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GENERALIZED THEOREM OF HARTMAN-GROBMAN ON MEASURE CHAINS

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Abstract

We will prove the Theorem of Hartman-Grobman in a very general form. It states the topological equivalence of the flow of a nonlinear non-autonomous differential or difference equation with critical component to the flow of a partially linearized equation. The critical spectrum has not necessarily to be contained in the imaginary axis or the unit circle respectively. Further on we will employ the so-called calculus on measure chains within dynamical systems theory. Within this calculus the usual one dimensional time scales can be replaced by measure chains which are essentially closed subsets of $\mathbb{R}$. The paper can be understood without knowledge of this calculus.

So our main theorem will be valid even for equations defined on very strange time scales such as sequences of closed intervals. This is especially interesting for applications within the theory of differential-difference equations or within numerical analysis of qualitative phenomena of dynamical systems.

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1. Introduction

This paper pursues essentially two goals. Firstly we will prove the Theorem of Hartman-Grobman in a very general form. It states the topological equivalence of the flow of a nonlinear differential or difference equation with critical component and the flow of a decoupled and partially linearized equation.

Secondly we want to demonstrate the possibilities of employing the so-called calculus on measure chains, which was developed in [7, 8]. This calculus allows us to generalize ordinary differential calculus and difference calculus for functions of

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one variable. The usual one dimensional time scales $\mathbb{R}$ (differential calculus) or $\mathbb{Z}$ (difference calculus) can be replaced by measure chains. Each closed subset of $\mathbb{R}$ bears the structure of a measure chain in a natural way. So, via calculus on measure chains, one can perform differential calculus for functions defined on, say, arbitrary discrete subsets of $\mathbb{R}$ or even a Cantor set.

For the reading and understanding of this paper it is not necessary to know calculus on measure chains. In Section 2 we will give more information on that calculus.

Now we are going to give a more accurate description of the generalized Theorem of Hartman-Grobman (cf. Theorem 4.5). Let an equation of the form

$$\begin{aligned} u^\Delta &= A^-(t)u + B^-(t,u,v,w), \\ v^\Delta &= A^\circ(t)v \ + B^\circ(t,u,v,w), \qquad (\mathrm{E}_B) \\ w^\Delta &= A^+(t)w + B^+(t,u,v,w), \end{aligned}$$

with time scale $\mathbb{T}$ (for instance $\mathbb{T} = \mathbb{R}$ or $h\mathbb{Z}$) and state space $X = U \times V \times W$ be given. The operation $^\Delta$ means 'generalized differentiation' which is ordinary differentiation (time scale $\mathbb{R}$) or forward differences (time scale $h\mathbb{Z}$). We assume that this equation fulfills some conditions on the linear part of the right hand side (growth conditions, cf. (I), Section 3.2) and on the nonlinear part (small Lipschitz constant, cf. (II_B), Section 3.2) and a certain kind of quasi-boundedness (cf. (III_B), Section 4.2).

Then there are continuous functions

$$\overrightarrow{\mathscr{G}}, \overleftarrow{\mathscr{G}} : \mathbb{T} \times X \to X \quad \text{with} \quad \sup_{x\in X}\{|\overrightarrow{\mathscr{G}}(t,x) - x|\} < \infty, \ \sup_{x\in X}\{|\overleftarrow{\mathscr{G}}(t,x) - x|\} < \infty,$$

for fixed time t inverse to each other, such that for any solution $b : \mathbb{T} \to X$ of (E_B) the transferred function $\overrightarrow{\mathscr{G}}(\cdot, b(\cdot))$ is a solution of the decoupled and partially linearized equation:

$$\begin{aligned} u^\Delta &= A^-(t)u, \\ v^\Delta &= A^\circ(t)v + B^\circ(t, \overleftarrow{\mathscr{H}}(t,0,v)), \qquad (\mathrm{E}_R) \\ w^\Delta &= A^+(t)w. \end{aligned}$$

The middle component describes the flow on the so-called center integral manifold (parametrized by $\overleftarrow{\mathscr{H}}$). So the essential statement of this theorem is that solutions of (E_B) can be uniquely characterized by solutions of the much simpler equation (E_R).

In the first and third component of $\overrightarrow{\mathscr{G}}(\cdot, b(\cdot))$ one saves the information about the exponential behaviour of the given solution $b(\cdot)$ in terms of solutions of the linearized equation. (This is the content of the classical Theorem of Hartman-Grobman for the case $V = \{0\}$.) The middle component of $\overrightarrow{\mathscr{G}}(\cdot, b(\cdot))$ is related to $b(\cdot)$ by a certain asymptotic relation, it is roughly called 'asymptotic phase'. We will describe this relation more accurately later (cf. Remark after Theorem 3.9). Our proof of the generalized theorem of Hartman and Grobman essentially follows the ideas presented by Kirchgraber and Palmer in [12]. The construction of invariant manifolds and of solution transferring maps is based on the principle of asymptotic equivalence. Whereas Kirchgraber and Palmer illustrate the geometric background of their considerations, we give a more algebraically rigorous but nevertheless transparent proof. In this compact presentation there are several generalizations of the results in [12] which are listed here:

- As state spaces we admit arbitrary Banach spaces. This generalization does not cause any additional effort in the proofs.
- We cover the general case of non-autonomous equations. One always observes that the presentation of the non-autonomous case provides an adequate framework for the development of the whole theory of topological equivalence.
- Palmer and Kirchgraber postulate that the eigenvalues of the center (critical) component of the linear operator in (E_B) lie on the imaginary axis or the unit circle, respectively. In this paper this condition is replaced by condition (I) (cf. Section 3.2.). In the autonomous case this is a condition on the seperation of the spectrum of the linear operator in vertical strips (time scale $\mathbb{T} = \mathbb{R}$) or in annuli (time scale $\mathbb{T} = h\mathbb{Z}$).
- As mentioned above we are going to prove the generalized Theorem of Hartman and Grobman for arbitrary time scales. This is especially interesting for the investigation of numerical methods with respect to qualitative features such as invariant manifolds and topological equivalence. Due to the fact that calculus on measure chains treats arbitrary time scales in a unified and systematic manner it is comparatively easy to look at the effects of time scale changing on properties of dynamical systems. In a future paper we will present results of this kind, which will continue the work of Beyn, Kloeden and Lorenz (cf. [13, 3, 4]).

2. Some calculus on measure chains

Calculus on measure chains is a generalization of the usual differential calculus of one variable. Essentially the basic range ($\mathbb{R}$) of definition of the functions to be differentiated can be replaced by any closed subset of $\mathbb{R}$.

If one is interested in the cases $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$, one only has to take into account the notations listed in the table below. Using this table as a dictionary one can read Sections 3 and 4 of this paper without problems. In the *proof* of Theorem 3.1 exclusively, we will apply a few more elementary notions and results from calculus on measure chains.

The *content* of this theorem can be understood again without difficulty. For $\mathbb{T} = \mathbb{R}$

or $\mathbb{T} = h\mathbb{Z}$ it is a standard result in ODE, respectively OΔE theory. So one can disregard its proof presented here in the general framework of calculus on measure chains.

If one wants to understand the paper for more general measure chains, one has additionally to become familiar with some notions and formulas presented after the table. These additional notions and formulas are applied exclusively in the proof of Theorem 3.1 which can be considered as a preparatory result. As was mentioned before, all considerations in the main part of this paper will not make further explicit use of calculus on measure chains.

Let X be a Banach space.

Measure chain	$\mathbb{R}$	$h\mathbb{Z}$
rd-continuous function $f : \mathbb{T} \to X$	continuous function	arbitrary function
Differentiation $f^\Delta(t) \in X$	$f^\Delta(t) = f'(t) = \frac{df}{dt}(t)$	$f^\Delta(t) = \frac{f(t+h)-f(t)}{h}$
Dynamical equation	Differential equation	Difference equation
Generalized real part axis $\mathscr{R}_{\overline{\mu}} \subseteq \mathbb{R}$	$\mathscr{R}_{\overline{\mu}} = \mathbb{R}$	$\mathscr{R}_{\overline{\mu}} =]-\frac{1}{h}, \infty[$
Exponential function $e_\alpha(r, s)$	$e^{\alpha(r-s)}$	$(1+\alpha h)^{(r-s)/h}$

Definition of measure chains A *measure chain* $(\mathbb{T}, \leq, \mu)$ consists of the following data:

- $(\mathbb{T}, \leq)$ is a *chain*, that is a linearly (totally) ordered set. $\mathbb{T}$ is then equipped with the order topology generated by the open intervals.
- $(\mathbb{T}, \leq)$ is a *conditionally complete chain*, that is each nonvoid subset which is bounded above has a lowest upper bound (in $\mathbb{T}$).
- There is a continuous function $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ (the *growth calibration*) with the properties

$$\mu(r, s) > 0 \iff r > s,$$
$$\mu(r, s) + \mu(s, t) = \mu(r, t)$$

  for all $r, s, t \in \mathbb{T}$.

Each closed subset of $\mathbb{R}$ with the natural ordering $\leq$ and $\mu(r, s) = r - s$ bears the structure of a measure chain in a natural way. The most prominent examples are $\mathbb{R}$ and $\mathbb{Z}$.

Jump operator and grainyness We define the following function:

$$\sigma : \begin{cases} \mathbb{T} & \to \mathbb{T}, \\ t & \mapsto \inf\{s \in \mathbb{T} : s > t\} \end{cases} \tag{1}$$

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is called the *jump operator*. If the set in (1) is void, then we set $\sigma(t) = t = \max \mathbb{T}$. A point $t \in \mathbb{T}$ is called *right-dense* or *right-scattered*, if $\sigma(t) = t$ respectively $\sigma(t) > t$. The definitions of *left-dense* and *left-scattered* points are obvious. The width of jumps can be measured by the grainyness function

$$\mu^* : \begin{cases} \mathbb{T} & \to \mathbb{R}_0^+, \\ t & \mapsto \mu(\sigma(t), t). \end{cases}$$

For deriving the results in this paper we need the following additional axioms on measure chains:

- The growth calibration $\mu(\cdot, \tau)$, τ fixed, is neither bounded above nor bounded below.
- The grainyness μ^* is bounded on $\mathbb{T}$: $\mu^*(t) \leq \overline{\mu}$, $t \in \mathbb{T}$.

The generalized real part axis $\mathscr{R}_{\overline{\mu}}$ is then defined by $\mathscr{R}_{\overline{\mu}} := \{r \in \mathbb{R} : 1 + r\overline{\mu} > 0\}$. We are going to define the natural properties of functions defined on $\mathbb{T}$.

Rd-continuous functions A function $f : \mathbb{T} \to X$ is called *rd-continuous*, if it is continuous in right-dense points and has a left-sided limit in all left-dense-right-scattered (ldrs-)points. The grainyness is rd-continuous but not continuous in general (consider $\mathbb{T} = [0, 1] \cup \mathbb{N} \subseteq \mathbb{R}$). For $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ rd-continuity coincides with continuity.

Differentiation A function $f : \mathbb{T} \to X$ is called *differentiable at* $t \in \mathbb{T}$, if the *derivative*

$$f^{\Delta}(t) := \lim_{s \to t, \sigma(s) \neq t} \frac{f(\sigma(s)) - f(t)}{\mu(\sigma(s), t)}$$

exists; f is *differentiable*, if it is differentiable at each $t \in \mathbb{T}$. A differentiable function is continuous and therefore rd-continuous. Differentiation is linear and fulfills the generalized product rule (for simplicity: f and g are real valued)

$$(f \cdot g)^{\Delta}(t) = f(\sigma(t)) \cdot g^{\Delta}(t) + f^{\Delta}(t) \cdot g(t). \tag{2}$$

Existence of antiderivatives If $b: \mathbb{T} \to X$ is rd-continuous, then there is an antiderivative

$$B(t) = \int_{\tau}^{t} b(s)\, \Delta s \tag{3}$$

with the properties

$$B^{\Delta}(t) = b(t), \ t \in \mathbb{T}, \quad B(\tau) = 0, \tag{4}$$

by which it is uniquely determined.

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For $\mathbb{T} = \mathbb{R}$ this integral coincides with the ordinary integral for Banach valued functions, defined by Riemann sums. For $\mathbb{T} = \mathbb{Z}$ we have

$$\int_\tau^t b(s)\,\Delta s = \operatorname{sgn}(t-\tau) \cdot \sum_{i=\min\{t,\tau\}}^{\max\{t,\tau\}-1} b(i).$$

Exponential function For a rd-continuous function $\alpha : \mathbb{T} \to \mathbb{R}$, $\alpha(t)\mu^*(t)+1 > 0$, one can define the (generalized) exponential function $e_\alpha(\cdot, \tau)$, $\tau \in \mathbb{T}$, as a solution of the IVP in $\mathbb{R}$:

$$x^\Delta = \alpha(t)x, \quad x(\tau) = 1. \tag{5}$$

If $e_\alpha(\cdot, \tau)$ and $e_\beta(\cdot, \tau)$ are two exponential functions then

$$e_\alpha(t, \tau) \cdot e_\beta(t, \tau) = e_{\alpha\oplus\beta}(t, \tau), t \in \mathbb{T},$$

where the group operation $\oplus$ is defined by

$$(\alpha \oplus \beta)(t) = \alpha(t) + \beta(t) + \alpha(t)\beta(t)\mu^*(t), \tag{6}$$

$$(\alpha \ominus \beta)(t) = \frac{\alpha(t) - \beta(t)}{1 + \beta(t)\mu^*(t)}. \tag{7}$$

In the proof of Theorem 3.1 we will need some further properties of the exponential function. The corresponding results (from Hilger [8]) are quoted there.

3. Solution transferring mappings

We first introduce some abbreviating notations. For a measure chain $\mathbb{T}$ and $\tau \in \mathbb{T}$ we define the infinite intervals

$$\mathbb{T}^\tau := \{t \in \mathbb{T} : t \leq \tau\}, \ \mathbb{T}_\tau := \{t \in \mathbb{T} : t \geq \tau\}.$$

For functions $x : \mathbb{T} \to X$, X Banach space, we define seminorms, using exponential functions as weight functions:

$${}_\alpha\|x\|^\tau := \sup_{t\in\mathbb{T}^\tau} |x(t)e_\alpha(\tau, t)|, \quad {}_\beta\|x\|_\tau := \sup_{t\in\mathbb{T}_\tau} |x(t)e_\beta(\tau, t)|.$$

It is always assumed that $\alpha, \beta \in \mathscr{R}_{\overline{\mu}}$ (cf. Table in Section 2) as soon as these seminorms are used. If for two functions $x, y : \mathbb{T} \to X$ we have ${}_\alpha\|x - y\|^\tau < \infty$, then we say that x is α-asymptotic to y on $\mathbb{T}^\tau$. With respect to a given nonlinear dynamical equation

$$x^\Delta = A(t)x + B(t, x), \qquad (\mathrm{E}_B)$$

we define the so-called solution defect operator $\mathscr{L}\mathscr{D}_B$ acting on rd-continuously differentiable functions $x(\cdot)$ by:

$$\mathscr{L}\mathscr{D}_B x(\cdot) := A(\cdot)x(\cdot) + B(\cdot, x(\cdot)) - x^\Delta(\cdot). \tag{8}$$

The index B emphasizes the dependence on the nonlinear function B in (E_B). It is worth stating the following equivalence

$$b(\cdot) \quad \text{is solution of} \quad (\mathrm{E}_B) \quad \text{if and only if} \quad \mathscr{L}\mathscr{D}_B b(\cdot) \equiv 0. \tag{9}$$

3.1. Bounded solutions of inhomogeneous equations

THEOREM 3.1 (Bounded solutions of inhomogeneous equations). *We consider the inhomogeneous dynamical equation on* $\mathbb{T}_\tau$*:*

$$x^\Delta = A(t)x + B(t),$$

where $A : \mathbb{T}_\tau \to \mathscr{L}(X)$ *and* $B : \mathbb{T}_\tau \to X$ *are rd-continuous functions. By* $\Phi_A(r, s)$ *we denote the corresponding transition operator. Furthermore let* $d : \mathbb{T}_\tau \to X$ *be an rd-continuously differentiable function. For* $\beta \in \mathscr{R}_{\overline{\mu}}$ *let the following condition hold:*

$$_\beta\|\mathscr{L}\mathscr{D}_B d(\cdot)\|_\tau < \infty.$$

Let $\gamma > 0$ *be a constant such that* $\beta - \gamma \in \mathscr{R}_{\overline{\mu}}$.

(i) *(The* I*-Problem) If* $|\Phi_A(r, s)| \le K \cdot e_{\beta-\gamma}(r, s)$ *for* $\tau \le s \le r$ *then for each solution* $x(\cdot)$ *of* (E_B) *we have*

$$_\beta\|x(\cdot) - d(\cdot)\|_\tau \le \frac{K}{\gamma} \cdot {}_\beta\|\mathscr{L}\mathscr{D}_B d(\cdot)\|_\tau + K \cdot |x(\tau) - d(\tau)|.$$

(ii) *(The* B*-Problem) If* $|\Phi_A(r, s)| \le K \cdot e_{\beta+\gamma}(r, s)$ *for* $\tau \le r \le s$ *then there is exactly one solution* $x(\cdot)$ *of* (E_B) *with* $_\beta\|x(\cdot) - d(\cdot)\|_\tau < \infty$*. It satisfies*

$$_\beta\|x(\cdot) - d(\cdot)\|_\tau \le \frac{K}{\gamma} \cdot {}_\beta\|\mathscr{L}\mathscr{D}_B d(\cdot)\|_\tau.$$

Let P *be a metric (parameter) space,* $B : \mathbb{T}_\tau \times X \times P \to X$ *and* $d : \mathbb{T}_\tau \times P \to X$ *depend on* $p \in P$*, such that* $\mathscr{L}\mathscr{D}_B d : \mathbb{T}_\tau \times P \to X$ *is rd-continuous (which essentially means that* $\mathscr{L}\mathscr{D}_B d(\cdot, \cdot)$ *is jointly rd-continuous in* $\mathbb{T}_\tau$ *and continuous in* P*). Then the following statements hold:*

(iii) *The solution function* $(t; \tau, \eta, p) \mapsto x(t; \tau, \eta, p)$ *for the* I*-problem*

$$x^\Delta = A(t)x + B(t, p), \qquad x(\tau) = \eta$$

is continuous.

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(iv) *The solution function* $(t, p) \mapsto x(t, p)$ *for the* B*-problem*

$$x^\Delta = A(t)x + B(t, p), \qquad {}_\beta\|x(\cdot) - d(\cdot, p)\|_\tau < \infty$$

is continuous.

REMARKS. 1. As is outlined in Section 1, we will use some calculus on measure chains exclusively in the proof of this theorem. For the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$ one can read the proof easily by replacing the integrals, exponential functions and $\oplus$ and $\ominus$-signs according to the definitions in Section 2.

2. There is also a version of this theorem for nonlinear equations $x^\Delta = A(t)x + B(t, x)$ with $B(t, x)$ being small in a certain sense. We will not need this generalization subsequently.

3. An accurate analysis of the proof can reveal the fact that the constant β can be replaced by an rd-continuous function $\beta : \mathbb{T} \to \mathbb{R}$, subject to the condition

$$1 + (\beta(t) - \gamma)\mu^*(t) \geq \text{const.} > 0.$$

The corresponding exponential function $e_\beta(t, \tau)$ is defined in (5). In this case one can drop the postulate of boundedness of the grainyness. One only has to replace the condition $\beta - \gamma \in \mathscr{R}_{\overline{\mu}}$ by the condition that $1 + (\beta - \gamma)\mu^*(t)$ is bounded away from zero.

PROOF. (a) We first consider the case $d(\cdot) \equiv 0$; the condition on $d(\cdot)$ then reads as

$$Q := {}_\beta\|\mathscr{L}\mathscr{D}_B d(\cdot)\|_\tau = {}_\beta\|B(\cdot)\|_\tau < \infty.$$

(b) In order to prove (i) we define the function $\xi : \mathbb{T}_\tau \to \mathbb{R}$ by $\xi := \beta \ominus (\beta - \gamma)$ and calculate (cf.(7)):

$$\begin{aligned} \xi(t) = [\beta \ominus (\beta - \gamma)](t) &= \frac{\beta - (\beta - \gamma)}{1 + (\beta - \gamma)\mu^*(t)} = \frac{\gamma}{1 + (\beta - \gamma)\mu^*(t)} \\ &\geq \frac{\gamma}{1 + (\beta - \gamma)\overline{\mu}} = \text{const.} > 0 \end{aligned}$$

for all $t \in \mathbb{T}_t$. Hence ξ is positively bounded away from zero and thus (by Hilger [8, Theorem 7.4(i)]) $e_\xi(\tau, t) \leq 1$ for $t \in \mathbb{T}_\tau$. By the variation of constants formula (Hilger [8, Theorem 6.4(ii)]) we have

$$x(t) = \Phi_A(t, \tau)x(\tau) + \int_\tau^t \Phi_A(t, \sigma(s))B(s)\,\Delta s.$$

Using the estimates given in the theorem we obtain

$$\begin{aligned}|x(t)| &\leq |\Phi_A(t,\tau)||x(\tau)| + \int_\tau^t |\Phi_A(t,\sigma(s))||B(s)|\,\Delta s \\ &\leq K\cdot e_{\beta-\gamma}(t,\tau)|x(\tau)| + e_{\beta-\gamma}(t,\tau)\cdot\int_\tau^t K\cdot e_{\beta-\gamma}(\tau,\sigma(s))\cdot Q e_\beta(s,\tau)\,\Delta s \\ &= K\cdot e_{\beta-\gamma}(t,\tau)|x(\tau)| \\ &\quad + e_{\beta-\gamma}(t,\tau)\cdot\frac{KQ}{\gamma}\cdot\big[e_{\beta-\gamma}(\tau,t)\cdot e_\beta(t,\tau) - e_{\beta-\gamma}(\tau,\tau)\cdot e_\beta(\tau,\tau)\big].\end{aligned}$$

In the last equation we used a formula given in Hilger [8, Theorem 6.2(vi)]. (This is integration of the exponential function in case $\mathbb{T}=\mathbb{R}$ and geometric summation in case $\mathbb{T}=h\mathbb{Z}$.) Multiplication of both sides with $e_\beta(\tau,t)$ yields

$$|x(t)|e_\beta(\tau,t) \leq K\cdot e_\xi(\tau,t)|x(\tau)| + \frac{KQ}{\gamma}\cdot\big[1-e_\xi(\tau,t)\big] \leq K\cdot|x(\tau)| + \frac{KQ}{\gamma}.$$

This is the desired estimate for (i) in the case that $d\equiv 0$.

(c) We now prove (ii). Here we define the function ξ by $\xi := \beta\ominus(\beta+\gamma)$ and calculate (as above)

$$\xi(t) = \frac{-\gamma}{1+(\beta+\gamma)\mu^*(t)} \leq \frac{-\gamma}{\max\{1, 1+(\beta+\gamma)\overline{\mu}\}} = \text{const.} < 0$$

for all $t\in\mathbb{T}_\tau$. So ξ is negatively bounded away from zero and thus (by Hilger [7, Theorem 9.1]) $1\geq e_\xi(t,\tau)\to 0$, as $t\to\infty$.

We show the uniqueness: The difference $y = x_1 - x_2$ of two different solutions of the B-problem is a solution of the corresponding homogeneous B-problem

$$x^\Delta = A(t)x,\ {}_\beta\|x(\cdot)\|_\tau < \infty.$$

So we have:

$$\begin{aligned}|y(\tau)| &= |\Phi_A(\tau,t)y(t)| \leq K e_{\beta+\gamma}(\tau,t)|y(t)| \leq K e_{\beta+\gamma}(\tau,t)\cdot{}_\beta\|y(\cdot)\|_\tau\cdot e_\beta(t,\tau) \\ &= K e_\xi(t,\tau)\cdot{}_\beta\|y(\cdot)\|_\tau \overset{t\to\infty}{\longrightarrow} 0,\end{aligned}$$

and $y(\tau)$ and hence $y(\cdot)$ vanishes identically.

We define the solution $x(\cdot)$ formally as a solution of the IVP

$$(\mathrm{E}_B),\qquad x(\infty)=0$$

by the variation of constants formula of Hilger [8, Theorem 6.4(ii)]:

$$x^*(t) = \int_\infty^t \Phi_A(t,\sigma(s))B(s)\,\Delta s = \lim_{n\to\infty}\underbrace{\int_{t_n}^t \Phi_A(t,\sigma(s))B(s)\,\Delta s}_{=:x_n(t)}. \tag{10}$$

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Here $(t_n)_{n\in\mathbb{N}}$ is an arbitrary increasing sequence in $\mathbb{T}_\tau$ with $\lim_{n\to\infty} t_n = \infty$. By means of preassumptions in the theorem and by the formula of Hilger [8, Theorem 6.2(vi)] we estimate for $p, q, r \in \mathbb{T}_\tau$ with $p \le q \le r$:

$$\begin{aligned}\left|\int_r^q \Phi_A(p,\sigma(s))B(s)e_\beta(\tau,p)\,\Delta s\right| &\le \int_q^r |\Phi_A(p,\sigma(s))||B(s)|e_\beta(\tau,s)e_\beta(s,p)\,\Delta s\\ &\le \int_q^r Ke_\delta(p,\sigma(s))\cdot Qe_\beta(s,p)\,\Delta s\\ &= \frac{KQ}{\gamma}\cdot\left[e_\xi(q,p)-e_\xi(r,p)\right].\end{aligned}$$

We use the last inequality in order to estimate for $m \le n$ and fixed $t \in \mathbb{T}_\tau$:

$$\begin{aligned}\left|x_n(t)-x_m(t)\right| &= \left|\int_{t_n}^{t_m} \Phi_A(t,\sigma(s))B(s)\,\Delta s\right|\\ &\le |\Phi_A(t,\tau)|e_\beta(t,\tau)\cdot\frac{KQ}{\gamma}\cdot[e_\xi(t_m,\tau)-e_\xi(t_n,\tau)].\end{aligned}$$

The bracket tends to zero for $n, m \to \infty$. Therefore $\left(x_n(t)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and thus convergent. It is obvious that this limit does not depend on the sequence $(t_n)_{n\in\mathbb{N}}$ chosen in (10).

For $t, t_n \in \mathbb{T}_\tau$ with $t \le t_n$ we have by (c):

$$\begin{aligned}|x_n(t)e_\beta(\tau,t)| &= \left|\int_{t_n}^{t} \Phi_A(t,\sigma(s))B(s)e_\beta(\tau,t)\,\Delta s\right|\\ &\le \frac{KQ}{\gamma}\cdot\left[e_\xi(t,t)-e_\xi(t_n,t)\right] \le \frac{KQ}{\gamma}.\end{aligned}$$

The limit process $n \to \infty$ yields the estimate of the theorem.

To see x is a solution of the dynamical equation: From the representation

$$x(t) = \Phi_A(t,\tau)\cdot\left[\int_\infty^\tau \Phi_A(\tau,\sigma(s))B(s)\,\Delta s + \int_\tau^t \Phi_A(\tau,\sigma(s))B(s)\,\Delta s\right]$$

one can see the differentiability. By the help of the product rule (2) one easily calculates:

$$x^\Delta(t) = \Phi_A(\tau,\sigma(t))B(t) + A(t)\Phi(t,\tau)\cdot\int_\infty^t \Phi_A(t,\sigma(s))B(s)\,\Delta s = B(t)+A(t)x(t).$$

(d) Finally we consider the case $d \not\equiv 0$ and discuss the parameter dependence. By a transformation $\mathcal{T}$ in the space of rd-continuous functions $\mathbb{T}_\tau \to X$, defined by

$$\mathcal{T}x(\cdot) := x(\cdot) - d(\cdot, p)$$

the inhomogeneous dynamical equation (E_B) becomes

$$y^\Delta = A(t)y + \underbrace{B(t,p) + A(t)d(t,p) - d^\Delta(t,p)}_{\mathscr{L}\mathscr{D}_B d(t,p)},$$

where the inhomogeneous part is replaced by the solution defect of $d(\cdot, p)$. The postulate ${}_\beta\|x(\cdot) - d(\cdot,p)\|_\tau < \infty$ has to be replaced by the postulate ${}_\beta\|y(\cdot)\|_\tau < \infty$. So one can reduce this more general problem to the one already treated.

The parameter $p \in P$ is now contained in the new inhomogeneous part $\mathscr{L}\mathscr{D}_B d(\cdot, p)$. Thus one has to reconsider the proof of (ii) with respect to an inhomogeneous part depending continuously on a parameter p. It will turn out that the continuous dependence goes through to the unique solution $x(\cdot; t, \eta, p)$ of the I-problem, respectively to the solution $x(\cdot, p)$ of the B-problem. One has to apply elementary lemmas on continuous parameter dependence of integrals and limits, which are valid for measure chains as well as for $\mathbb{R}$- and $h\mathbb{Z}$-calculus. An important fact hereby is that intervals of measure chains are compact if they are closed and bounded (cf. Hilger [8, Theorem 1.4]).

3.2. The BIB-Problem Now let the state space X have a decomposition

$$X = U \times V \times W = X^- \times X^\circ \times X^+, \quad X^{\overline{o}} := X^- \times X^\circ, \quad X^{\overset{o}{+}} := X^\circ \times X^+.$$

From now on we consider the dynamical equation:

$$x^\Delta = A(t)x + B(t,x), \qquad (\mathrm{E_B})$$

with components:

$$\begin{aligned} u^\Delta &= A^-(t)u + B^-(t,u,v,w), \\ v^\Delta &= A^\circ(t)v + B^\circ(t,u,v,w), \qquad (\mathrm{E_B}) \\ w^\Delta &= A^+(t)w + B^+(t,u,v,w). \end{aligned}$$

For the nonlinearity we define the optimal Lipschitz constant L_B by

$$L_B := \sup\left\{\frac{|B(t,x_1) - B(t,x_2)|}{|x_1 - x_2|} : x_1, x_2 \in X,\ x_1 \neq x_2,\ t \in \mathbb{T}\right\} \in [0, \infty].$$

We assume that each IVP associated with (E_B) has exactly one solution on the whole time axis $\mathbb{T}$. This solution depends continuously on a parameter $p \in P$, if the right hand side of the equation is rd-continuous (see Theorem 3.1(iii)) on $\mathbb{T} \times X \times P$. In case $\mathbb{T} = \mathbb{R}$ this postulate is met if one requires $L_B < \infty$ (as is done in (II_B) below). For $\mathbb{T} = h\mathbb{Z}$ one has to pay attention to the fact that there might be none or infinitely many solutions in backward time direction. One can exclude this pathology by the postulate

$$x \mapsto x + h \cdot [A(t)x + B(t,x)] \quad \text{is an invertible mapping} \quad X \to X.$$

Using constants $K \geq 1$, $\gamma > 0$ and $\alpha, \beta \in \mathscr{R}_{\overline{\mu}}$, such that $\alpha \leq \beta$ and $\alpha - \gamma \in \mathscr{R}_{\overline{\mu}}$ we are going to formulate the following conditions for (E_B):

$$\text{(I)} \qquad \begin{aligned} |\Phi_-(r,s)| &\leq K \cdot e_{\alpha-\gamma}(r,s) \quad \text{for} \quad s \leq r, \\ |\Phi_{\overset{o}{+}}(r,s)| &\leq K \cdot e_{\alpha+\gamma}(r,s) \quad \text{for} \quad r \leq s, \\ |\Phi_{\overline{o}}(r,s)| &\leq K \cdot e_{\beta-\gamma}(r,s) \quad \text{for} \quad s \leq r, \\ |\Phi_+(r,s)| &\leq K \cdot e_{\beta+\gamma}(r,s) \quad \text{for} \quad r \leq s, \end{aligned}$$

where the function $\Phi_-(\cdot, s)$ (etc.) is the evolution operator for the corresponding linear equation, that is the solution of the IVP

$$u^\Delta = A^-(t)u, \; u(s) = \mathrm{id}_U$$

with state space $\mathscr{L}(U, U)$. The second condition is

$$(\mathrm{II}_B) \qquad L_B < \frac{\gamma}{K(K+2)}.$$

REMARK. Because of (II_B) the matrix

$$(11) \qquad \mathscr{M}_B := \frac{K}{M_3} \cdot \begin{pmatrix} M_1 M_2 & M M_2 & K M^2 & K M M_1 \\ M(M_2 + K^2) & M_1 M_2 & K M M_1 & K M_1^2 \\ K M_1^2 & K M M_1 & M_1 M_2 & M(M_2 + K^2) \\ K M M_1 & K M^2 & M M_2 & M_1 M_2 \end{pmatrix},$$

which we will need in the theorem below, with

$$\begin{aligned} M &:= M_B = K L_B / \gamma \geq 0, \\ M_1 &:= 1 - M > 0, \\ M_2 &:= (1-M)^2 - M^2 = 1 - 2M > 0, \\ M_3 &:= [(1-M)^2 - M^2]^2 - K^2 M^2 = (1 - 2M + KM)(1 - (K+2)M) > 0, \end{aligned}$$

is not negative and has positive diagonal elements.

THEOREM 3.2 (The BIB-Problem). *For* (E_B) *let the conditions* (I) *and* (II_B) *be fulfilled. Let* $\tau \in \mathbb{T}$ *be arbitrary and fixed.*

(i) *If one has given* $\eta \in V$ *and functions* $c, d : \mathbb{T} \to X$ *with*

$$_\alpha\|\mathscr{L}\mathscr{D}_B c(\cdot)\|^\tau < \infty, \; _\beta\|\mathscr{L}\mathscr{D}_B d(\cdot)\|_\tau < \infty,$$

then the boundedness–initial value–boundedness-problem (BIB-*problem*)

$$(\mathrm{E}_B), \qquad _\alpha\|u(\cdot) - c^-(\cdot)\|^\tau < \infty, \quad v(\tau) = \eta, \quad _\beta\|w(\cdot) - d^+(\cdot)\|_\tau < \infty$$

has exactly one solution $b(\cdot) = b(\cdot; \tau, \eta|c, d)$. *It satisfies:*

$$\begin{pmatrix} _\alpha\|b^-(\cdot) - c^-(\cdot)\|^\tau \\ _\alpha\|b^{\overset{o}{+}}(\cdot) - c^{\overset{o}{+}}(\cdot)\|^\tau \\ _\beta\|b^{\overline{o}}(\cdot) - d^{\overline{o}}(\cdot)\|_\tau \\ _\beta\|b^+(\cdot) - d^+(\cdot)\|_\tau \end{pmatrix} \le \mathscr{M}_B \cdot \left[\begin{pmatrix} 0 \\ |\eta - c^\circ(\tau)| \\ |\eta - d^\circ(\tau)| \\ 0 \end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix} _\alpha\|\mathscr{L}\mathscr{D}_B^- c(\cdot)\|^\tau \\ _\alpha\|\mathscr{L}\mathscr{D}_B^{\overset{o}{+}} c(\cdot)\|^\tau \\ _\beta\|\mathscr{L}\mathscr{D}_B^{\overline{o}} d(\cdot)\|_\tau \\ _\beta\|\mathscr{L}\mathscr{D}_B^+ d(\cdot)\|_\tau \end{pmatrix} \right] < \infty.$$

(ii) *Let* P_1, P_2 *be metric (parameter) spaces and* $c : \mathbb{T} \times P_1 \to X, d : \mathbb{T} \times P_2 \to X$ *depend on* $p_1 \in P_1$ *respectively* $p_2 \in P_2$, *such that the functions* $\mathscr{L}\mathscr{D}_B c : \mathbb{T} \times P_1 \to X$ *and* $\mathscr{L}\mathscr{D}_B d : \mathbb{T} \times P_2 \to X$ *are rd-continuous (cf. Theorem* 3.1(iii),(iv)). *Then the mapping* $(t; \tau, \eta, p_1, p_2) \mapsto b(t; \tau, \eta|c(\cdot, p_1), d(\cdot, p_2))$ *is continuous.*

In particular it is globally Lipschitz with respect to $\eta \in V$:

$$\begin{pmatrix} _\alpha\|b^-(\cdot; \tau, \eta|c, d) - b^-(\cdot; \tau, \underline{\eta}|c, d)\|^\tau \\ _\alpha\|b^{\overset{o}{+}}(\cdot; \tau, \eta|c, d) - b^{\overset{o}{+}}(\cdot; \tau, \underline{\eta}|c, d)\|^\tau \\ _\beta\|b^{\overline{o}}(\cdot; \tau, \eta|c, d) - b^{\overline{o}}(\cdot; \tau, \underline{\eta}|c, d)\|_\tau \\ _\beta\|b^+(\cdot; \tau, \eta|c, d) - b^+(\cdot; \tau, \underline{\eta}|c, d)\|_\tau \end{pmatrix} \le \mathscr{M}_B \cdot \begin{pmatrix} 0 \\ |\eta - \underline{\eta}| \\ |\eta - \underline{\eta}| \\ 0 \end{pmatrix}.$$

PROOF. In steps (a) to (e) we show item (i) of the theorem.

(a) We define the spaces

$$S_1 := \{u : \mathbb{T}^\tau \to X^-, \text{continuous: } _\alpha\|u - c^-\|^\tau < \infty\},$$
$$S_2 := \{z : \mathbb{T}^\tau \to X^{\overset{o}{+}}, \text{continuous: } _\alpha\|z - c^{\overset{o}{+}}\|^\tau < \infty\},$$
$$S_3 := \{y : \mathbb{T}_\tau \to X^{\overline{o}}, \text{continuous: } _\beta\|y - d^{\overline{o}}\|_\tau < \infty\},$$
$$S_4 := \{w : \mathbb{T}_\tau \to X^+, \text{continuous: } _\beta\|w - d^+\|_\tau < \infty\}.$$

On the product space $S := S_1 \times S_2 \times S_3 \times S_4$ we define the iteration mapping

$$T : \begin{cases} S & \to S \\ (u, z, y, w) & \mapsto (U, Z, Y, W) \end{cases}$$

as follows:

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- U is the unique solution of the inhomogeneous B-problem on $\mathbb{T}^\tau$:

$$u^\Delta = A^-(t)u + B^-(t, u(t), z(t)), \ {}_\alpha\|u(\cdot) - c^-(\cdot)\|^\tau < \infty.$$

- Z is the unique solution of the inhomogeneous I-problem on $\mathbb{T}^\tau$:

$$z^\Delta = A^{\overset{o}{+}}(t)z + B^{\overset{o}{+}}(t, u(t), z(t)), \ z(\tau) = (\eta, w(\tau)).$$

- Y is the unique solution of the inhomogeneous I-problem on $\mathbb{T}_\tau$:

$$y^\Delta = A^{\overline{o}}(t)y + B^{\overline{o}}(t, y(t), w(t)), \ y(\tau) = (u(\tau), \eta).$$

- W is the unique solution of the inhomogeneous B-problem on $\mathbb{T}_\tau$:

$$w^\Delta = A^+(t)w + B^+(t, y(t), w(t)), \ {}_\beta\|w(\cdot) - d^+(\cdot)\|_\tau < \infty.$$

(b) One easily proves that the conditions of Theorem 3.1 (i) and (ii) and its dual version (for $\mathbb{T}^\tau$) are fulfilled and T is thus well defined. In particular, for the respective solution defect operators we have:

$$\begin{aligned} {}_\alpha\|c^\Delta(\cdot) - A(\cdot)c(\cdot) - B(\cdot, u(\cdot), z(\cdot))\|^\tau &\le L_B \cdot {}_\alpha\|(u(\cdot), z(\cdot)) - c(\cdot)\|^\tau \\ &\quad + {}_\alpha\|\mathscr{L}\mathscr{D}_B c(\cdot)\|^\tau < \infty, \\ {}_\beta\|d^\Delta(\cdot) - A(\cdot)d(\cdot) - B(\cdot, y(\cdot), w(\cdot))\|_\tau &\le L_B \cdot {}_\beta\|(y(\cdot), w(\cdot)) - d(\cdot)\|_\tau \\ &\quad + {}_\beta\|\mathscr{L}\mathscr{D}_B d(\cdot)\|_\tau < \infty. \end{aligned}$$

From Theorem 3.1 we conclude:

$$\begin{aligned} \begin{pmatrix} {}_\alpha\|U(\cdot) - c^-(\cdot)\|^\tau \\ {}_\alpha\|Z(\cdot) - c^{\overset{o}{+}}(\cdot)\|^\tau \\ {}_\beta\|Y(\cdot) - d^{\overline{o}}(\cdot)\|_\tau \\ {}_\beta\|W(\cdot) - d^+(\cdot)\|_\tau \end{pmatrix} &\le \frac{K}{\gamma} \cdot \left[\begin{pmatrix} L_B & L_B & 0 & 0 \\ L_B & L_B & 0 & 0 \\ 0 & 0 & L_B & L_B \\ 0 & 0 & L_B & L_B \end{pmatrix} \begin{pmatrix} {}_\alpha\|u(\cdot) - c^-(\cdot)\|^\tau \\ {}_\alpha\|z(\cdot) - c^{\overset{o}{+}}(\cdot)\|^\tau \\ {}_\beta\|y(\cdot) - d^{\overline{o}}(\cdot)\|_\tau \\ {}_\beta\|w(\cdot) - d^+(\cdot)\|_\tau \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} {}_\alpha\|\mathscr{L}\mathscr{D}_B^- c(\cdot)\|^\tau \\ {}_\alpha\|\mathscr{L}\mathscr{D}_B^{\overset{o}{+}} c(\cdot)\|^\tau \\ {}_\beta\|\mathscr{L}\mathscr{D}_B^{\overline{o}} d(\cdot)\|_\tau \\ {}_\beta\|\mathscr{L}\mathscr{D}_B^+ d(\cdot)\|_\tau \end{pmatrix} \right] + K \cdot \begin{pmatrix} 0 \\ |(\eta, w(\tau))^T - d^{\overset{o}{+}}(\tau)| \\ |(u(\tau), \eta)^T - c^{\overline{o}}(\tau)| \\ 0 \end{pmatrix} \\ &\le \begin{pmatrix} M_B & M_B & 0 & 0 \\ M_B & M_B & 0 & K \\ K & 0 & M_B & M_B \\ 0 & 0 & M_B & M_B \end{pmatrix} \begin{pmatrix} {}_\alpha\|u(\cdot) - c^-(\cdot)\|^\tau \\ {}_\alpha\|z(\cdot) - c^{\overset{o}{+}}(\cdot)\|^\tau \\ {}_\beta\|y(\cdot) - d^{\overline{o}}(\cdot)\|_\tau \\ {}_\beta\|w(\cdot) - d^+(\cdot)\|_\tau \end{pmatrix} \\ &\quad + K \left[\begin{pmatrix} 0 \\ |\eta - c^\circ(\tau)| \\ |\eta - d^\circ(\tau)| \\ 0 \end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix} {}_\alpha\|\mathscr{L}\mathscr{D}_B^- c(\cdot)\|^\tau \\ {}_\alpha\|\mathscr{L}\mathscr{D}_B^{\overset{o}{+}} c(\cdot)\|^\tau \\ {}_\beta\|\mathscr{L}\mathscr{D}_B^{\overline{o}} d(\cdot)\|_\tau \\ {}_\beta\|\mathscr{L}\mathscr{D}_B^+ d(\cdot)\|_\tau \end{pmatrix} \right]. \end{aligned} \tag{12}$$

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(c) One sees that $(u^*, z^*, y^*, w^*) \in S$ is a fixed point of T, if and only if the mapping $b: \mathbb{T} \to X$ defined by

$$b(t) := \begin{cases} \begin{pmatrix} u^*(t) \\ z^*(t) \end{pmatrix}, & \text{if } t \in \mathbb{T}^\tau, \\ \begin{pmatrix} y^*(t) \\ w^*(t) \end{pmatrix}, & \text{if } t \in \mathbb{T}_\tau, \end{cases}$$

is a solution of the BIB-problem. Observe that due to the initial values in the definition of T the functions $(u^*, z^*)^T : \mathbb{T}^\tau \to X$ and $(y^*, w^*)^T : \mathbb{T}_\tau \to X$ fit together at $\tau \in \mathbb{T}$.

(d) We are going to prove that T has exactly one fixed point in S. To this end for $i = 1, 2$ choose $(u_i, z_i, y_i, w_i) \in S$ and define $(U_i, Z_i, Y_i, W_i) := T(u_i, z_i, y_i, w_i)$. Then we can state:

- By Theorem 3.1(ii) $U_1 - U_2$ is the unique solution on $\mathbb{T}^\tau$ of the B-problem:

$$u^\Delta = A^-(t)u + B^-(t, u_1(t), z_1(t)) - B^-(t, u_2(t), z_2(t)), \; {}_\alpha\|u(\cdot)\|^\tau < \infty.$$

- $Z_1 - Z_2$ is the solution on $\mathbb{T}^\tau$ of the I-problem:

$$z^\Delta = A^{\overset{o}{+}}(t)z + B^{\overset{o}{+}}(t, u_1(t), z_1(t)) - B^{\overset{o}{+}}(t, u_2(t), z_2(t)), \; z(\tau) = (0, w_1(\tau) - w_2(\tau)).$$

- $Y_1 - Y_2$ is the solution on $\mathbb{T}_\tau$ of the I-problem:

$$y^\Delta = A^{\overline{o}}(t)y + B^{\overline{o}}(t, y_1(t), w_1(t)) - B^{\overline{o}}(t, y_2(t), w_2(t)), \; y(\tau) = (u_1(\tau) - u_2(\tau), 0).$$

- By Theorem 3.1(ii) $W_1 - W_2$ is the unique solution on $\mathbb{T}_\tau$ of the B-problem:

$$w^\Delta = A^+(t)w + B^+(t, y_1(t), w_1(t)) - B^+(t, y_2(t), w_2(t)), \; {}_\beta\|w(\cdot)\|_\tau < \infty.$$

A repetition of the estimate in (b) for this similar situation yields:

(13)

$$T\begin{pmatrix} {}_\alpha\|u_1 - u_2\|^\tau \\ {}_\alpha\|z_1 - z_2\|^\tau \\ {}_\beta\|y_1 - y_2\|_\tau \\ {}_\beta\|w_1 - w_2\|_\tau \end{pmatrix} = \begin{pmatrix} {}_\alpha\|U_1 - U_2\|^\tau \\ {}_\alpha\|Z_1 - Z_2\|^\tau \\ {}_\beta\|Y_1 - Y_2\|_\tau \\ {}_\beta\|W_1 - W_2\|_\tau \end{pmatrix} \le \underbrace{\begin{pmatrix} M_B & M_B & 0 & 0 \\ M_B & M_B & 0 & K \\ K & 0 & M_B & M_B \\ 0 & 0 & M_B & M_B \end{pmatrix}}_{\mathscr{K}_B} \begin{pmatrix} {}_\alpha\|u_1 - u_2\|^\tau \\ {}_\alpha\|z_1 - z_2\|^\tau \\ {}_\beta\|y_1 - y_2\|_\tau \\ {}_\beta\|w_1 - w_2\|_\tau \end{pmatrix}.$$

It can be shown that $\mathscr{M}_B = K \cdot (\mathscr{I} - \mathscr{K}_B)^{-1}$ (cf. (11)). The vector $(q_1, q_2, q_3, q_4)^T := \mathscr{M}_B \cdot \mathbf{I}$ with $\mathbf{I} := (1, 1, 1, 1)^T$ is positive, so the matrix $\mathscr{Q} := \operatorname{diag}(q_1, q_2, q_3, q_4)$ is

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invertible. By the metric

$$d_S\left(\begin{pmatrix}u_1\\z_1\\y_1\\w_1\end{pmatrix}\begin{pmatrix}u_2\\z_2\\y_2\\w_2\end{pmatrix}\right) := \left|\mathcal{Q}^{-1}\cdot\begin{pmatrix}{}_\alpha\|u_1-u_2\|^\tau\\ {}_\alpha\|z_1-z_2\|^\tau\\ {}_\beta\|y_1-y_2\|_\tau\\ {}_\beta\|w_1-w_2\|_\tau\end{pmatrix}\right|_\infty$$

($|\cdot|_\infty$: sup norm) S becomes a complete metric space. Now from inequality (13) we derive:

$$d_S\left(\begin{pmatrix}U_1\\Z_1\\Y_1\\W_1\end{pmatrix},\begin{pmatrix}U_2\\Z_2\\Y_2\\W_2\end{pmatrix}\right) \le |\mathcal{Q}^{-1}\cdot\mathcal{K}_B\cdot\mathcal{Q}|_\infty\cdot d_S\left(\begin{pmatrix}u_1\\z_1\\y_1\\w_1\end{pmatrix},\begin{pmatrix}u_2\\z_2\\y_2\\w_2\end{pmatrix}\right).$$

The inequality $|\mathcal{Q}^{-1}\cdot\mathcal{K}_B\cdot\mathcal{Q}|_\infty < 1$ remains to be shown. This follows from the estimate (to be read componentwise):

$$\begin{aligned}(0,0,0,0)^T &\le \mathcal{Q}^{-1}\mathcal{K}_B\mathcal{Q}\cdot\mathbb{1} = \mathcal{Q}^{-1}(\mathcal{I}-K\mathcal{M}_B^{-1})\mathcal{Q}\cdot\mathbb{1} = \mathbb{1}-K\mathcal{Q}^{-1}\mathcal{M}_B^{-1}\cdot\underline{q}\\ &= \mathbb{1}-K\mathcal{Q}^{-1}\cdot\mathbb{1} < \mathbb{1}.\end{aligned}$$

The line sums of $\mathcal{Q}^{-1}\cdot\mathcal{K}_B\cdot\mathcal{Q}$ are all in the interval $[0,1[$, hence $|\mathcal{Q}^{-1}\cdot\mathcal{K}_B\cdot\mathcal{Q}|_\infty < 1$. According to the Banach fixed point theorem the mapping $T: S\to S$ has exactly one fixed point. We have thus shown the existence and uniqueness of the solution of the BIB-problem.

(e) On both sides of inequality (12) we insert the fixed point (u^*, z^*, y^*, w^*) of the mapping T. A resolution of this inequality with respect to the fixed point vector — this essentially means a multiplication with $(\mathcal{I}-\mathcal{K}_B)^{-1} = K^{-1}\mathcal{M}_B$ — yields inequality in (i).

(f) In order to show the continuity of the fixed point we have to define the iteration mapping T on a product space of functions which depend on parameters,

$$S_1 := \{u : \mathbb{T}^\tau\times P_1\times P_2\to X^-, \text{continuous: } \sup_{p_1,p_2}{}_\alpha\|u-c^-\|^\tau < \infty\},\quad S_2 := \ldots$$

Additionally one has to take into account that $T: S\to S$ depends continuously on parameters, such as τ and η. One can perform the whole proof given before for this modified situation as well. An important fact here is that the image $(U, Z, Y, W) = T(u, z, y, w)$ is again contained in the space S of vectors of continuously parameter dependent functions. This property is assured by the continuous dependence on the parameter $p = (p_1, p_2)$ of the solutions of the I-problem and the B-problem (cf. Theorem 3.1(iii),(iv)). A more thorough discussion of these technical details can be found in Hilger [7].

(g) We have to show the Lipschitz property. Abbreviating, we set $b(\cdot) := b(\cdot;\tau,\eta|c,d)$ and $\underline{b}(\cdot) := \underline{b}(\cdot;\tau,\underline{\eta}|c,d)$. First it is clear that $b(\cdot) - \underline{b}(\cdot)$ solves the nonlinear BIB-problem

$$x^\Delta = A(t)x + \underbrace{B(t, x + \underline{b}(t)) - B(t, \underline{b}(t))}_{S(t,x)}, \tag{14}$$

$$_\alpha\|u\|^\tau < \infty, \qquad v(\tau) = \eta - \underline{\eta}, \qquad _\beta\|w\|_\tau < \infty,$$

with nonlinear part $S(t,x)$. For the corresponding solution defect operators of the zero function it can be seen that $\mathscr{L}\mathscr{D}_S 0 = 0$. So for this BIB-problem the condition in (i) of this theorem is fulfilled. The estimate given in (i) applied to this new situation yields the estimate in (ii). Hence the theorem is proven.

REMARK. Without emphasizing this fact in step (d) we used the special case $n = 4$ of the following generalization of the Banach fixed point theorem:

Let $T: S \to S$ ($S = S_1 \times \cdots \times S_n$, S_i complete metric spaces) be a generalized contraction, that is

$$\begin{pmatrix} d_1((Tx)_1, (Ty)_1) \\ \vdots \\ d_n((Tx)_n, (Ty)_n) \end{pmatrix} \le \mathscr{K} \cdot \begin{pmatrix} d_1(x_1, y_1) \\ \vdots \\ d_n(x_n, y_n) \end{pmatrix}$$

with a matrix $\mathscr{K}$, subject to the condition that all principal minors of the complementary matrix $\mathscr{I} - \mathscr{K}$ are positive. Then T has exactly one fixed point. The proof can be found in Hilger [7]. The positivity condition just ensures that $(\mathscr{I} - \mathscr{K})^{-1} \cdot \mathbf{I}$ is strictly positive. As one can see in the above proof this is essential for the construction of the equivalent metric on S. There the principal minors of $\mathscr{I} - \mathscr{K}_B$ are given by the constants M_1, M_2, M_3 (cf. (11)).

3.3. Main theorem on solution transfer Besides the dynamical equation (E_B) we consider two further dynamical equations (E_C) and (E_D).

$$x^\Delta = A(t)x + C(t,x), \qquad (\mathrm{E}_C)$$

$$x^\Delta = A(t)x + D(t,x), \qquad (\mathrm{E}_D)$$

which have the same form as (E_B). For reasons to become clear in Theorem 3.3 below we call these two equations comparison equations.

We are going to relate the solutions of (E_B) to the solutions of (E_C) and (E_D) according to Theorem 3.2. To this end we first define a (trivial) fiber bundle over $\mathbb{T}$

$$\mathscr{D}\mathscr{H}_{BCD} := \Big\{(\tau, c_0, d_0) \in \mathbb{T} \times X \times X : {}_\alpha\|\mathscr{L}\mathscr{D}_B c(\cdot;\tau,c_0)\|^\tau < \infty, \\ {}_\beta\|\mathscr{L}\mathscr{D}_B d(\cdot;\tau,d_0)\|_\tau < \infty\Big\}, \tag{15}$$

where $c(\cdot;\tau,c_0)$ (e.g.) is the solution of the IVP (E_C), $x(\tau)=c_o$. The estimate ${}_\alpha\|\mathscr{L}\mathscr{D}_B c(\cdot;\tau,c_0)\|^\tau<\infty$ (for example) also contains the fact that $c(\cdot;\tau,c_0)$ exists for all $t\le\tau$. The fiber bundle over $\tau\in\mathbb{T}$ exactly contains the pairs (c_0,d_0) of points through which pass the solutions of (E_C), respectively (E_D) with finite (E_B)-solution defect.

Now we define the mapping

$$(16)\qquad \mathscr{H}_{BCD}:\begin{cases}\mathscr{D}\mathscr{H}_{BCD}\times V & \to X\\ (\tau,c_0,d_0,\eta) & \mapsto b\big(\tau;\tau,\eta|c(\cdot;\tau,c_0),d(\cdot;\tau,d_0)\big),\end{cases}$$

where $b\big(\cdot\,;\tau,\eta|c(\cdot;\tau,c_0),d(\cdot;\tau,d_0)\big)$ is the solution of the BIB-problem (cf. Theorem 3.2)

$$(17)\qquad (\mathrm{E}_B),\quad {}_\alpha\|u(\cdot)-c^-(\cdot;\tau,c_0)\|^\tau<\infty,\quad v(\tau)=\eta,$$
$${}_\beta\|w(\cdot)-d^+(\cdot;\tau,d_0)\|_\tau<\infty.$$

The importance of the mapping $\mathscr{H}_{BCD}$ becomes apparent in the following

THEOREM 3.3 (Main theorem on solution transfer).

(i) *The mapping $\mathscr{H}_{BCD}$ satisfies the estimate*

$$\begin{pmatrix}|\mathscr{H}^-_{BCD}(\tau,c_0,d_0,\eta)-c_0^-|\\ |\mathscr{H}^+_{BCD}(\tau,c_0,d_0,\eta)-c_0^+|\\ |\mathscr{H}^-_{BCD}(\tau,c_0,d_0,\eta)-d_0^-|\\ |\mathscr{H}^+_{BCD}(\tau,c_0,d_0,\eta)-d_0^+|\end{pmatrix}\le\mathscr{M}_B\cdot\left[\begin{pmatrix}0\\ |\eta-c_0^\circ|\\ |\eta-d_0^\circ|\\ 0\end{pmatrix}+\frac{1}{\gamma}\begin{pmatrix}{}_\alpha\|\mathscr{L}\mathscr{D}_B^-c(\cdot;\tau,c_0)(\cdot)\|^\tau\\ {}_\alpha\|\mathscr{L}\mathscr{D}_B^{\overset{o}{+}}c(\cdot;\tau,c_0)(\cdot)\|^\tau\\ {}_\beta\|\mathscr{L}\mathscr{D}_B^{\overline{o}}d(\cdot;\tau,d_0)(\cdot)\|_\tau\\ {}_\beta\|\mathscr{L}\mathscr{D}_B^+d(\cdot;\tau,d_0)(\cdot)\|_\tau\end{pmatrix}\right]$$

Observe that we have $\mathscr{H}^\circ_{BCD}(\tau,c_0,d_0,\eta)=\eta$ for all $(\tau,c_0,d_0)\in\mathscr{D}\mathscr{H}_{BCD}$.

(ii) *The mapping $\mathscr{H}_{BCD}$ is continuous and globally Lipschitz with respect to $\eta\in V$:*

$$\begin{pmatrix}|\mathscr{H}^-_{BCD}(\tau,c_0,d_0,\eta)-\mathscr{H}^-_{BCD}(\tau,c_0,d_0,\underline{\eta})|\\ |\mathscr{H}^+_{BCD}(\tau,c_0,d_0,\eta)-\mathscr{H}^+_{BCD}(\tau,c_0,d_0,\underline{\eta})|\\ |\mathscr{H}^-_{BCD}(\tau,c_0,d_0,\eta)-\mathscr{H}^-_{BCD}(\tau,c_0,d_0,\underline{\eta})|\\ |\mathscr{H}^+_{BCD}(\tau,c_0,d_0,\eta)-\mathscr{H}^+_{BCD}(\tau,c_0,d_0,\underline{\eta})|\end{pmatrix}\le\mathscr{M}_B\cdot\begin{pmatrix}0\\ |\eta-\underline{\eta}|\\ |\eta-\underline{\eta}|\\ 0\end{pmatrix}.$$

(iii) *Let $(\tau,c_0,d_0,\eta)\in\mathscr{D}\mathscr{H}_{BCD}\times V$ be fixed and $c(\cdot):=c(\cdot;\tau,c_0)$, $d(\cdot):=d(\cdot;\tau,d_0)$ be solutions of* (E_C) *and* (E_D) *with respect to the initial points (τ,c_0) and (τ,d_0), respectively.*

Then the following statements on the mapping $b:\mathbb{T}\to X$ are equivalent:

(A) *$b(\cdot):=\mathscr{H}_{BCD}(\cdot,c(\cdot),d(\cdot),v(\cdot))$ and $v(\cdot)$ is a solution of the* IVP

$$v^\Delta=A^\circ(t)v+B^\circ(t,\mathscr{H}_{BCD}(t,c(t),d(t),v)),\qquad v(\tau)=\eta.$$

(B) $b(\cdot)$ *is a solution of the* BIB-*problem*

$$(\mathrm{E}_B), \quad {}_\alpha\|u(\cdot) - c^-(\cdot)\|^\tau < \infty, \quad v(\tau) = \eta, \quad {}_\beta\|w(\cdot) - d^+(\cdot)\|_\tau < \infty.$$

(C) $b(\cdot)$ *is a solution of the problem*

$$(\mathrm{E}_B), \quad {}_\alpha\|x(\cdot) - c(\cdot)\|^\tau < \infty, \quad v(\tau) = \eta, \quad {}_\beta\|x(\cdot) - d(\cdot)\|_\tau < \infty.$$

(D) $b(\cdot)$ *is a solution of the* IVP

$$(\mathrm{E}_B), \qquad x(\tau) = \mathscr{H}_{BCD}(\tau, c_0, d_0, \eta).$$

(E) $b(\cdot)$ *is a solution of the nonlinear* IVP

$$x^\Delta = A(t)x + B(t, \mathscr{H}_{BCD}(t, c(t), d(t), v)), \qquad x(\tau) = \mathscr{H}_{BCD}(\tau, c(\tau), d(\tau), \eta).$$

REMARK. The important property of $\mathscr{H}_{BCD}$ is given by the equivalence of (A) and(B) in (iii). If $c(\cdot)$ and $d(\cdot)$ are solutions of (E_C) and (E_D) respectively, then the mapping $\mathscr{H}_{BCD}$ transforms these two functions into a solution of (E_B) which is α-asymptotic to $c(\cdot)$ on $\mathbb{T}^\tau$ and β-asymptotic to $d(\cdot)$ on $\mathbb{T}_\tau$. Due to the fact that $b(\cdot)$ is not uniquely defined by these two properties one has additionally to insert the solution of the IVP (A) into $\mathscr{H}_{BCD}$. It contains information on the X°-component of $b(\cdot)$.

PROOF. (i) and (ii) directly follow from the corresponding items in Theorem 3.2. One only has to observe the definition of $\mathscr{H}_{BCD}$. An essential fact hereby is that for a function $x : \mathbb{T} \to X$ the following estimates hold:

$$|x(\tau)| \le {}_\alpha\|x(\cdot)\|^\tau \qquad \text{and} \qquad |x(\tau)| \le {}_\beta\|x(\cdot)\|_\tau.$$

The continuity of $\mathscr{H}_{BCD}$ is inherited from the continuity of the solution $b(\cdot\,; \tau, \eta|$ $c(\cdot; \tau, c_0), d(\cdot; \tau, d_0))$ of the BIB-problem (17). One has to take notice of the fact that the functions $c(\cdot; \tau, c_0)$ and $d(\cdot; \tau, d_0)$ are continuous with respect to inital data.

The equivalence of (B) and (C) in (iii) directly follows from the definition of $\mathscr{H}_{BCD}$ and Theorem 3.3(i).

Let $\widehat{b}(\cdot)$ be the function described in (B). For arbitrary and fixed $s \in \mathbb{T}$ we have the identity

$$\widehat{b}(\cdot) = b(\cdot; s, \widehat{b}^\circ(s)|c(\cdot), d(\cdot)),$$

because both these functions solve the BIB-problem

$$(\mathrm{E}_B), \qquad {}_\alpha\|u - c^-\|^\tau < \infty, \quad v(\tau) = \eta, \quad {}_\beta\|w - d^+\|_\tau < \infty,$$

the left one by definition, the right one by the fact that one can replace τ by s in these three conditions due to the compactness of $[\tau, s]$, respectively $[s, \tau]$. By Theorem 3.2

this BIB-problem has a unique solution. From the definition of $\mathscr{H}_{BCD}$ we derive for arbitrary and fixed $s \in \mathbb{T}$:

(18)
$$\begin{aligned}\widehat{b}(s) &= b(s; s, \widehat{b}^{\circ}(s) | c(\cdot), d(\cdot)) = b(s; s, \widehat{b}^{\circ}(s) | c(\cdot; s, c(s)), d(\cdot; s, d(s))) \\ &=: \mathscr{H}_{BCD}(s, c(s), d(s), \widehat{b}^{\circ}(s)),\end{aligned}$$

further

$$(\widehat{b}^{\circ})^{\Delta}(s) = A^{\circ}(t)\widehat{b}^{\circ}(t) + B^{\circ}(t, \widehat{b}(t)) =: A^{\circ}(t)\widehat{b}^{\circ}(t) + B^{\circ}(t, \mathscr{H}_{BCD}(t, c(t), d(t), \widehat{b}^{\circ}(t))).$$

Therefore $\widehat{b}^{\circ}$ solves the IVP given in (A). This observation and the identity (18) imply the equivalence of (A) and (B). This equivalence contains the following properties of $\widehat{b}(\cdot)$:

$$\widehat{b} \text{ is a solution of } (\mathrm{E}_B), \qquad \widehat{b}(\cdot) = \mathscr{H}_{BCD}(\cdot, c(\cdot), d(\cdot), b^{\circ}(\cdot)), \qquad \widehat{b}^{\circ}(\tau) = \eta.$$

So $\widehat{b}(\cdot)$ is the (unique) solution of the IVPs given in (D) and (E).

For each fixed $(\tau, c_0, d_0) \in \mathscr{D}\mathscr{H}_{BCD}$ by $\mathscr{H}_{BCD}$ is defined a so-called *integral manifold*

$$\mathscr{M}\mathscr{H}_{BCD}(\tau, c_0, d_0) := \{(t, \mathscr{H}_{BCD}(t, c(t; \tau, c_0), d(t; \tau, d_0), v) : (t, v) \in \mathbb{T} \times V\} \subseteq \mathbb{T} \times X$$

with parametrization space $\mathbb{T} \times V$; $\mathscr{M}\mathscr{H}_{BCD}(\tau, c_0, d_0)$ is globally Lipschitz with respect to $v \in V$, because the defining map $\mathscr{H}_{BCD}$ is. This property is directly connected to the fact that the nonlinearity B in (E_C) is globally Lipschitz (by condition (II_B)). For the autonomous case and $c_0 = 0$ and $d_0 = 0$ it is known that the (classical) center manifold $\mathscr{M}\mathscr{H}_{BLL}(0, 0)$ asymptotic to the zero function is $\mathscr{C}^k$, if the nonlinearity $B : X \to X$ is $\mathscr{C}^k$ (L is the symbol for the linear equation). The proof of this property requires sophisticated methods involving so-called scales of Banach spaces. This idea was proposed in [15]. In [9] the author of this paper gave a proof of the $\mathscr{C}^k$-property which covers the 'homogeneous' time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$. Here calculus on measure chains is also employed.

The following theorem gives some results on these manifolds.

THEOREM 3.4. *Let $c(\cdot) = c(\cdot; \tau, c_0)$ and $d(\cdot) = d(\cdot; \tau, d_0)$ be fixed solutions of* (E_C) *respectively* (E_D). *Then the integral manifolds $\mathscr{M}\mathscr{H}_{BCD}(s, c(s), d(s))$ are identical for all $s \in \mathbb{T}$. $\mathscr{M}\mathscr{H}_{BCD}(\tau, c_0, d_0)$ is invariant under the flow of* (E_B) *in the following sense:*

$$(t_0, b_0) \in \mathscr{M}\mathscr{H}_{BCD}(\tau, c_0, d_0) \Longrightarrow (t, b(t; t_0, b_0)) \in \mathscr{M}\mathscr{H}_{BCD}(\tau, c_0, d_0) \text{ for all } t \in \mathbb{T}.$$

This invariant manifold contains exactly all solutions of (E_B) *which are α-asymptotic on $\mathbb{T}^{\tau}$ to $c(\cdot)$ and β-asymptotic on $\mathbb{T}_{\tau}$ to $d(\cdot)$.*

PROOF. (a) We have :

$$\mathcal{M}\mathcal{H}_{BCD}(s, c(s), d(s)) = \{(t, \mathcal{H}_{BCD}(t, c(t; s, c(s)), d(t; s, d(s)), v) : (t, v) \in \mathbb{T} \times V\}$$
$$= \{(t, \mathcal{H}_{BCD}(t, c(t), d(t), v) : (t, v) \in \mathbb{T} \times V\}.$$

This expression is independent of s.

(b) Let $(t_0, b_0) \in \mathcal{M}\mathcal{H}_{BCD}(\tau, c_0, d_0)$. Then $b(\cdot) := b(\cdot; t_0, b_0)$ solves the IVP

$$(\mathrm{E}_B), \qquad x(t_0) = b_0 = \mathcal{H}_{BCD}(t_0, c(t_0), d(t_0), b_0^\circ).$$

Because of the equivalence of (D) and (A) in Theorem 3.3(iii) — replace τ by t_0 — we then have for all $t \in \mathbb{T}$:

$$b(\cdot) = \mathcal{H}_{BCD}(\cdot, c(\cdot), d(\cdot), b^\circ(\cdot)).$$

Therefore $(t, b(t)) \in \mathcal{M}\mathcal{H}_{BCD}(\tau, c_0, d_0)$ for all $t \in \mathbb{T}$. The last statement is a direct consequence of Theorem 3.3.

3.4. Solution transfer at lacking critical component We are going to formulate Theorem 3.3 for the special case that the middle component in the equations (E_B), (E_C) and (E_D) is lacking (i.e. $X^\circ = V = \{0\}$). To this end let the nonlinear equation

$$\begin{aligned} u^\Delta &= A^-(t)u + B^-(t, u, w), \qquad (\mathrm{H}_B)\\ w^\Delta &= A^+(t)w + B^+(t, u, w), \end{aligned}$$

and two further nonlinear equations (H_C) and (H_D) with nonlinearities C and D of the same form be given.

For this situation and $\delta := \alpha = \beta$ the conditions (I) and (II_B) read as follows:

$$\text{(I)} \qquad \begin{aligned} |\Phi_-(r, s)| &\le K \cdot e_{\delta-\gamma}(r, s) \quad \text{for} \quad s \le r,\\ |\Phi_+(r, s)| &\le K \cdot e_{\delta+\gamma}(r, s) \quad \text{for} \quad r \le s, \end{aligned}$$

$$(\mathrm{II}_B) \qquad L_B < \frac{\gamma}{K(K+2)}.$$

We define $\mathcal{D}\mathcal{J}_{BCD}$ by

$$(19) \qquad \mathcal{D}\mathcal{J}_{BCD} := \Big\{(\tau, c_0, d_0) \in \mathbb{T} \times X \times X : {}_\delta\|\mathcal{L}\mathcal{D}_B c(\cdot; \tau, c_0)\|^\tau < \infty,\ {}_\delta\|\mathcal{L}\mathcal{D}_B d(\cdot; \tau, d_0)\|_\tau < \infty\Big\}.$$

The relations

$$(20) \qquad \mathcal{J}_{BCD} : \begin{cases} \mathcal{D}\mathcal{J}_{BCD} & \to X \cong X^- \times X^+,\\ (\tau, c_0, d_0) & \mapsto \mathcal{H}_{BCD}(\tau, c_0, d_0, 0) \end{cases}$$

give the counterpart of $\mathcal{H}_{BCD}$ for this situation. Theorem 3.3 reads for this special case:

THEOREM 3.5 (Solution transfer at lacking critical component). *We consider the mapping* $\mathscr{J}_{BCD}$ *with respect to the situation described above. Let the conditions* (I) *and* (II_B) *be fulfilled.*

(i) *The mapping* $\mathscr{J}_{BCD}$ *satisfies the estimate*

$$\begin{pmatrix} |\mathscr{J}_{BCD}(\tau, c_0, d_0) - c_0| \\ |\mathscr{J}_{BCD}(\tau, c_0, d_0) - d_0| \end{pmatrix} \le \frac{K}{\gamma M_3} \cdot \begin{pmatrix} M_2 + K^2 M & K \\ K & M_2 + K^2 M \end{pmatrix} \cdot \begin{pmatrix} {}_\delta\|\mathscr{L}\mathscr{D}_B c(\cdot; \tau, c_0)\|^\tau \\ {}_\delta\|\mathscr{L}\mathscr{D}_B d(\cdot; \tau, d_0)\|_\tau \end{pmatrix}.$$

(ii) *The mapping* $\mathscr{J}_{BCD}$ *is continuous.*

(iii) *Let* $(\tau, c_0, d_0) \in \mathscr{D}\mathscr{J}_{BCD}$ *and* $c(\cdot) := c(\cdot; \tau, c_0)$, $d(\cdot) := d(\cdot; \tau, d_0)$.

Then the following statements on the mappings $b : \mathbb{T} \to X$ *are equivalent:*

(A) $b(\cdot) := \mathscr{J}_{BCD}(\cdot, c(\cdot), d(\cdot))$.

(B) $b(\cdot)$ *is a solution of the* BB*-problem*

$$(\text{H}_B), \qquad {}_\delta\|u(\cdot) - c^-(\cdot)\|^\tau < \infty, \qquad {}_\delta\|w(\cdot) - d^+(\cdot)\|_\tau < \infty.$$

(*One can also list the corresponding properties* (C) – (E) *from Theorem* 3.3 (iii). *We omit these because they are not necessary for our further considerations.*)

PROOF. We only have to show, how one can derive the estimate in (i) via linear algebra. Directly from Theorem 3.3 we see that

$$\begin{pmatrix} |\mathscr{J}^-_{BCD}(\tau, c_0, d_0) - c_0^-| \\ |\mathscr{J}^+_{BCD}(\tau, c_0, d_0) - c_0^+| \\ |\mathscr{J}^-_{BCD}(\tau, c_0, d_0) - d_0^-| \\ |\mathscr{J}^+_{BCD}(\tau, c_0, d_0) - d_0^+| \end{pmatrix} \le \frac{1}{\gamma} \cdot \mathscr{M}_B \cdot \begin{pmatrix} {}_\delta\|\mathscr{L}\mathscr{D}^-_B\, c(\cdot; \tau, c_0)\|^\tau \\ {}_\delta\|\mathscr{L}\mathscr{D}^+_B\, c(\cdot; \tau, c_0)\|^\tau \\ {}_\delta\|\mathscr{L}\mathscr{D}^-_B\, d(\cdot; \tau, d_0)\|_\tau \\ {}_\delta\|\mathscr{L}\mathscr{D}^+_B\, d(\cdot; \tau, d_0)\|_\tau \end{pmatrix}.$$

In the two vectors we take the sup norm of the first and the second line and of the third and the fourth line, respectively. Accordingly we have to take line-sup-norms of the corresponding 2×2 submatrices of $\mathscr{M}_B$ (cf. (11)). This yields the estimate in (i).

4. Topological equivalence

We will now use the theorems of Section 3 in order to establish some theorems on the topological equivalence of the given system

$$x^\Delta = A(t)x + B(t, x) \qquad (\text{E}_B)$$

to systems of simpler forms. Let (E_B) satisfy conditions (I), (II_B) and let

$$x^\Delta = A(t)x \qquad (E_L)$$

be the associated linear equation with the 'nonlinearity' $L \equiv 0$.

The following theorems all follow directly from the very general Theorem 3.3 and its special case, Theorem 3.5.

4.1. Center integral manifolds For the fixed equation (E_B) we define:

$$\mathscr{D}\overleftarrow{\mathscr{H}} := \left\{(\tau, l_0) \in \mathbb{T} \times X : {}_\alpha\|B(\cdot, l(\cdot; \tau, l_0))\|^\tau < \infty, {}_\beta\|B(\cdot, l(\cdot; \tau, l_0))\|_\tau < \infty\right\}$$

($l(\cdot; \tau, l_0)$ is the solution of the IVP (E_L), $x(\tau) = l_0$). With (16) we define

$$\overleftarrow{\mathscr{H}}: \begin{cases} \mathscr{D}\overleftarrow{\mathscr{H}} \times V & \to X, \\ (\tau, l_0, \eta) & \mapsto \mathscr{H}_{BLL}(\tau, l_0, l_0, \eta) = b(\tau; \tau, \eta | l(\cdot; \tau, l_0), l(\cdot; \tau, l_0)). \end{cases}$$

So, instead of two arbitrary comparison equations (E_C) and (E_D) we consider here the linear equation (E_L) alone. The following theorem is a direct translation of Theorem 3.3 to this new situation.

THEOREM 4.1 (Center integral manifolds). *For* (E_B) *let the conditions* (I) *and* (II_B) *be fulfilled.*

(i) *The mapping* $\overleftarrow{\mathscr{H}}$ *satisfies the following estimate:*

$$\begin{pmatrix} |\overleftarrow{\mathscr{H}}^-(\tau, l_0, \eta) - l_0^-| \\ |\overleftarrow{\mathscr{H}}^+(\tau, l_0, \eta) - l_0^+| \end{pmatrix} \le (\mathscr{M}_B)_{14} \cdot \left[\begin{pmatrix} 0 \\ |\eta - l_0^\circ| \\ |\eta - l_0^\circ| \\ 0 \end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix} {}_\alpha\|\mathscr{L}\mathscr{D}_B^- l(\cdot; \tau, l_0)(\cdot)\|^\tau \\ {}_\alpha\|\mathscr{L}\mathscr{D}_B^{\overset{o}{+}} l(\cdot; \tau, l_0)(\cdot)\|^\tau \\ {}_\beta\|\mathscr{L}\mathscr{D}_B^{\overset{o}{-}} l(\cdot; \tau, l_0)(\cdot)\|_\tau \\ {}_\beta\|\mathscr{L}\mathscr{D}_B^+ l(\cdot; \tau, l_0)(\cdot)\|_\tau \end{pmatrix} \right].$$

Here $(\mathscr{M}_B)_{14}$ *is the matrix consisting of the first and the fourth line of* $\mathscr{M}_B$. *Observe that* $\overleftarrow{\mathscr{H}}^\circ(\tau, l_0, \eta) = \eta$.

(ii) *The mapping* $\overleftarrow{\mathscr{H}}$ *is continuous and continuous with respect to* $\eta \in V$:

$$\begin{pmatrix} |\overleftarrow{\mathscr{H}}^-(\tau, l_0, \eta) - \overleftarrow{\mathscr{H}}^-(\tau, l_0, \underline{\eta})| \\ |\overleftarrow{\mathscr{H}}^+(\tau, l_0, \eta) - \overleftarrow{\mathscr{H}}^+(\tau, l_0, \underline{\eta})| \end{pmatrix} \le (\mathscr{M}_B)_{14} \cdot \begin{pmatrix} 0 \\ |\eta - \underline{\eta}| \\ |\eta - \underline{\eta}| \\ 0 \end{pmatrix}.$$

In particular $\overleftarrow{\mathscr{H}}$ *satisfies a Lipschitz condition with respect to* η.

(iii) *Let* $(\tau, l_0, \eta) \in \mathscr{D}\overleftarrow{\mathscr{H}} \times V$ *and* $l(\cdot) := l(\cdot; \tau, l_0)$. *The following statements on a mapping* $b : \mathbb{T} \to X$ *are equivalent:*

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(A) $b(\cdot) := \overleftarrow{\mathscr{H}}(\cdot, l(\cdot), v(\cdot))$ *and* $v(\cdot)$ *is a solution of the* IVP

$$v^{\Delta} = A^{\circ}(t)v + B^{\circ}(t, \overleftarrow{\mathscr{H}}(t, l(t), v)), \qquad v(\tau) = \eta.$$

(B) $b(\cdot)$ *is a solution of the* BIB-*problem*

$$(\mathrm{E}_B), \qquad {}_{\alpha}\|u(\cdot) - l^{-}(\cdot)\|^{\tau} < \infty,\ v(\tau) = \eta,\ {}_{\beta}\|w(\cdot) - l^{+}(\cdot)\|_{\tau} < \infty.$$

(iv) *The integral manifold defined for each* $(\tau, l_0) \in \mathscr{D}\overleftarrow{\mathscr{H}}$ *by*

$$\mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_0) = \{(t, \overleftarrow{\mathscr{H}}(t, l(t; \tau, l_0), v) : (t, v) \in \mathbb{T} \times V\} = \mathscr{M}\mathscr{H}_{BLL}(\tau, l_0, l_0) \subseteq \mathbb{T} \times X$$

(*the so called center integral manifold* α-β-*asymptotic to* $l(\cdot; \tau, l_0)$) *is invariant under the flow of* (E_B) *in the following sense:*

$$(t_0, b_0) \in \mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_0) \Longrightarrow (t, b(t; t_0, b_0)) \in \mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_0) \quad \textit{for all} \quad t \in \mathbb{T}.$$

REMARKS. We consider some special cases of the above theorem:

1. Under the conditions

$$(21) \qquad {}_{\alpha}\|B(\cdot, 0)\|^{\tau} < \infty, \quad {}_{\beta}\|B(\cdot, 0)\|_{\tau} < \infty$$

we have $(\tau, 0) \in \mathscr{D}\overleftarrow{\mathscr{H}}$ and therefore get the center integral manifold α-β-asymptotic to the zero solution (of (E_B)). By the above theorem it contains all solutions $b(\cdot)$ of (E_B) which are subject to the condition

$${}_{\alpha}\|b(\cdot)\|^{\tau} < \infty,\ {}_{\beta}\|b(\cdot)\|_{\tau} < \infty.$$

2. In the case that $B(\cdot, 0) \equiv 0$, condition (21) is fulfilled for all pairs (α, β). One finds for each (α, β)-trichotomy of (E_B) the corresponding center integral manifold asymptotic to the zero solution. If $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$, then the corresponding center integral manifolds $\mathscr{M}\overleftarrow{\mathscr{H}}_1(\tau, 0)$ and $\mathscr{M}\overleftarrow{\mathscr{H}}_2(\tau, 0)$ fulfill the following inclusion:

$$\mathscr{M}\overleftarrow{\mathscr{H}}_2(\tau, 0) \subseteq \mathscr{M}\overleftarrow{\mathscr{H}}_1(\tau, 0).$$

This is the idea of the hierarchy of manifolds proposed in Aulbach [1].

3. If $B(\cdot, 0)$ is bounded (for example $B(\cdot, 0)$ constant or periodic), then (21) is fulfilled for all pairs

$$\alpha \leq 0 \leq \beta.$$

In autonomous or periodic systems under the conditions (I_B) and (II_B) one gets the classical center manifold.

Now the question arises: For which (τ, l_0) is a given solution $b(\cdot)$ of (E_B) located on $\mathcal{M}\overleftarrow{\mathcal{H}}(\tau, l_0)$? This can also be answered by the help of Theorem 3.3. We define:

$$\mathcal{D}\overrightarrow{\mathcal{H}} := \Big\{(\tau, b_0) \in \mathbb{T} \times X : {}_\alpha\|B(\cdot, b(\cdot;\tau, b_0))\|^\tau < \infty, {}_\beta\|B(\cdot, b(\cdot;\tau, b_0))\|_\tau < \infty\Big\},$$

$$\overrightarrow{\mathcal{H}}: \begin{cases} \mathcal{D}\overrightarrow{\mathcal{H}} \times V & \to X, \\ (\tau, b_0, \eta) & \mapsto \mathcal{H}_{LBB}(\tau, b_0, b_0, \eta) = l(\tau;\tau, \eta | b(\cdot;\tau, b_0), b(\cdot;\tau, b_0)). \end{cases}$$

The mapping $\mathcal{H}_{LBB}$ is defined in 16, one has to replace (E_B) by (E_L) and (E_C), (E_D) by (E_B). So here we study the linear equation (E_L). The nonlinear equation (E_B) plays the role of the comparison equations (E_C) and (E_D).

$l(\cdot;\tau, \eta | b(\cdot;\tau, b_0), b(\cdot;\tau, b_0))$ is the solution of the BIB-problem

$$(E_L), \qquad {}_\alpha\|u(\cdot) - b^-(\cdot)\|^\tau < \infty, \qquad v(\tau) = \eta, \qquad {}_\beta\|w(\cdot) - b^+(\cdot)\|_\tau < \infty.$$

The following theorem holds.

THEOREM 4.2 (Linearization map). *Let the conditions* (I) *and* (II_B) *for* (E_B) *be fulfilled.*

(i) *The mapping* $\overrightarrow{\mathcal{H}}$ *satisfies the following estimate:*

$$\begin{pmatrix} |\overrightarrow{\mathcal{H}}^-(\tau, b_0, \eta) - b_0^-| \\ |\overrightarrow{\mathcal{H}}^+(\tau, b_0, \eta) - b_0^+| \end{pmatrix} \le \frac{K}{\gamma} \begin{pmatrix} {}_\alpha\|\mathcal{L}\mathcal{D}_L^- b(\cdot;\tau, b_0)\|^\tau \\ {}_\beta\|\mathcal{L}\mathcal{D}_L^+ b(\cdot;\tau, b_0)\|_\tau \end{pmatrix}.$$

Again observe $\overrightarrow{\mathcal{H}}^\circ(\tau, b_0, \eta) = \eta$.

(ii) *The mapping* $\overrightarrow{\mathcal{H}}$ *is continuous. If* $(E_{\underline{B}})$ *is again another equation of the same form, then for the corresponding mapping* $\underline{\overrightarrow{\mathcal{H}}}$ *we have:*

$$\begin{pmatrix} \big|[\overrightarrow{\mathcal{H}}^-(\tau, b_0, \eta) - b_0^-] - [\underline{\overrightarrow{\mathcal{H}}}^-(\tau, \underline{b}_0, \underline{\eta}) - \underline{b}_0^-]\big| \\ \big|[\overrightarrow{\mathcal{H}}^+(\tau, b_0, \eta) - b_0^+] - [\underline{\overrightarrow{\mathcal{H}}}^+(\tau, \underline{b}_0, \underline{\eta}) - \underline{b}_0^+]\big| \end{pmatrix}$$
$$\le \frac{K}{\gamma} \begin{pmatrix} {}_\alpha\|B^-(\cdot, b(\cdot;\tau, b_0)) - \underline{B}^-(\cdot, \underline{b}(\cdot;\tau, \underline{b}_0))\|^\tau \\ {}_\beta\|B^+(\cdot, b(\cdot;\tau, b_0)) - \underline{B}^+(\cdot, \underline{b}(\cdot;\tau, \underline{b}_0))\|_\tau \end{pmatrix}.$$

(iii) *Let* $(\tau, b_0, \eta) \in \mathcal{D}\overrightarrow{\mathcal{H}} \times V$ *and* $b(\cdot) := b(\cdot;\tau, b_0)$.

Then the following statements on the mapping $l : \mathbb{T} \to X$ *are equivalent:*

(A) $l(\cdot) := \overrightarrow{\mathcal{H}}(\cdot, b(\cdot), v(\cdot))$ *and* $v(\cdot)$ *is a solution of the linear* IVP

$$v^\Delta = A^\circ(t)v, \qquad v(\tau) = \eta.$$

(B) *$l(\cdot)$ is a solution of the linear* BIB-*problem*

$$(\mathrm{E}_L), \quad {}_\alpha\|u(\cdot)-b^-(\cdot)\|^\tau<\infty, \quad v(\tau)=\eta, \quad {}_\beta\|w(\cdot)-b^+(\cdot)\|_\tau<\infty.$$

(C) *$l(\cdot)$ is a solution of the linear problem*

$$(\mathrm{E}_L) \quad {}_\alpha\|x(\cdot)-b(\cdot)\|^\tau<\infty, \quad v(\tau)=\eta, \quad {}_\beta\|x(\cdot)-b(\cdot)\|_\tau<\infty.$$

(D) *$l(\cdot)$ is a solution of the linear* IVP

$$(\mathrm{E}_L), \quad x(\tau)=\overrightarrow{\mathscr{H}}(\tau,b_0,\eta).$$

(iv) *For all $(\tau,b_0,l_0^\circ)\in\mathscr{D}\overrightarrow{\mathscr{H}}\times V$ we have:*

$$(22) \quad (\tau,\overrightarrow{\mathscr{H}}(\tau,b_0,l_0^\circ))\in\mathscr{D}\overleftarrow{\mathscr{H}} \quad \text{and} \quad \overleftarrow{\mathscr{H}}(\tau,\overrightarrow{\mathscr{H}}(\tau,b_0,l_0^\circ),b_0^\circ)=b_0.$$

For all $(\tau,l_0,b_0^\circ)\in\mathscr{D}\overleftarrow{\mathscr{H}}\times V$ we have:

$$(23) \quad (\tau,\overleftarrow{\mathscr{H}}(\tau,l_0,b_0^\circ))\in\mathscr{D}\overrightarrow{\mathscr{H}} \quad \text{and} \quad \overrightarrow{\mathscr{H}}(\tau,\overleftarrow{\mathscr{H}}(\tau,l_0,b_0^\circ),l_0^\circ)=l_0.$$

(In the absence of the critical component ($V=\{0\}$) this is the classical Theorem of Hartman-Grobman.)

PROOF. (i) The estimates directly follow from the corresponding ones in Theorem 3.3, if one takes into account that the Lipschitz constant $L_L=0$ and therefore $M_L=0$ and then $(\mathscr{M}_L)_{14}=\begin{pmatrix}K&0&0&0\\0&0&0&K\end{pmatrix}$ (cf. (11)).

In order to prove (iv) it is sufficient to consider the first partial statement. To this end let $b(\cdot):=b(\cdot;\tau,b_0)$ and $l(\cdot)$ be a solution of (E_L) with $l^\circ(\tau)=l_0^\circ$. Then we have

$$\begin{aligned}{}_\alpha\|\mathscr{L}\mathscr{D}_B\overrightarrow{\mathscr{H}}(\cdot,b(\cdot),l^\circ(\cdot))\|^\tau &= {}_\alpha\|B(\cdot,\overrightarrow{\mathscr{H}}(\cdot,b(\cdot),l^\circ(\cdot)))\|^\tau\\ &\le {}_\alpha\|B(\cdot,\overrightarrow{\mathscr{H}}(\cdot,b(\cdot),l^\circ(\cdot)))-B(\cdot,b(\cdot))\|^\tau+{}_\alpha\|B(\cdot,b(\cdot))\|^\tau\\ &\le L_B\cdot{}_\alpha\|\overrightarrow{\mathscr{H}}(\cdot,b(\cdot),l^\circ(\cdot))-b(\cdot)\|^\tau+{}_\alpha\|\mathscr{L}\mathscr{D}_L b(\cdot)\|^\tau<\infty.\end{aligned}$$

The last estimate follows from (i) of this theorem and the relation $(\tau,b_0)=(\tau,b(\tau))\in\mathscr{D}\overleftarrow{\mathscr{H}}$. So we have proven the left identity in (22). By Theorem 4.2 (iii) the mapping $b(\cdot)$, by Theorem 4.1(iii) the mapping $\overleftarrow{\mathscr{H}}(\cdot,\overrightarrow{\mathscr{H}}(\cdot,b(\cdot),l_0^\circ(\cdot)),b^\circ(\cdot))$ is a solution of the BIB-problem

$$(24) \quad {}_\alpha\|u(\cdot)-\overrightarrow{\mathscr{H}}^-(\cdot,b(\cdot),l^\circ(\cdot))\|^\tau<\infty,$$

$$(25) \quad (\mathrm{E}_B), \quad v(\tau)=b^\circ(\tau),$$

$$(26) \quad {}_\beta\|w(\cdot)-\overrightarrow{\mathscr{H}}^+(\cdot,b(\cdot),l^\circ(\cdot))\|_\tau<\infty.$$

Therefore the two functions coincide. The evaluation at time $t=\tau$ yields the right relation in (22).

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4.2. Horizontal fibration In addition to (I) and (II_B) let the following condition hold from now on

(III_B) For all solutions $x(\cdot)$ of (E_B) or (E_L) : ${}_\alpha\|B(\cdot, x(\cdot))\|^\tau < \infty$, ${}_\beta\|B(\cdot, x(\cdot))\|_\tau < \infty$.

As a consequence we have for all solutions $l(\cdot)$ of (E_L):

$$(27) \qquad {}_\alpha\|\mathscr{L}\mathscr{D}_B l(\cdot)\|^\tau < \infty, \quad {}_\beta\|\mathscr{L}\mathscr{D}_B l(\cdot)\|_\tau < \infty$$

and for all solutions $b(\cdot)$ of (E_B):

$$(28) \qquad {}_\alpha\|\mathscr{L}\mathscr{D}_L b(\cdot)\|^\tau < \infty, \quad {}_\beta\|\mathscr{L}\mathscr{D}_L b(\cdot)\|_\tau < \infty.$$

This condition does not depend on $\tau \in \mathbb{T}$. For example, it is a consequence of another condition

$$\sup_{x\in X} {}_\alpha\|B(\cdot, x)\|^\tau < \infty, \qquad \sup_{x\in X} {}_\beta\|B(\cdot, x)\|_\tau < \infty,$$

which is somewhat more transparent, but stronger. From (III_B) we immediately get for the mappings defined in the last section: $\mathscr{D}\overleftarrow{\mathscr{H}} = \mathscr{D}\overrightarrow{\mathscr{H}} = \mathbb{T} \times X$. Now we have:

THEOREM 4.3 (Horizontal fibration). *For* (E_B) *let the conditions* (I), (II_B) *and* (III_B) *be fulfilled. For fixed* $\tau \in \mathbb{T}$ *we get the following fibration*

$$(29) \qquad \mathbb{T} \times X = \dot{\bigcup_{l_0 \in U\times\{0\}\times W}} \mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_0)$$

of $\mathbb{T} \times X$ *(disjoint union). It is known as a horizontal fibration.*

PROOF. (a) Disjointness: Let $l_{01}, l_{02} \in U \times \{0\} \times W$ and assume there exists $(t, x) \in \mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_{01}) \cap \mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_{02})$. We set $l_1(\cdot) := l(\cdot; \tau, l_{01})$ and $l_2(\cdot) := l(\cdot; \tau, l_{02})$. By definition of $\mathscr{M}\overleftarrow{\mathscr{H}}$ we have:

$$x = b(t; t, v|l_1(\cdot), l_1(\cdot)) = b(t; t, v|l_2(\cdot), l_2(\cdot)).$$

We conclude (regarding the uniqueness of the solution of the IVP):

$$b(\cdot) := b(\cdot; t, v|l_1(\cdot), l_1(\cdot)) = b(\cdot; t, v|l_2(\cdot), l_2(\cdot)).$$

By definition of $b(\cdot; t, v|l_1(\cdot), l_1(\cdot))$ as a solution of the BIB-problem we have:

$${}_\alpha\|l_1^-(\cdot) - l_2^-(\cdot)\|^t \le {}_\alpha\|l_1^-(\cdot) - b^-(\cdot)\|^t + {}_\alpha\|b^-(\cdot) - l_2^-(\cdot)\|^t < \infty.$$

This is only possible for $l_{01}^- = l_{02}^-$. In the same way one derives $l_{01}^+ = l_{02}^+$ and thus $l_{01} = l_{02}$.

(b) The covering property: Let $(t_0, b_0) \in \mathbb{T} \times X$ and let $b(\cdot) := b(\cdot; t_0, b_0)$ be the solution of the IVP (E_B), $x(t_0) = b_0$. Further on we define $l(\cdot) := \overrightarrow{\mathscr{H}}(\cdot, b(\cdot), 0)$ (cf. Theorem 3.7) and $l_0 := l(\tau)$. Then (by Theorem 3.7(iii), (A) $\Longleftrightarrow$ (B)) we conclude

$$_\alpha\|l^-(\cdot) - b^-(\cdot)\|^\tau < \infty, \qquad l(\tau) = 0, \qquad _\beta\|l^+(\cdot) - b^+(\cdot)\|_\tau < \infty,$$

and $b(\cdot)$ is a solution of the BIB-problem:

$$(\mathrm{E}_B), \qquad _\alpha\|u(\cdot) - l^-(\cdot)\|^{t_0} < \infty, \qquad v(t_0) = b_0^\circ, \qquad _\beta\|u(\cdot) - l^-(\cdot)\|_{t_0} < \infty.$$

So we have $b(\cdot) = b(\cdot; t_0, b_0^\circ | l(\cdot), l(\cdot))$, especially

$$\begin{aligned} b_0 = b(t_0) &= b(t_0; t_0, b_0^\circ | l(\cdot), l(\cdot)) = b(t_0; t_0, b_0^\circ | l(\cdot; t_0, l(t_0)), l(\cdot; t_0, l(t_0))) \\ &=: \overleftarrow{\mathscr{H}}(t_0, l(t_0), b_0^\circ)) = \overleftarrow{\mathscr{H}}(t_0, l(t_0; \tau, l_0), b_0^\circ)). \end{aligned}$$

This just means that $(t_0, b_0) \in \mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_0)$.

4.3. Asymptotic phase (– vertical fibration) In this section we are going to construct mappings, which transfer solutions from one integral center manifold of (E_B) to another. To this end we consider the dynamical equation (E_B) in two different forms

$$\begin{aligned} u^\Delta &= A^-(t)u + B^-(t, u, z), \qquad (\mathrm{E}_B^\alpha), \\ z^\Delta &= \overset{o}{A^+}(t)z + \overset{o}{B^+}(t, u, z), \end{aligned}$$

and

$$\begin{aligned} y^\Delta &= A^{\overline{o}}(t)y + B^{\overline{o}}(t, y, w), \qquad (\mathrm{E}_B^\beta) \\ w^\Delta &= A^+(t)w + B^+(t, y, w). \end{aligned}$$

For these two equations we consider the mappings $\mathscr{J}$ introduced in Section 3.4 for equations of the form (H_B). The role of the δ-dichotomy (in Section 3.4) is here played once by the α-dichotomy and once by the β-dichotomy of (E_B). Instead of the comparison equations (H_C) and (H_D) we will choose the linearized equation (E_L) and the equation (E_B) itself.

Looking at (19) and (20) we define:

$$\begin{aligned} \mathscr{D}\mathscr{J}^\alpha_{BLB} &= \{(\tau, l_0, b_0) \in \mathbb{T} \times X \times X : {}_\alpha\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|^\tau < \infty\} \overset{*}{=} \mathbb{T} \times X \times X, \\ \mathscr{D}\mathscr{J}^\beta_{BBL} &= \{(\tau, b_0, l_0) \in \mathbb{T} \times X \times X : {}_\beta\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|_\tau < \infty\} \overset{*}{=} \mathbb{T} \times X \times X. \end{aligned}$$

The identities $*$ follow from (9) and (27) and

$$\begin{aligned} \mathscr{J}^\alpha_{BLB} &= \mathscr{J}^\alpha_{BLB} : \mathbb{T} \times X \times X \to X \cong U \times Z \quad \text{w.r.t. } (\mathrm{E}_B^\alpha) \text{ and } \alpha\text{-dichotomy}, \\ \mathscr{J}^\beta_{BBL} &= \mathscr{J}^\beta_{BBL} : \mathbb{T} \times X \times X \to X \cong Y \times W \quad \text{w.r.t. } (\mathrm{E}_B^\beta) \text{ and } \beta\text{-dichotomy}. \end{aligned}$$

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Now we combine these two mappings as follows:

$$\overrightarrow{\mathscr{J}}: \begin{cases} \mathbb{T} \times X \times X & \to X, \\ (\tau, l_0, b_0) & \mapsto \mathscr{J}^{\beta}_{BBL}(\tau, \mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0), l_0), \end{cases}$$

$$\overleftarrow{\mathscr{J}}: \begin{cases} \mathbb{T} \times X \times X & \to X, \\ (\tau, l_0, b_0) & \mapsto \mathscr{J}^{\alpha}_{BLB}(\tau, l_0, \mathscr{J}^{\beta}_{BBL}(\tau, b_0, l_0)). \end{cases}$$

The following theorem describes the properties of these mappings:

THEOREM 4.4 (Double asymptotic phase).

(i) *The mappings* $\overleftarrow{\mathscr{J}}$ *and* $\overrightarrow{\mathscr{J}}$ *fulfill the following estimates:*

$$\begin{pmatrix} |\overleftarrow{\mathscr{J}}(\tau, l_0, b_0) - l_0| \\ |\overrightarrow{\mathscr{J}}(\tau, l_0, b_0) - l_0| \end{pmatrix} \leq \frac{K(M_2 + K^2 M)}{\gamma M_3} \cdot \begin{pmatrix} {}_{\alpha}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|^{\tau} \\ {}_{\beta}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|_{\tau} \end{pmatrix}$$

$$\begin{pmatrix} |\overleftarrow{\mathscr{J}}(\tau, l_0, b_0) - b_0| \\ |\overrightarrow{\mathscr{J}}(\tau, l_0, b_0) - b_0| \end{pmatrix}$$
$$\leq \frac{K^2}{\gamma M_3} \cdot \begin{pmatrix} {}_{\alpha}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, \mathscr{J}^{\beta}_{BBL}(\tau, b_0, l_0))\|^{\tau} + {}_{\beta}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, b_0)\|_{\tau} \\ {}_{\beta}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, \mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0))\|_{\tau} + {}_{\alpha}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, b_0)\|^{\tau} \end{pmatrix}.$$

(ii) *The mappings* $\overrightarrow{\mathscr{J}}$ *and* $\overleftarrow{\mathscr{J}}$ *are continuous.*

(iii) *If* $l(\cdot)$ *and* $b(\cdot)$ *are solutions of* (E_L) *resp.* (E_B), *then*

$$\overrightarrow{\mathscr{J}}(\cdot, l(\cdot), b(\cdot)) \qquad \textit{and} \qquad \overleftarrow{\mathscr{J}}(\cdot, l(\cdot), b(\cdot))$$

are solutions of (E_B).

(iv) *Furthermore for* $(\tau, l_0, b_0) \in \mathbb{T} \times X \times X$ *the following identities hold:*

(30)
$$\overrightarrow{\mathscr{J}}(\tau, l_0, b_0) = \overleftarrow{\mathscr{H}}(\tau, l_0, \overrightarrow{\mathscr{J}}^{\circ}(\tau, l_0, b_0)), \text{ i.e. } \overrightarrow{\mathscr{J}}(\tau, l_0, b_0) \in \mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_0),$$

(31)
$$\overleftarrow{\mathscr{J}}(\tau, l_0, b_0) = \overleftarrow{\mathscr{H}}(\tau, l_0, \overleftarrow{\mathscr{J}}^{\circ}(\tau, l_0, b_0)), \text{ i.e. } \overleftarrow{\mathscr{J}}(\tau, l_0, b_0) \in \mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_0),$$

(32) $$l_0 = \overrightarrow{\mathscr{H}}(\tau, \overrightarrow{\mathscr{J}}(\tau, l_0, b_0), l_0^{\circ}),$$

(33) $$l_0 = \overrightarrow{\mathscr{H}}(\tau, \overleftarrow{\mathscr{J}}(\tau, l_0, b_0), l_0^{\circ}),$$

(34) $$b_0 = \overleftarrow{\mathscr{J}}(\tau, \overrightarrow{\mathscr{H}}(\tau, b_0, 0), \overrightarrow{\mathscr{J}}(\tau, l_0, b_0)),$$

(35) $$b_0 = \overrightarrow{\mathscr{J}}(\tau, \overrightarrow{\mathscr{H}}(\tau, b_0, 0), \overleftarrow{\mathscr{J}}(\tau, l_0, b_0)).$$

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REMARKS. 1. If $l(\cdot)$ is a solution of (E_L) and $b(\cdot)$ is a solution of (E_B), then (iii) and (iv) of the preceding theorem tell us that $\overrightarrow{\mathscr{J}}(\cdot, l(\cdot), b(\cdot))$ is a solution of (E_B) on the center integral manifold $\mathscr{M}\overleftarrow{\mathscr{H}}(\tau, l_0)$ which is α-β-asymptotic to $l(\cdot; \tau, l_0)$.

2. As we will see in the subsequent proof, there is a solution $b_2(\cdot)=\mathscr{J}^{\alpha}_{BLB}(\cdot, l(\cdot), b(\cdot))$ of (E_B) with the following property:

$$_{\alpha}\|\overset{o}{b}{}^{+}_{2}(\cdot) - \overset{o}{b}{}^{+}_{1}(\cdot)\|_{\tau} < \infty, \qquad _{\beta}\|b^{\bar{o}}_{2}(\cdot) - \overrightarrow{\mathscr{J}}^{\bar{o}}(\cdot, l(\cdot), b(\cdot))\|^{\tau} < \infty.$$

$b_2(\cdot)$ is called the positive asymptotic phase for $b(\cdot)$ and $\overrightarrow{\mathscr{J}}(\cdot, l(\cdot), b(\cdot))$ is called the negative asymptotic phase for $b_2(\cdot)$. Therefore one can call $\overrightarrow{\mathscr{J}}(\cdot, l(\cdot), b(\cdot))$ the negative-positive asymptotic phase of $b(\cdot)$ on the center integral manifold α-β-asymptotic to $l(\cdot)$. Consequently one can call $\overrightarrow{\mathscr{J}}(\cdot, l(\cdot), b(\cdot))$ the positive-negative asymptotic phase of $b(\cdot)$ on the center integral manifold α-β-asymptotic to $l(\cdot)$. The identity (34) just says that the effect of $\overrightarrow{\mathscr{J}}$ on a solution $b(\cdot)$ of (E_B) is reversed by $\overleftarrow{\mathscr{J}}$ in a certain sense. The argument $\overrightarrow{\mathscr{H}}(\tau, b_0, 0)$ in $\overleftarrow{\mathscr{J}}$ contains the information about the original center integral manifold α-β-asymptotic to $b(\cdot)$ in terms of the linearizing homeomorphism $\overrightarrow{\mathscr{H}}$.

PROOF. (ii) and (iii) are direct consequences of the definitions of $\overrightarrow{\mathscr{J}}$ and $\overleftarrow{\mathscr{J}}$ and Theorem 3.5(ii), (iii).

(i) We only show the second lines in both inequalities, the others are dual.

First, using the estimates given in Theorem 3.5(i) and observing (9) we have

$$\begin{aligned}
&\begin{pmatrix} |\mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0) - l_0| \\ |\mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0) - b_0| \end{pmatrix} \\
&\quad \le \frac{K}{\gamma M_3} \cdot \begin{pmatrix} M_2 + K^2 M & K \\ K & M_2 + K^2 M \end{pmatrix} \begin{pmatrix} _{\alpha}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|^{\tau} \\ _{\beta}\|\mathscr{L}\mathscr{D}_B b(\cdot; \tau, b_0)\|_{\tau} \end{pmatrix} \\
&\quad = \frac{K}{\gamma M_3} \cdot \begin{pmatrix} M_2 + K^2 M \\ K \end{pmatrix} \cdot {}_{\alpha}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|^{\tau}.
\end{aligned}$$

By duality we also have

$$\begin{pmatrix} |\mathscr{J}^{\beta}_{BBL}(\tau, b_0, l_0) - b_0| \\ |\mathscr{J}^{\beta}_{BBL}(\tau, b_0, l_0) - l_0| \end{pmatrix} \le \frac{K}{\gamma M_3} \cdot \begin{pmatrix} K \\ M_2 + K^2 M \end{pmatrix} \cdot {}_{\beta}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|_{\tau}.$$

Now, inserting the definition of $\overrightarrow{\mathscr{J}}$, we see that

$$\begin{aligned}
|\overrightarrow{\mathscr{J}}(\tau, l_0, b_0) - l_0| &= |\mathscr{J}^{\beta}_{BBL}(\tau, \mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0), l_o) - l_0| \\
&\le \frac{K(M_2 + K^2 M)}{\gamma M_3} \cdot {}_{\beta}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|_{\tau}
\end{aligned}$$

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and

$$
\begin{aligned}
&|\overrightarrow{\mathscr{J}}(\tau, l_0, b_0) - b_0| \\
&\quad \le |\mathscr{J}^{\beta}_{BBL}(\tau, \mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0), l_o) - \mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0)| + |\mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0) - b_0| \\
&\quad \le \frac{K^2}{\gamma M_3} \cdot \Big({}_{\beta}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, \mathscr{J}^{\alpha}_{BLB}(\tau, l_0, b_0))\|_{\tau} + {}_{\alpha}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, b_0)\|^{\tau} \Big).
\end{aligned}
$$

(iv) We only show the identities (27), (29) and (31). The others are dual.
Let $b_1(\cdot) := b(\cdot; \tau, b_0)$ and $l_1(\cdot) := l(\cdot; \tau, l_0)$, $l_2(\cdot) := \overrightarrow{\mathscr{H}}(\cdot, b_1(\cdot), 0)$.
For $i = 2, \ldots, 5$ we define the mappings $b_i \colon \mathbb{T} \to X$ by

$$
\begin{aligned}
&b_2 \colon t \mapsto \mathscr{J}^{\alpha}_{BLB}(t, l_1(t), b_1(t)), \\
&b_3 \colon t \mapsto \mathscr{J}^{\beta}_{BBL}(t, b_2(t), l_1(t)) = \overrightarrow{\mathscr{J}}(t, l_1(t), b_1(t)), \\
&b_4 \colon t \mapsto \mathscr{J}^{\beta}_{BBL}(t, b_3(t), l_2(t)), \\
&b_5 \colon t \mapsto \mathscr{J}^{\alpha}_{BLB}(t, l_2(t), b_4(t)) = \overleftarrow{\mathscr{J}}(t, l_2(t), b_3(t)).
\end{aligned}
$$

From Theorem 3.5(iii) we know that all these functions are solutions of (E_B). and that there are the following estimates:

$$
\begin{aligned}
&\text{for} \quad b_2(\cdot): {}_{\alpha}\|b_2^-(\cdot) - l_1^-(\cdot)\|^{\tau} < \infty, {}_{\alpha}\|b_2^{\overset{o}{+}}(\cdot) - b_1^{\overset{o}{+}}(\cdot)\|_{\tau} < \infty, \\
&\text{for} \quad b_3(\cdot): {}_{\beta}\|b_3^{\overline{o}}(\cdot) - b_2^{\overline{o}}(\cdot)\|^{\tau} < \infty, {}_{\beta}\|b_3^{+}(\cdot) - l_1^{+}(\cdot)\|_{\tau} < \infty, \\
&\text{for} \quad b_4(\cdot): {}_{\beta}\|b_4^{\overline{o}}(\cdot) - b_3^{\overline{o}}(\cdot)\|^{\tau} < \infty, {}_{\beta}\|b_4^{+}(\cdot) - l_2^{+}(\cdot)\|_{\tau} < \infty, \\
&\text{for} \quad b_5(\cdot): {}_{\alpha}\|b_5^-(\cdot) - l_2^-(\cdot)\|^{\tau} < \infty, {}_{\alpha}\|b_5^{\overset{o}{+}}(\cdot) - b_4^{\overset{o}{+}}(\cdot)\|_{\tau} < \infty.
\end{aligned}
$$

From the first two lines we see:

$$
{}_{\beta}\|b_3^{+}(\cdot) - l_1^{+}(\cdot)\|_{\tau} < \infty
$$

and

$$
\begin{aligned}
{}_{\alpha}\|b_3^-(\cdot) - l_1^-(\cdot)\|^{\tau} &\le {}_{\alpha}\|b_3^-(\cdot) - b_2^-(\cdot)\|^{\tau} + {}_{\alpha}\|b_2^-(\cdot) - l_1^-(\cdot)\|^{\tau} \\
&\le {}_{\beta}\|b_3^{\overline{o}}(\cdot) - b_2^{\overline{o}}(\cdot)\|^{\tau} + {}_{\alpha}\|b_2^-(\cdot) - l_1^-(\cdot)\|^{\tau} < \infty.
\end{aligned}
$$

Therefore $b_3(\cdot) = \overrightarrow{\mathscr{J}}(\cdot, l_1(\cdot), b_1(\cdot))$ lies on the α-β-center manifold around $l_1(\cdot)$. Theorem 4.1 (iii) then implies:

$$
\overrightarrow{\mathscr{J}}(\cdot, l_0, b_0) = \overleftarrow{\mathscr{H}}(\cdot, l_0, \overrightarrow{\mathscr{J}}^{\circ}(\cdot, l_0, b_0)).
$$

Inserting τ yields (30).

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Application of the mapping $\overrightarrow{\mathscr{H}}$ into equation (27) yields, when observing Theorem 4.2 (iii) (A) $\Longleftrightarrow$ (B):

$$l_1(\cdot) = \overrightarrow{\mathscr{H}}(\cdot, \overleftarrow{\mathscr{H}}(\cdot, l_1(\cdot), \overrightarrow{\mathscr{J}}^\circ(\cdot, l_1(\cdot), b(\cdot)), l_1^\circ(\cdot))) = \overrightarrow{\mathscr{H}}(\cdot, \overrightarrow{\mathscr{J}}(\cdot, l_1(\cdot), b(\cdot)), l_1^\circ(\cdot)),$$

which is just identity (32).

For $b_2(\cdot)$ we have (when observing the definition of $l_2(\cdot)$ and Theorem 4.2 (iii))

$$\begin{aligned} {}_\beta\|b_2^+(\cdot) - l_2^+(\cdot)\|_\tau &\le {}_\beta\|b_2^+(\cdot) - b_1^+(\cdot)\|_\tau + {}_\beta\|b_1^+(\cdot) - l_2^+(\cdot)\|_\tau \\ &\le {}_\alpha\|b_2^{\overset{o}{+}}(\cdot) - b_1^{\overset{o}{+}}(\cdot)\|_\tau + {}_\beta\|b_1^+(\cdot) - l_2^+(\cdot)\|_\tau < \infty. \end{aligned}$$

Therefore $b_2(\cdot)$ and $b_4(\cdot)$ are solutions of the following BB-problem which is uniquely solvable (cf. Theorem 3.5):

$$(\mathrm{E}_B), \qquad {}_\beta\|y(\cdot) - b_3^{\overline{o}}(\cdot)\|^\tau < \infty, \qquad {}_\beta\|w(\cdot) - l_2^+(\cdot)\|_\tau < \infty,$$

hence they are identical.

Furthermore $b_1(\cdot)$ and $b_5(\cdot)$ are solutions of the BB-problem

$$(\mathrm{E}_B), \qquad {}_\alpha\|u(\cdot) - l_2^-(\cdot)\|^\tau < \infty, \qquad {}_\alpha\|z(\cdot) - b_2^{\overset{o}{+}}(\cdot)\|_\tau < \infty,$$

they are identical as well. If one here inserts the definitions of $b_1(\cdot)$ and $b_5(\cdot)$ and then evaluates these two functions at the point $t = \tau$, then one arrives at the identity (34).

4.4. Generalized Theorem of Hartman-Grobman In addition to the equation (E_B) and the corresponding linear homogeneneous equation (E_L) we consider the so-called reduced equation

$$\begin{aligned} u^\Delta &= A^-(t)u, \\ v^\Delta &= A^\circ(t)v + B^\circ(t, \overleftarrow{\mathscr{H}}(t, 0, v)), \qquad (\mathrm{E}_R) \\ w^\Delta &= A^+(t)w. \end{aligned}$$

Let again the conditions (I), (II_B) and (III_B) be fulfilled.

We define the two mappings

$$\overrightarrow{\mathscr{G}}: \begin{cases} \mathbb{T} \times X & \to X, \\ (\tau, b_0) & \mapsto (\overrightarrow{\mathscr{H}}^-(\tau, b_0, 0), \overrightarrow{\mathscr{J}}^\circ(\tau, 0, b_0), \overrightarrow{\mathscr{H}}^+(\tau, b_0, 0)), \end{cases}$$

and

$$\overleftarrow{\mathscr{G}}: \begin{cases} \mathbb{T} \times X & \to X, \\ (\tau, r_0) & \mapsto \overleftarrow{\mathscr{J}}(\tau, r_0, \overleftarrow{\mathscr{H}}(\tau, 0, r_0^\circ)). \end{cases}$$

Then the following generalization of the Hartman-Grobman-Theorem holds:

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THEOREM 4.5 (Generalized Hartman-Grobman-Theorem).

(i) *The mappings* $\overrightarrow{\mathscr{G}}$ *and* $\overleftarrow{\mathscr{G}}$ *fulfill the following estimates:*

$$\begin{pmatrix} |\overrightarrow{\mathscr{G}}^{-}(\tau, b_0) - b_0^{-}| \\ |\overrightarrow{\mathscr{G}}^{\circ}(\tau, b_0) - b_0^{\circ}| \\ |\overrightarrow{\mathscr{G}}^{+}(\tau, b_0) - b_0^{+}| \end{pmatrix} \leq \frac{K}{\gamma} \cdot \begin{pmatrix} {}_{\alpha}\|\mathscr{L}\mathscr{D}_L^{-} b(\cdot; \tau, b_0)\|^{\tau} \\ \frac{K}{M_3} \cdot \left({}_{\beta}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, \mathscr{J}_{BLB}^{\alpha}(\tau, l_0, b_0))\|_{\tau} + {}_{\alpha}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, b_0)\|^{\tau}\right) \\ {}_{\beta}\|\mathscr{L}\mathscr{D}_L^{+} b(\cdot; \tau, b_0)\|_{\tau} \end{pmatrix},$$

$$|\overleftarrow{\mathscr{G}}(\tau, r_0) - r_0| \leq \frac{K(M_2 + K^2 M)}{\gamma M_3} \cdot {}_{\beta}\|\mathscr{L}\mathscr{D}_B l(\cdot; \tau, l_0)\|_{\tau}.$$

(ii) $\overrightarrow{\mathscr{G}}$ *and* $\overleftarrow{\mathscr{G}}$ *are continuous functions.*

(iii) *If* $b(\cdot)$ *is a solution of* (E_B), *then* $r(\cdot) := \overrightarrow{\mathscr{G}}(\cdot, b(\cdot))$ *a is solution of* (E_R) *with*

$${}_{\alpha}\|u(\cdot) - b^{-}(\cdot)\|^{\tau} < \infty, \qquad {}_{\beta}\|w(\cdot) - b^{+}(\cdot)\|_{\tau} < \infty.$$

If $r(\cdot)$ *is a solution of* (E_R), *then* $b(\cdot) := \overleftarrow{\mathscr{G}}(\cdot, r(\cdot))$ *is a solution of* (E_B) *with*

$${}_{\alpha}\|u(\cdot) - r^{-}(\cdot)\|^{\tau} < \infty, \qquad {}_{\beta}\|w(\cdot) - r^{+}(\cdot)\|_{\tau} < \infty.$$

For $(\tau, x) \in \mathbb{T} \times X$ *we have:*

$$\overleftarrow{\mathscr{G}}(\tau, \overrightarrow{\mathscr{G}}(\tau, x)) = x \qquad \text{and} \qquad \overrightarrow{\mathscr{G}}(\tau, \overleftarrow{\mathscr{G}}(\tau, x)) = x. \tag{36}$$

$\overleftarrow{\mathscr{G}}$ *and* $\overrightarrow{\mathscr{G}}$ *are fiber homeomorphisms, that is for each fixed* $\tau \in \mathbb{T}$ $\overleftarrow{\mathscr{G}}(\tau, \cdot)$ *and* $\overrightarrow{\mathscr{G}}(\tau, \cdot)$ *are homeomorphisms inverse to each other.*

REMARK. As already outlined in the introduction, the mapping $\overrightarrow{\mathscr{G}}$ conveys a certain information about the solution $b(\cdot)$ of (E_B). The function $\overrightarrow{\mathscr{G}}^{\mp}(\cdot, b(\cdot))$ — the first and third component of $\overrightarrow{\mathscr{G}}(\cdot, b(\cdot))$ — is a solution of the linear equations in (E_R). These solutions characterize the center integral manifold for $b(\cdot)$ via its α-β-asymptotic behaviour. In the center component of $\overrightarrow{\mathscr{G}}(\cdot, b(\cdot))$we conserve the information about the asymptotic phase of $b(\cdot)$ on the center integral manifold α-β-asymptotic to the zero function. So this center component solves the (middle) equation of (E_R), which is called the reduced equation (cf. Theorem 3.3). It describes the flow on the center integral manifold of (E_B) α-β-asymptotic to the zero function. From the fact that $\overrightarrow{\mathscr{G}}$ and $\overleftarrow{\mathscr{G}}$ are inverse to each other (as described in (33)) we see that these two features (manifold and asymptotic phase) are sufficient to uniquely characterize solutions of (E_B). For the transformation via $\overleftarrow{\mathscr{G}}$ the asymptotic phase is needed to single the original solution of (E_B) out of its center integral manifold.

PROOF. (ii) is a consequence of the continuity of the mappings $\overrightarrow{\mathscr{H}}, \overleftarrow{\mathscr{H}}, \overrightarrow{\mathscr{J}}$ and $\overleftarrow{\mathscr{J}}$. (i) directly follows from the corresponding estimates for the mappings $\overrightarrow{\mathscr{H}}, \overleftarrow{\mathscr{H}}, \overrightarrow{\mathscr{J}}$ and $\overleftarrow{\mathscr{J}}$. One has to observe the definitions of $\overrightarrow{\mathscr{G}}$ respectively $\overleftarrow{\mathscr{G}}$.

(iii) We first consider the following diagram:

$$\begin{array}{ccccc}
\mathbb{T} \times X \longleftrightarrow & & \overbrace{\mathscr{M}\overleftarrow{\mathscr{H}}(t_0, 0) \times_{\mathbb{T}} (\mathbb{T} \times X^{\mp})}^{=:\mathscr{M}\mathscr{P}} & & \longleftrightarrow \mathbb{T} \times X \\
(\tau, b_0) & \overset{\mathscr{G}_1}{\mapsto} & (\tau, \overrightarrow{\mathscr{J}}(\tau, 0, b_0), \overrightarrow{\mathscr{H}}(\tau, b_0, 0)) = (\tau, x_0, l_0) & \overset{\mathscr{G}_2}{\mapsto} & (\tau, l_0^-, x_0^\circ, l_0^+) \\
\overleftarrow{\mathscr{J}}(\tau, l_0, x_0) & \overset{\mathscr{G}_4}{\leftarrow} & (\tau, x_0, l_0) = (\tau, \overleftarrow{\mathscr{H}}(\tau, 0, r_0^\circ), (r_0^-, 0, r_0^+)) & \overset{\mathscr{G}_3}{\leftarrow} & (\tau, r_0)
\end{array}$$

$\mathscr{M}\mathscr{P}$ is defined as the fiber bundle product of the two fiber bundles $\mathscr{M}\overleftarrow{\mathscr{H}}(t_0, 0)$ (α-β-integral center manifold asymptotic to the zero function, t_0 is chosen fixed) and $\mathbb{T} \times X^{\mp}$ (trivial bundle of the 'hyperbolic part' over $\mathbb{T}$). By definition we have $\overrightarrow{\mathscr{G}} = \mathscr{G}_2 \circ \mathscr{G}_1$ and $\overleftarrow{\mathscr{G}} = \mathscr{G}_4 \circ \mathscr{G}_3$. We have to show: $\mathscr{G}_1$ and $\mathscr{G}_4$ are bundle homeomorphisms inverse to each other. The same with $\mathscr{G}_2$ and $\mathscr{G}_3$.

The identity (34) (with $l_0 = 0$) yields for arbitrary $(\tau, b_0) \in \mathbb{T} \times X$:

$$(\mathscr{G}_4 \circ \mathscr{G}_1)(\tau, b_0) = (\tau, \overleftarrow{\mathscr{J}}(\tau, \overrightarrow{\mathscr{H}}(\tau, b_0, 0), \overrightarrow{\mathscr{J}}(\tau, 0, b_0)) = (\tau, b_0),$$

so we have $\mathscr{G}_4 \circ \mathscr{G}_1 = \mathrm{id}|_{\mathbb{T}\times X}$. For arbitrary $(\tau, x_0, l_0) \in \mathscr{M}\mathscr{P}(\tau, 0)$ we have:

$$(\mathscr{G}_1 \circ \mathscr{G}_4)(\tau, x_0, l_0) = (\tau, \overrightarrow{\mathscr{J}}(\tau, 0, \overleftarrow{\mathscr{J}}(\tau, l_0, x_0), \overrightarrow{\mathscr{H}}(\tau, \overleftarrow{\mathscr{J}}(\tau, l_0, x_0), 0)) \overset{*}{=} (\tau, x_0, l_0).$$

Thus it follows that: $\mathscr{G}_1 \circ \mathscr{G}_4 = \mathrm{id}|_{\mathscr{M}\mathscr{P}}$. For the second and third component in $*$ the following considerations were necessary.

Because of $(\tau, x_0) \in \mathscr{M}\overleftarrow{\mathscr{H}}(t_0, 0)$ (center manifold around the zero function) we have $\overrightarrow{\mathscr{H}}(\tau, x_0, 0) = 0$. Identity (35) just yields $*$ for the second component. Because of $l_0^\circ = 0$ we get the equality of the third components in $*$ by (33).

Let $(\tau, x_0, l_0) \in \mathscr{M}\mathscr{P}$. Then we have:

$$(\mathscr{G}_3 \circ \mathscr{G}_2)(\tau, x_0, l_0) = (\tau, \overleftarrow{\mathscr{H}}(\tau, 0, x_0^\circ), (l_0^-, 0, l_0^+)) = (\tau, x_0, l_0)$$

because of $x_0 = \overleftarrow{\mathscr{H}}(\tau, 0, x_0^\circ) \iff (\tau, x_0) \in \mathscr{M}\overleftarrow{\mathscr{H}}(t_0, 0)$ and $l_0 = (l_0^-, 0, l_0^+) \iff (\tau, l_0) \in \mathbb{T} \times X^{\mp}$. We conclude $\mathscr{G}_3 \circ \mathscr{G}_2 = \mathrm{id}|_{\mathscr{M}\mathscr{P}}$.

We have

$$\mathscr{G}_2 \circ \mathscr{G}_3(\tau, r_0) = (\tau, r_0^-, \overleftarrow{\mathscr{H}}^\circ(\tau, 0, r_0^\circ), r_0^+) = (\tau, r_0);$$

thus $\mathscr{G}_2 \circ \mathscr{G}_3 = \mathrm{id}|_{\mathbb{T}\times X}$.

References

[1] B. Aulbach, 'Hierarchies of invariant manifolds', *J. Nigerian Math. Soc.* **6** (1987), 71–89.

[2] B. Aulbach and S. Hilger, 'Linear dynamic processes with inhomogeneous time scales', in: *Nonlinear dynamics and quantum dynamical systems* (Akademie, Berlin, 1990).

[3] W.-J. Beyn, 'On the numerical approximation of phase portraits near stationary points', *SIAM J. Numer. Anal.* **24**, (1987), 1095-1113.

[4] W.-J. Beyn and J. Lorenz, 'Center manifolds of dynamical systems under discretization', *Numer. Funct. Anal. Optim.* **9** (1987), 381-414.

[5] F. R. Gantmacher, *Matrizentheorie* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1986).

[6] D. Henry, *Invariant manifolds* (Lecture Notes, Dept. Nat. Aplicada, Universidad de Sao Paulo, 1983).

[7] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten* (Doctoral Dissertation, Univ. Würzburg, 1988).

[8] ———, 'Analysis on measure chains — A unified approach to continuous and discrete calculus', *Resultate Math.* **18** (1990), 18–56.

[9] ———, 'Smoothness of invariant manifolds', *J. Funct. Anal.* **106** (1992), 95–129.

[10] A. Kelley, 'The stable, center-stable, center, center-unstable, unstable manifolds', *J. Differential Equations* **3** (1967), 546-570.

[11] U. Kirchgraber, 'The geometry in the neighborhood of an invariant manifold and topological conjugacy', *Research Report* 83-01, *Sem. Ang. Math.*, ETH *Zürich* (1983).

[12] U. Kirchgraber and K. J. Palmer, *Geometry in the neighborhood of invariant manifolds of maps and flows and linearization* (Pitman, London, 1991).

[13] P. E. Kloeden and J. Lorenz, 'Stable attracting sets in dynamical systems and their one-step discretization', *SIAM J. Numer. Anal.* **23** (1986), 986–995.

[14] K. J. Palmer, 'Qualitative behavior of a system of ODE near an equilibrium point – a generalization of the Hartman-Grobman-Theorem', *Preprint* 372, *Inst. Ang. Math., Univ. Bonn*, 1980.

[15] A. Vanderbauwhede and S. A Van Gils, 'Center manifolds and contractions on a scale of Banach spaces', *J. Funct. Anal.* **72** (1987), 209-224.

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