϕ) is a homotopy-associative H′-space and let Y be a space. Suppose that $f_1, \ldots, f_n$ are maps from X to Y. Let us define $\phi_n: X \to \vee_{i=1}^{n} X$ as follows. Put $\phi_2 = \phi$. Suppose that ϕ_n has been defined, let $\phi_{n+1} = (\phi \vee 1)\phi_n$, where 1 is the identity map of $\vee_{i=1}^{n-1}$. Then we have $\nabla T_{n-1}(f_1, \ldots, f_n)\phi_n: X \to Y$, where ∇ is the generalized folding map. It is easily seen that $\nabla T_{n-1}(f_1, \ldots, f_n)\phi_n = f_1 + \ldots + f_n$ in $[X, Y]$. By the above methods, we see that we have that

$$\Omega\{T_{n-1}(f_1, \ldots, f_n)\phi_n\} = (\Omega u_{n-1})\Omega\{F_{n-1}(f_1, \ldots, f_n)\}H_{n-1}{}^{n}(\phi_n) + \sum_{k=1}^{n} \Omega\{\iota_k p_k j_{n-1} T_{n-1}(f_1, \ldots, f_n)\phi_n\}.$$

Hence,

$$\begin{aligned}\Omega(f_1 + \ldots + f_n) &= \Omega\{\nabla T_{n-1}(f_1, \ldots, f_n)\phi_n\} \\ &= \Omega(\nabla u_{n-1})\Omega\{F_{n-1}(f_1, \ldots, f_n)\}H_{n-1}{}^{n}(\phi_n) + \sum_{k=1}^{n} \Omega\{\nabla \iota_k p_k j_{n-1} T_{n-1}(f_1, \ldots, f_n)\phi_n\} \\ &= \Omega(\nabla u_{n-1})\Omega\{F_{n-1}(f_1, \ldots, f_n)\}H_{n-1}{}^{n}(\phi_n) + \sum_{k=1}^{n} \Omega f_k.\end{aligned}$$

Thus, we have the following theorem.

THEOREM 3. *Let (X, ϕ) be a homotopy associative* H′*-space and let $f_1, \ldots, f_n$ be maps from X to Y, where Y is some space. Then*

$$\Omega(f_1 + \ldots + f_n) = \Omega(\nabla u_{n-1})\Omega\{F_{n-1}(f_1, \ldots, f_n)\}H_{n-1}{}^{n}(\phi_n) + \sum_{k=1}^{n} \Omega f_k.$$

COROLLARY. *Let (X, ϕ) be a homotopy-associative* H′*-space and let $n_X \in [X, X]$ be $1_X + \ldots + 1_X$ (n summands). Then*

$$\Omega n_X = \Omega(\nabla u_{n-1})H_{n-1}{}^{n}(\phi_n) + n_{\Omega X}.$$

Remark. We have seen that if $\Omega(\nabla u_{n-1}) \simeq *: \Omega F_{n-1}(X, \ldots, X) \to \Omega X$, then $\Omega(\nabla i) \simeq *: \Omega(X \flat X) \to \Omega X$, and hence ΩX is homotopy-commutative. We observe here that if (X, ϕ) is an H′-space and ΩX is homotopy-commutative, then X is actually also an H-space. For by Stasheff's criterion (see **8**), since ΩX is homotopy-commutative, the map $e\nabla: \Sigma\Omega X \vee \Sigma\Omega X \to X$ extends to a map $f: \Sigma\Omega X \times \Sigma\Omega X \to X$. Since X is an H′-space, there is a map $s: X \to \Sigma\Omega X$ such that $es \simeq 1_X$. Now consider the following diagram, where j denotes the various natural inclusions:

$$\begin{array}{ccccc} X \vee X & \xrightarrow{s \vee s} & \Sigma\Omega X \vee \Sigma\Omega X & \xrightarrow{e\nabla} & X \\ \downarrow j & & \downarrow j & \nearrow f & \\ X \times X & \xrightarrow[s \times s]{} & \Sigma\Omega X \times \Sigma\Omega X & & \end{array}$$

We can define a multiplication $m = f(s \times s): X \times X \to X$. Then $mj = f(s \times s)j = fj(s \vee s) \simeq e\nabla(s \vee s) = es\nabla \simeq \nabla$. Hence, m provides an H-structure on X.

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