QUASI-MULTIPLICATIVE MAPS ON BAER SEMIGROUPS

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1. Introduction. A *Baer semigroup* is a semigroup S with 0 and 1 in which, for each $x \in S$, the left annihilator $L(x) = \{y \in S : yx = 0\}$ of x is a principal left ideal generated by an idempotent and the right annihilator $R(x) = \{y \in S : xy = 0\}$ of x is a principal right ideal generated by an idempotent. Baer semigroups are of interest because (see [**5**]) the left annihilators of the elements of a Baer semigroup S, $\mathscr{L}(S) = \{L(x) : x \in S\}$, form a bounded lattice and (see [**4**]) every bounded lattice arises in this manner. In this note we look at a type of map ϕ on a Baer semigroup S which has the property that $S\phi$ is a Baer semigroup. (The homomorphic image of a Baer semigroup need not be a Baer semigroup. For a case where it is, see [**7**].) When the Baer semigroup is specialized to a Boolean algebra, this type of map generalizes Halmos's notion of a quantifier.

2. Quasi-multiplicative maps.

DEFINITION 1. A map ϕ of a semigroup S into itself is *quasi-multiplicative* if $((x\phi)y)\phi = (x(y\phi))\phi = (x\phi)(y\phi)$ for all $x, y \in S$.

THEOREM 2. *If S is a Baer semigroup and $\phi : S \to S$ is a zero-preserving, idempotent, quasi-multiplicative map, then $S\phi$ is a subsemigroup of S and a Baer semigroup in its own right.*

Proof. It is clear that $S\phi$ is a subsemigroup of S. If $x \in S$, let $(x\phi)^l$ be the idempotent that generates $L(x\phi)$, i.e., $L(x\phi) = S(x\phi)^l$. We claim that, in $S\phi$, we have $L(x\phi) = (S\phi)((x\phi)^l\phi)$, and that $(x\phi)^l\phi$ is an idempotent. From $((x\phi)^l\phi)(x\phi) = ((x\phi)^l x\phi)\phi = 0\phi = 0$ it follows that $(S\phi)((x\phi)^l\phi) \subseteq L(x\phi)$. It also follows that $(x\phi)^l\phi$ is idempotent, since $(y\phi)(x\phi) = 0 \Rightarrow y\phi = (y\phi)(x\phi)^l \Rightarrow y\phi = y\phi\phi = ((y\phi)(x\phi)^l)\phi = (y\phi)((x\phi)^l\phi)$. This last argument also shows that, in $S\phi$, $L(x\phi) \subseteq (S\phi)((x\phi)^l\phi)$. A dual argument shows that, in $S\phi$, we have $R(x\phi) = ((x\phi)^r\phi)(S\phi)$, where $(x\phi)^r$ is the idempotent that generates $R(x\phi)$ in S, and that $(x\phi)^r\phi$ is an idempotent.

We have as a corollary the following result from [**1**].

COROLLARY 3. *If S is a Baer semigroup and $e = e^2 \in S$, then eSe is a Baer semigroup.*

Proof. The map $x \mapsto exe$ is idempotent, zero-preserving and quasi-multiplicative.

REMARK. To see that a zero-preserving, idempotent, quasi-multiplicative map ϕ on a Baer semigroup S need not be of the form $x\phi = (1\phi)x(1\phi)$, notice that any map $\phi : S \to S$ such that range $\phi = \{0, 1\}$, $0\phi = 0$, and $1\phi = 1$ is idempotent and quasi-multiplicative.

THEOREM 4. *Any quasi-multiplicative map that preserves* 1 *is idempotent. If ϕ is any zero-preserving, idempotent, quasi-multiplicative map on a Baer semigroup S, then $\phi = \theta\psi$,*

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where there is an idempotent $e \in S$ such that $x\theta = exe$ and where ψ, considered as a map on $S\theta = eSe$, is zero-preserving, quasi-multiplicative and 1*-preserving.*

Proof. To see the first part, notice that $(x\phi)\phi = ((x\phi)1)\phi = (x\phi)(1\phi) = (x\phi)1 = x\phi$. For the second assertion let $e = 1\phi$. We have $(1\phi)(1\phi) = ((1\phi)1)\phi = 1\phi\phi = 1\phi$. Now let $\psi = \phi|_{eSe}$. ψ clearly preserves 1ϕ, which is the unit element of eSe. Finally, $((1\phi)x(1\phi))\phi = (1\phi)(x(1\phi))\phi = (1\phi)((x\phi)1)\phi = (1\phi)(x\phi) = (1(x\phi))\phi = x\phi\phi = x\phi$, which shows that $\phi = \theta\psi$ when $x\theta = exe$.

REMARK. It is shown in [1] that $\mathscr{L}(eSe)$ is isomorphic to the set of fixed points of a certain map on $\mathscr{L}(S)$.

3. Quasi-multiplicative residuated maps on a Boolean algebra. A Boolean algebra, considered as a semigroup with respect to the meet operation, is a Baer semigroup. In this lattice context the condition in Definition 1 becomes $(x \wedge y\phi)\phi = x\phi \wedge y\phi$. A map ϕ on a Boolean algebra L which has this property and which is zero-preserving and increasing ($x \leqq x\phi$ for all $x \in L$) is what Halmos calls [3, p. 220] a quantifier. Quantifiers were studied in the more general context of an orthomodular lattice by M. F. Janowitz in [6]. It turns out that quantifiers are residuated maps and that there is a notion of adjoint available for residuated maps on a Boolean algebra (or any other involution partially ordered set). On using this adjoint, it follows from Theorem 3 of [3] that the quantifiers on a Boolean algebra are precisely the increasing projections (ϕ being a projection if $\phi = \phi^2 = \phi^*$). We shall use this adjoint to characterize quasi-multiplicative residuated maps on a Boolean algebra.

A map ϕ of a lattice L into itself is *residuated* if the inverse image of a principal ideal is again a principal ideal or, equivalently, if ϕ is isotone and there is another isotone map ϕ^+ (called a *residual* map) of L into itself such that $x\phi^+\phi \leqq x \leqq x\phi\phi^+$ for all $x \in L$. Residuated maps are join homomorphisms and preserve the lattice zero if it has one. The semigroup of residuated maps on a lattice L will be denoted by $S(L)$. Under the pointwise partial order, $S(L)$ is a join semilattice with, for $\phi, \psi \in S(L)$, $\phi \vee \psi$ and $(\phi \vee \psi)^+$ given by $x(\phi \vee \psi) = x\phi \vee x\psi$ and $x(\phi \vee \psi)^+ = x\phi^+ \wedge x\psi^+$.

LEMMA 5. *If L is a Boolean algebra and $\phi \in S(L)$, then ϕ^*, given by $x\phi^* = (x'\phi^+)'$ for all $x \in L$, is residuated and the map $\phi \to \phi^*$ is a semigroup involution on $S(L)$.*

Proof. This follows from Lemma 22 of [5].

LEMMA 6. *Let L be a Boolean algebra. An isotone map $\phi : L \to L$ is residuated if and only if there exists an isotone map $\psi : L \to L$ such that, for each $x \in L$,*

(i) $(x\phi)'\psi \leqq x'$,
(ii) $(x\psi)'\phi \leqq x'$.

If such a ψ exists, it is unique and equals ϕ^*.

Proof. This follows from a proof on p. 651 of [2].

COROLLARY 7. *If L is a Boolean algebra and $\phi, \psi \in S(L)$, then $(\phi \vee \psi)^* = \phi^* \vee \psi^*$.*

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Proof. $(x(\phi \vee \psi))'(\phi^* \vee \psi^*) = (x\phi' \wedge x\psi')(\phi^* \vee \psi^*) = (x\phi' \wedge x\psi')\phi^* \vee (x\phi' \wedge x\psi')\psi^* \leqq x\phi'\phi^* \vee x\psi'\psi^* \leqq x'$ and similarly $(x(\phi^* \vee \psi^*))'(\phi \vee \psi) \leqq x'$.

LEMMA 8. *A quasi-multiplicative residuated map ϕ on a lattice L is zero-preserving (if L has a zero) and is idempotent.*

Proof. Since ϕ is residuated, it preserves the zero and joins; ϕ is idempotent since $x\phi\phi = (x\phi \wedge (x \vee x\phi))\phi = x\phi \wedge (x \vee x\phi)\phi = x\phi \wedge (x\phi \vee x\phi\phi) = x\phi$.

LEMMA 9. *If L is a Boolean algebra and $\phi \in S(L)$, then ϕ is quasi-multiplicative if $\phi^2 \leqq \phi$ and $\phi\phi^* \leqq \phi$.*

Proof. Notice first that $y\phi\phi^* \leqq y\phi \Rightarrow ((y\phi)'\phi^+)' \leqq y\phi \Rightarrow (y\phi)' \leqq (y\phi)'\phi^+ \Rightarrow (y\phi)'\phi \leqq (y\phi)' \Rightarrow y\phi \leqq ((y\phi)'\phi)'$. Notice also that, since $x\phi \leqq (x \vee (y\phi)')\phi = [(x \wedge y\phi) \vee (y\phi)']\phi = (x \wedge y\phi)\phi \vee (y\phi)'\phi$, we have $x\phi \wedge ((y\phi)'\phi)' \leqq (x \wedge y\phi)\phi$. Now $x\phi \wedge y\phi \leqq x\phi \wedge ((y\phi)'\phi)' \leqq (x \wedge y\phi)\phi$. On the other hand, $(x \wedge y\phi)\phi \leqq x\phi$ and $(x \wedge y\phi)\phi \leqq y\phi\phi \leqq y\phi$; so $(x \wedge y\phi)\phi \leqq x\phi \wedge y\phi$. This shows that $(x \wedge y\phi)\phi = x\phi \wedge y\phi$.

LEMMA 10. *If L is a Boolean algebra and $\phi \in S(L)$ is quasi-multiplicative, then $\psi \in S(L)$, given by $x\psi = x\phi \wedge (0\phi^+)'$, is a projection.*

Proof. $\psi \in S(L)$ since it is the composition of two residuated maps. (In a Boolean algebra $x \mapsto x \wedge e$ is residuated with $x \mapsto x \vee e'$ as its residual.) To show that $\psi = \psi^*$ it will suffice, by Lemma 6, to show that $(x\psi)'\psi \leqq x'$ for all $x \in L$. Now $(x\psi)'\psi = (x\phi \wedge (0\phi^+)')'\phi \wedge (0\phi^+)' = ((x\phi)' \vee 0\phi^+)\phi \wedge (0\phi^+)' = (x\phi)'\phi \wedge (0\phi^+)'$; so we have $(x\psi)'\psi \leqq x' \Leftrightarrow (x\phi)'\phi \wedge (0\phi^+)' \leqq x' \Leftrightarrow (x\phi)'\phi \wedge (0\phi^+)' \wedge x = 0 \Leftrightarrow (x\phi)'\phi \wedge x \leqq 0\phi^+ \Leftrightarrow ((x\phi)'\phi \wedge x)\phi = 0$. But $((x\phi)'\phi \wedge x)\phi = (x\phi)'\phi \wedge x\phi = ((x\phi)' \wedge x\phi)\phi = 0\phi = 0$; so we have $\psi = \psi^*$. To show that $\psi^2 = \psi$ it will be sufficient to show that $\psi^2 \leqq \psi$, for then we shall have $\psi\psi^* = \psi^2 \leqq \psi$, making ψ quasi-multiplicative by Lemma 9 and hence idempotent by Lemma 8. But $x\psi\psi = (x\phi \wedge (0\phi^+)')\phi \wedge (0\phi^+)' \leqq x\phi\phi \wedge (0\phi^+)' = x\phi \wedge (0\phi^+)' = x\psi$. This completes the proof.

REMARK. The converse of Lemma 10 is false. Consider an 8-element Boolean algebra with atoms a, b, and c. The map ψ_a given by

$$x\psi_a = \begin{cases} 0 & \text{if} \quad x \leqq a, \\ x \vee a & \text{if} \quad x \nleqq a \end{cases}$$

is residuated with ψ_a^+ given by

$$x\psi_a^+ = \begin{cases} x & \text{if} \quad a \leqq x, \\ a & \text{if} \quad a \nleqq x. \end{cases}$$

The map ψ_a is not quasi-multiplicative, since $(c' \wedge b'\psi_a)\psi_a = a\psi_a = 0$, whereas $c'\psi_a \wedge b'\psi_a = c' \wedge b' = a$. However,

$$x\psi_a \wedge (0\psi_a^+)' = x\psi_a \wedge a' = \begin{cases} 0 \wedge a' & \text{if} \quad x \leqq a, \\ (x \vee a) \wedge a' & \text{if} \quad x \nleqq a \end{cases} = \begin{cases} x \wedge a' & \text{if} \quad x \leqq a, \\ x \wedge a' & \text{if} \quad x \nleqq a \end{cases} = x \wedge a';$$

so ψ is a projection.

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THEOREM 11. *Let L be a Boolean algebra and $\phi \in S(L)$. The following are equivalent.*

(a) *ϕ is quasi-multiplicative.*

(b) *$\phi = \phi_1 \vee \phi_2$, where $\phi_1, \phi_2 \in S(L)$, ϕ_1 is a projection, $\phi_1\phi_2 = \phi_2$, $\phi_2\phi_1 = 0$, and $\phi_2\phi_2^* \leqq \phi_1$.*

(c) *$\phi^2 \leqq \phi$ and $\phi\phi^* \leqq \phi$.*

Proof. (a)$\Rightarrow$(b). Let ϕ_1 and ϕ_2 be given by $x\phi_1 = x\phi \wedge (0\phi^+)'$ and $x\phi_2 = x\phi \wedge 0\phi^+$. It is clear that $x\phi = x\phi_1 \vee x\phi_2$ for all $x \in L$; $\phi_1, \phi_2 \in S(L)$ since they are compositions of residuated maps. ϕ_1 is a projection by Lemma 10.

$\phi_1\phi_2 = \phi_2$ since $x\phi_2 = x\phi \wedge 0\phi^+ = (x\phi)\phi \wedge 0\phi^+ = [(x\phi \wedge 0\phi^+) \vee (x\phi \wedge (0\phi^+)')]\phi \wedge 0\phi^+ = [(x\phi \wedge 0\phi^+)\phi \vee (x\phi \wedge (0\phi^+)')\phi] \wedge 0\phi^+ = (x\phi \wedge (0\phi^+)')\phi \wedge 0\phi^+ = x\phi_1\phi_2$ for all $x \in L$.

$\phi_2\phi_1 = 0$ since $x\phi_2\phi_1 = (x\phi \wedge 0\phi^+)\phi \wedge (0\phi^+)' \leqq 0\phi^+\phi = 0$ for all $x \in L$.

Notice that ϕ_2^+ is given by $x\phi_2^+ = (x \vee (0\phi^+)')\phi^+$. Thus $x\phi_2\phi_2^* = (x\phi \wedge 0\phi^+)\phi_2^* = ((x\phi \wedge 0\phi^+)'\phi_2^+)' = (((x\phi)' \vee (0\phi^+)')\phi_2^+)' = (((x\phi)' \vee (0\phi^+)')\phi^+)'$. We thus have $x\phi_2\phi_2^* \leqq ((0\phi^+)'\phi^+)'$ and $x\phi_2\phi_2^* \leqq ((x\phi)'\phi^+)'$. Now $0\phi^+\phi = 0 \leqq (0\phi^+)' \Rightarrow 0\phi^+ \leqq (0\phi^+)'\phi^+ \Rightarrow ((0\phi^+)'\phi^+)' \leqq (0\phi^+)' \Rightarrow x\phi_2\phi_2^* \leqq (0\phi^+)'$. In addition, $(x\phi)'\phi \wedge x\phi = ((x\phi)' \wedge x\phi)\phi = 0 \Rightarrow (x\phi)'\phi \leqq (x\phi)' \Rightarrow (x\phi)' \leqq (x\phi)'\phi^+ \Rightarrow ((x\phi)'\phi^+)' \leqq x\phi \Rightarrow x\phi_2\phi_2^* \leqq x\phi$. This shows that $x\phi_2\phi_2^* \leqq x\phi \wedge (0\phi^+)' = x\phi_1$ for all $x \in L$, and so $\phi_2\phi_2^* \leqq \phi_1$.

(b)$\Rightarrow$(c). Suppose that $\phi = \phi_1 \vee \phi_2$, where ϕ_1 and ϕ_2 are as in (b). Then $x\phi^2 = (x\phi_1 \vee x\phi_2)(\phi_1 \vee \phi_2) = x\phi_1\phi_1 \vee x\phi_2\phi_1 \vee x\phi_1\phi_2 \vee x\phi_2\phi_2$. Since $\phi_1\phi_1 = \phi_1$, $\phi_2\phi_1 = 0$, $\phi_1\phi_2 = \phi_2$ and $\phi_2\phi_2 = \phi_2\phi_1\phi_2 = 0$, we have $x\phi^2 = x\phi_1 \vee x\phi_2 = x\phi$. By Corollary 7 we have that $(\phi_1 \vee \phi_2)^* = \phi_1^* \vee \phi_2^*$. Thus $x\phi\phi^* = (x\phi_1 \vee x\phi_2)(\phi_1^* \vee \phi_2^*) = x\phi_1\phi_1^* \vee x\phi_2\phi_1^* \vee x\phi_1\phi_2^* \vee x\phi_2\phi_2^*$. Since $\phi_1\phi_1^* = \phi_1$, $\phi_2\phi_1^* = \phi_2\phi_1 = 0$, $\phi_1\phi_2^* = (\phi_2\phi_1^*)^* = (\phi_2\phi_1)^* = 0^* = 0$, and $\phi_2\phi_2^* \leqq \phi_1$, we have $x\phi\phi^* \leqq x\phi_1 \leqq x\phi_1 \vee x\phi_2 = x\phi$. Thus $\phi\phi^* \leqq \phi$.

(c)$\Rightarrow$(a). This is Lemma 9.

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