A Theorem on the Integral of Stieltjes.

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§ 1. *Introduction.*

In a recent paper * Mr J. M. Whittaker has given the following Theorem :—

If $\psi(x)$ *be the indefinite Riemann integral of a bounded positive function* $g(x)$, *and if* $f(x)$ *be any bounded function, then the equation*

$$\int_a^b f(x) \; g(x) \; dx = \int_a^b f(x) \; d\psi(x)$$

is true whenever either side exists.

It was suggested to me by Mr E. T. Copson that a similar result would probably hold if $\psi(x)$ is the Stieltjes integral with respect to a monotone function of a positive bounded function. The Theorem here proved includes Whittaker's Theorem as a special case, and is much more general than that suggested by Copson, viz. :—

Theorem : *Data*: (i) *The function* $\psi(x) = \int_a^x g(x) \; d\phi(x)$ *is well-defined* † *in* (a, b)

(ii) $f(x)$ *is bounded in* (a, b), *and* $-F \leq f(x) \leq F$.

Result: $\int_a^b f(x) \, d\psi(x) = \int_a^b f(x) \, g(x) \, d\phi(x)$ *whenever either integral exists.*

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* *Proc. Lond. Math. Soc.*, Ser. II, Vol. 25 (1926), p. 213.

† It is only necessary to postulate the existence of $\int_a^b g(x) \, d\phi(x)$, since that of $\int_a^x g(x) \, d\phi(x)$ may then be deduced. [See Pollard: *Quarterly Jo.*, Vol 49 (1923), p. 76 (II)].

Simple cases of the Theorem are already known: thus when $f(x)$, $g(x)$ are continuous, and $\phi(x)$ is of bounded variation, no trouble arises as to the existence of the integrals concerned, and their equality is readily proved.*

§ 2. *The Stieltjes Integral.*

Two definitions of the Stieltjes integral are recognised. That adopted in the present paper was given by Stieltjes,† and may be stated as follows:

Let $g(x)$, $\phi(x)$ be any two real functions defined in the interval (a, b); and let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be a finite set of intervals which together make up (a, b). $\Delta_r \phi$ denotes the "increment" of $\phi(x)$ in Δ_r. Let ξ_r be any point of Δ_r.

Form the sum $S \equiv \sum_{r=1}^{r=n} g(\xi_r) \Delta_r \phi$.

Suppose that, given any ϵ, we can assign an η, such that every sum like S differs from a fixed constant L by less than ϵ, provided only that, for all values of r concerned, $\Delta_r < \eta$. Then L is defined to be the value of the Stieltjes integral $\int_a^b g(x)\, d\phi(x)$.

The second definition is similar to that of Darboux for the Riemann integral. It is applicable only if $\phi(x)$ is monotone, though capable of extensions. With certain restrictions, this definition is equivalent—wherever applicable—to the one given above. Such an equivalent definition is used in Whittaker's paper.‡

Pollard§ has discussed a more general form of the "Darboux" definition, but we do not use his work here.

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* *Cf.* CARLEMANN: *Equations Intégrales Singulières à Noyau Réel et Symétrique* (*Uppsala* 1923), p. 11.

† *Ann. Fac. Sc. Toulouse*, VIII (1894).

‡ See HOBSON: *Functions of a Real Variable* (Second Ed.), Vol. I, p. 506, sqq., along with *Addendum* to p. 508 in Vol. II, p. 774.

§ *Quarterly Jo.*, Vol. 49 (1923), p. 73.

§ 3. *The Theorem Proved (Case I).*

We now prove the Theorem for the case when $\phi(x)$ is monotone—say non-decreasing—in (a, b) and $g(x)$ is bounded.

Since $\psi(x)$ is well defined in (a, b), given any ϵ, we can assign an η, such that

$$\left| \int_a^b g(x)\, d\phi(x) - \sum_r g(\xi_r) \Delta_r \phi \right| \leqq \epsilon \text{ if only } \Delta_r < \eta \quad \ldots\ldots\ldots\ldots(1)$$

Suppose, then, any sum like S chosen in accordance with (1). In the sum we may clearly replace $g(\xi_r)$ by G_r, the upper bound, or by g_r, the lower bound of $g(x)$ in Δ_r.

Hence $$\sum_r (G_r - g_r) \Delta_r \phi \leqq 2\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now it is evident from definition that

$$G_r \Delta_r \phi \geqq \left\{ \begin{matrix} \Delta_r \psi \\ g(\xi_r) \Delta_r \phi \end{matrix} \right\} \geqq g_r \Delta_r \phi$$

Hence by (2), $$\sum_r \mid \Delta_r \psi - g(\xi_r) \Delta_r \phi \mid \leqq 2\epsilon \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

And so $$\mid \sum_r f(\xi_r) \Delta_r \psi - \sum_r f(\xi_r) g(\xi_r) \Delta_r \phi \mid$$

$$\leqq \sum_r \mid f(\xi_r) \mid \quad \mid \Delta_r \psi - g(\xi_r) \Delta_r \phi \mid \leqq 2\epsilon F \quad \ldots\ldots\ldots\ldots(4)$$

Suppose now that $\int_a^b f(x) \ g(x) \ d\phi(x)$ exists. Then we can choose η in inequality (1) so small that, in addition to the restriction already placed upon it,

$$\left| \int_a^b f(x)\, g(x)\, d\phi(x) - \sum_r f(\xi_r)\, g(\xi_r)\, \Delta_r \phi \right| < \epsilon F \quad \ldots\ldots(5)$$

Thus, by (4), $$\left| \int_a^b f(x)\, g(x)\, d\phi(x) - \sum_r f(\xi_r) \Delta_r \psi \right| < 3\epsilon F$$

provided only $\Delta_r < \eta$. ......(6)

Now ξ_r is *any* point in the interval Δ_r. Thus inequality (6) defines $\int_a^b f(x)\, d\psi(x)$ to have the value $\int_a^b f(x)\, g(x)\, d\phi(x)$.

Similarly if we postulate the existence of $\int_a^b f(x)\, d\psi(x)$, we can prove, from (4), that $\int_a^b f(x)\; g(x)\; d\phi(x)$ also exists and has the same value.

The above case reduces to Whittaker's theorem if $\phi(x) = x$ and $g(x)$ is positive.

§4. *The Theorem Proved* (*General Case*).

We now remove the restriction that $\phi(x)$ be monotone. All the above reasoning remains valid in the general case, except the mode of deriving inequality (3) We therefore only require to prove the Lemma :—

LEMMA : *Data*: $\psi(x) = \int_a^x g(x)\; d\phi(x), \qquad [a \leq x \leq b].$

Result: *Given any* ϵ, *we can assign an* η, *such that, for every finite set of intervals* Δ_r *into which* (a, b) *is divided, and every value of* ξ_r *within* Δ_r,

$Q \equiv \Sigma \mid \Delta_r \psi - g(\xi_r)\, \Delta_r \phi \mid < \epsilon$

if, for all values of r concerned, $\Delta_r < \eta$.

Suppose, if possible that the Lemma is false. Then there must exist a non-zero positive number α, and, corresponding to any given value of η, at least one sum Q^* like Q, such that $Q^* > \alpha$ but $\Delta_r < \eta$ (all r). ....(7)

Suppose η chosen to satisfy inequality (1), with $\epsilon = \frac{\alpha}{6}$; and consider any sum Q^* which satisfies inequality (7). Let P be the sum of all terms of Q^* for which $\Delta_r \psi \geq g(\xi_r)\, \Delta_r \phi$, and N the sum of those terms of Q^* for which $\Delta_r \psi < g(\xi_r)\, \Delta_r \phi$. Let $\Sigma_r^{(1)}$, $\Sigma_r^{(2)}$ denote summations with respect to r over values corresponding to terms of P, N respectively.

Now $P + N = Q^* > \alpha$. Hence either $P > \frac{\alpha}{2}$ or $N > \frac{\alpha}{2}$.

Suppose $P \equiv \Sigma_r^{(1)} \{\Delta_r \psi - g(\xi_r)\, \Delta_r \phi\} > \frac{\alpha}{2} > 3\epsilon$ ....(8)

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Since $\Delta_r \psi \equiv \int_{\Delta_r} g(x)\, d\phi(x)$ exists as a Stieltjes integral,† we may form for each Δ_r a sum S_r like S, [§ 2], such that

$$| \Delta_r \psi - S_r | < \frac{\epsilon}{m}$$

where m is the number of terms in the summation $\Sigma^{(1)}$.

Then $| \sum_r^{(1)} \Delta_r \psi - \sum_r^{(1)} S_r | < \epsilon.$

Hence, by (8), $\sum_r^{(1)} S_r - \sum_r^{(1)} g(\xi_r) \Delta_r \phi > 2\epsilon$

And so, by (1), $\left| \int_a^b g(x)\, d\phi(x) - \{ \sum_r^{(1)} S_r + \sum_r^{(2)} g(\xi_r) \Delta_r \phi \} \right| > \epsilon.$

But the bracket contains a sum like S, for which every subinterval concerned is of length less than η; and so we have a contradiction to inequality (1).

Similarly we obtain a contradiction if $N > \frac{\alpha}{2}$.

Thus the Lemma must be true, and the Theorem is proved.

It may be noted that we have not postulated the boundedness of $g(x)$ in the proof of the Lemma. We therefore dispense with this restriction in the statement of the Theorem, though the generality gained thereby is not great. The boundedness of $f(x)$ is essential to the truth of the Theorem.

§ 5. *An Application.*

The theorem proved has many applications. As an example we discuss very rapidly the Variation of $\psi(x)$, in the case when $\phi(x)$ is of Bounded Variation.

It is clear from definition of the integral that, for any sub-interval Δ of (a, b),

$$| \Delta\psi | \leq [\text{Upper Bound of } | g(x) | \text{ in } \Delta][\text{Var. } \phi(x) \text{ in } \Delta].$$

Applying this to each of any set of sub-intervals into which (a, b) is divided, we have easily:

$$\text{Var. } \psi(x) \Big]_a^b \leq \Big[\text{Upper Bd. of } | g(x) | \text{ in } (a, b) \Big] \times \text{Var. } \phi(x) \Big]_a^b \quad \text{......(9)}$$

† See footnote † p. 79.

We now show further that:

$$\text{Var. } \psi(x)\Big]_a^b \geq \Big[\text{Lower Bd. of } |\, g(x)\,| \text{ in } (a, b)\Big] \times \text{Var. } \phi(x)\Big]_a^b \quad \text{......(10)}$$

In fact, if the Lower Bound of $|\, g(x)\,|$ is zero, the inequality is obvious. If not, we may use the above Theorem, giving:

$$\int_a^x \frac{1}{g(x)}\, d\psi(x) = \int_a^x \frac{1}{g(x)} \cdot g(x)\, d\phi(x) = \phi(x) - \phi(a).$$

Hence, by inequality (9)

$$\text{Var. } \phi(x)\Big]_a^b \leq \Big[\text{Upper Bd. of } \left|\frac{1}{g(x)}\right| \text{ in } (a, b)\Big] \times \text{Var. } \psi(x)\Big]_a^b$$

and inequality (10) follows.

The argument may be carried a step further:

It may be shown that the existence of $\int_a^b g(x)\ d\phi(x)$ involves that of $\int_a^b |\, g(x)\,|\, d\Phi(x)$, where $\Phi(x)$ is the variation of $\phi(x)$ in (a, x). If, using this fact, we apply inequalities (9), (10) to each of a set of sub-intervals into which (a, b) is divided, we readily prove:

$$\text{Var. } \psi(x)\Big]_a^b = \int_a^b |\, g(x)\,|\, d\Phi(x).$$

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