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Projective metabelian nonfree groups

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This paper is concerned with projective metabelian $\underline{\underline{AA}}_n$-groups, where $\underline{\underline{A}}$ is the variety of all abelian groups, $\underline{\underline{A}}_n$ - of all abelian groups of exponent n. Let P be a projective $\underline{\underline{AA}}_n$-group. Since $\underline{\underline{AA}}_n \supset \underline{\underline{A}}$ the group P/P' is a free abelian group. Define rank P = rank P/P'. It is shown that for all numbers $r, n > 1$, except $r = n = 2$, there exists a projective nonfree $\underline{\underline{AA}}_n$-group of rank r with $r + 1$ generators.

1. Introduction

This paper is concerned with projective metabelian $\underline{\underline{AA}}_n$-groups, where $\underline{\underline{A}}$ is the variety of all abelian groups, $\underline{\underline{A}}_n$ - of all abelian groups of exponent n. Let P be a projective $\underline{\underline{AA}}_n$-group. Since $\underline{\underline{AA}}_n \supset \underline{\underline{A}}$, the group P/P' is a free abelian group. Define rank P = rank P/P'. We show that if $r, n \geq 2$, except $r = n = 2$, then there exists a projective nonfree $\underline{\underline{AA}}_n$-group of rank r with $r + 1$ generators. On the other hand, McIsaac [4] has proved that projective $\underline{\underline{AA}}_2$-groups of rank 2 are free. It is not difficult to prove that projective $\underline{\underline{AA}}_n$-groups of rank 1 are free.

2. Preliminaries

Let $C_{r,n}$ be a direct product of r cyclic groups $\{x_i\}$ of the same

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order n. Consider the group ring $ZC_{r,n}$. Let m be the augmentation ideal of $ZC_{r,n}$. For $y \in C_{r,n}$ define $n(y) = 1 + y + \ldots + y^{n-1}$.

PROPOSITION 1. *If* $N_{r,n} = \sum_{y \in C_{r,n}} y \in ZC_{r,n}$, *then*

$$N_{r,n} = \prod_{i=1}^{r} n(x_i) .$$

Proof. For all x_j, $1 \le j \le r$, we have $x_j \prod_{i=1}^{r} n(x_i) = \prod_{i=1}^{r} n(x_i)$. So $\prod_{i=1}^{r} n(x_i) = aN_{r,n}$, $a \in Z$, since only $ZN_{r,n}$ satisfies this property. But $aN_{r,n} \equiv an^r \pmod{m} \equiv \prod_{i=1}^{r} n(1) = n^r$ and, hence, $a = 1$.

Consider now the augmentation epimorphism

$$\varepsilon : ZC_{r,n} \to Z , \quad \varepsilon(x_i) = 1 , \quad \ker \varepsilon = m .$$

It induces an epimorphism

$$\varepsilon' : ZC_{r,n}/(N_{r,n}) \to Z/(n^r) ,$$

since $N \equiv n^r \pmod{m}$. In its turn ε' induces a homomorphism of groups of units

$$\varepsilon^* = \varepsilon_{r,n} : \left(ZC_{r,n}/(N_{r,n})\right)^* \to \left(Z/(n^r)\right)^* . \tag{1}$$

THEOREM 1. *Let* $r, n \ge 2$ *except* $r = n = 2$. *Then* $\varepsilon_{r,n}$, *from* (1), *is not an epimorphism.*

Proof. Suppose that for $r - 1$ the theorem is proved, and $K + (n^{r-1}) \in (Z/n^{r-1})^* \backslash \mathrm{Im}\, \varepsilon_{r-1,n}$. Then $K + (n^r) \in (Z/n^r)^*$. Suppose $K + (n^r) \in \mathrm{Im}\, \varepsilon_{r,n}$. Then $K \equiv f(x_1, \ldots, x_r) \pmod{(m, n^r)}$, where $f(x_1, \ldots, x_r) \in ZC_{r,n}$ and, for some $g(x_1, \ldots, x_r) \in ZC_{r,n}$,

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(2) $f(x_1, \ldots, x_r)g(x_1, \ldots, x_r) = 1 + aN_{r,n} = 1 + a \prod_{i=1}^{r} n(x_i)$.

Put $x_r = 1$ in (2). We have

$$f(x_1, \ldots, x_{r-1}, 1)g(x_1, \ldots, x_{r-1}, 1) = 1 + na \prod_{i=1}^{r-1} n(x_i) = 1 + naN_{r-1,n} .$$

So $f(x_1, \ldots, x_{r-1}, 1) \in (ZC_{r-1,n}/N_{r-1,n})^*$,

$$f(x_1, \ldots, x_{r-1}, 1) \equiv f(x_1, \ldots, x_r) \equiv K \left(\operatorname{mod}(m, n^r)\right)$$

and $K + (n^{r-1}) \in \operatorname{Im} \varepsilon_{r-1,n}$. This contradiction shows that $K + (n^r) \notin \operatorname{Im} \varepsilon_{r-1,n}$.

This remark shows that we need only to prove that $\varepsilon_{3,2}$ and $\varepsilon_{2,n}$, $n > 2$, are not epimorphisms.

LEMMA 1. *$\varepsilon_{3,2}$ is not an epimorphism.*

Proof. Let $A = A(x, y, z)$, $B = B(x, y, z) \in ZC_{3,2}$ and $AB = 1 + aN_{3,2}$, $a \in Z$. By Proposition 1, $A(-1, y, z)$ is a unit in $ZC_{2,2} \subset ZC_{3,2}$. But all units in $ZC_{2,2}$ are trivial (see [2], Theorem 6). Since $\varepsilon_{3,2}$ maps trivial units into ± 1 we can assume that $A(-1, y, z) = B(-1, y, z) = 1$. Then $x^2 = y^2 = z^2 = 1$ implies $A = 1 + (1+x)(a_0+a_1y+a_2z)$. So,

$$A(x, -1, z) = (1+a_0-a_1) + x(a_0-a_1) + a_2z + a_2xz .$$

Again, by Higman's Theorem, two of the three numbers $1 + a_0 - a_1$, $a_0 - a_1$, a_2 equal zero. But $1 + a_0 - a_1 \neq a_0 - a_1$. Hence $a_2 = 0$. Similarly, the consideration of $A(x, y, -1)$ implies $a_1 = 0$. Thus $A = 1 + a_0(1+x)$. The same argument shows $B = 1 + b_0(1+x)$. Hence $AB = 1 + (a_0+b_0+2a_0b_0)(1+x) = 1 + a(1+x)(1+y)(1+z)$ and this implies $a_0 + b_0 + 2a_0b_0 = 0$. If $a_0 = 0$ then $b_0 = 0$ and $A = 1$, $\varepsilon_{3,2}(A) = 1$.

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Otherwise $a_0 = b_0 = 1$ and $A = -x$, $\varepsilon_{3,2}(A) = -1$. So, in all cases, $\operatorname{Im} \varepsilon_{3,2} = \{1, -1\} \neq \{1, 3, 5, -1\} = (Z/8)^*$. This proves Lemma 1.

Suppose now $r = 2$, $n > 2$, $n = ab$. Then there is a natural epimorphism $f : ZC_{2,n} \to ZC_{2,b}$. It is clear that $f(n(x_i)) = a \cdot b(x_i)$; hence $f(N_{2,n}) = a^2 N_{2,b}$. The epimorphism f induces a homomorphism rings $f' : ZC_{2,n}/N_{2,n} \to ZC_{2,b}/N_{2,b}$ and a homomorphism of groups $f^* : (ZC_{2,n}/N_{2,n})^* \to (ZC_{2,b}/N_{2,b})^*$. Let j be the natural homomorphism $j : Z/n^2 \to Z/b^2$, $j(K+(n^2)) = K + (b^2)$.

LEMMA 2. *The following diagram*

$$\begin{array}{ccc} (ZC_{2,n}/N_{2,n})^* & \xrightarrow{\varepsilon_{2,n}} & (Z/n^2)^* \\ \downarrow f^* & & \downarrow j^* \\ (ZC_{2,b}/N_{2,b})^* & \xrightarrow[\varepsilon_{2,b}]{} & (Z/b^2)^* \end{array}$$

is commutative.

The proof is trivial, since all homomorphisms are induced by ring homomorphisms and for ring generators x_i we have

$j^* \varepsilon_{2,n} x_i = 1 + (b^2) = \varepsilon_{2,b} f^* x_i$.

COROLLARY. *If* $\varepsilon_{2,4}$ *is not epi, then* $\varepsilon_{2,2^k}$, $K \geq 2$, *is not epi either.*

The proof follows from Lemma 2, since j^* is epi.

LEMMA 3. $\varepsilon_{2,4}$ *is not epi.*

Proof. We shall show that $3 + (16) \notin \operatorname{Im} \varepsilon_{2,4}$. Let $A = A(x, y)$, $B = B(x, y) \in ZC_{2,4}$, and $AB = 1 + cN_{2,4}$, $c \in Z$; put $x = -1$. Since $(1+x) | N_{2,4}$ we have in $ZC_{1,4}$,

$$A(-1, y)B(-1, y) = 1 .$$

By Theorem 6 from [2], $ZC_{1,4}$ contains only trivial units, so, without loss of generality, we suppose $A(-1, y) = 1$;

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that is, $A(x, y) = 1 + (1+x)D(x, y)$ where $D(x, y) = E(x) + (1+y)J(x, y)$. By Theorem 6 from [2], $A(x, -1) = 1 + (1+x)E(x)$ is a trivial unit, $A(x, -1) = \pm x^t$. Thus $A(x, y) = \pm x^t\big(1+(1+x)(1+y)J(x, y)\big)$. Let $J(x, y) = J_1(x) + yJ_2(x) + \big(1+y^2\big)J_3(x, y)$. Then in $Z[i]C_{1,4}$,

$$(2')\qquad \pm x^{4-t}A(x, i) = 1 + (1+x)\big(J_1(x)-J_2(x)\big) + i(1+x)\big(J_1(x)+J_2(x)\big)$$

is a unit. But $Q[i]C_{1,4}$ is isomorphic to the direct sum of four copies of $Q[i]$ in which the generator of $C_{1,4}$ acts as a multiplication by ± 1, $\pm i$. So, since all units of $Q[i]$ have finite order, the same is true by Theorem 5 from [2] for $Z[i]C_{1,4}$, and by Theorem 3 (see [2]) all units of $Z[i]C_{1,4}$ are of the form $\pm x^k$, $\pm ix^k$. Hence, in (2'), $J_1 = J_2 = 0$, and $A(x, y) = \pm x^t\Big[1+(1+x)(1+y)\big(1+y^2\big)J_3(x, y)\Big]$. Similarly, $J_3(x, y) = \big(1+x^2\big)J_4$ and thus $A(x, y) \equiv \pm x^t \pmod{N_{2,4}}$, $\varepsilon_{2,4}(A) = \pm 1$.

Suppose now that $n = p^k d$, $k \geq 1$, p an odd prime, and we have already proved that for all $z \in \operatorname{Im} \varepsilon_{2,p}$,

$$(3)\qquad z^{2(p-1)^2} = 1 .$$

Then $1 + pd + \binom{n}{2} \notin \operatorname{Im} \varepsilon_{2,n}$. Indeed, if $1 + pd + \binom{n}{2} \in \operatorname{Im} \varepsilon_{2,n}$, then by Lemma 2, $1 + pd + \binom{p}{2} \in \operatorname{Im} \varepsilon_{2,p}$, so mod p^2,

$$(1+pd)^{2(p-1)^2} - 1 \equiv (1+2pd) - 1 \equiv 2pd \equiv 0 ;$$

this contradicts $(d, p) = (2, p) = 1$. Thus, we need only prove (3).

Let H be the group of all automorphisms of $ZC_{2,p} = Z[x, y]/\big(x^p-1, y^p-1\big)$ leaving invariant the cyclic subgroups $\{x\}, \{y\} \subset C_{2,p}$. It is clear that for $\alpha \in H$, $g \in ZC_{2,p}$,

$$\varepsilon\big(\alpha(g)\big) = \varepsilon(g)$$

and $|H| = (p-1)^2$. Note that the element $N = N_{2,p}$ is H-invariant;

that is, for all $\alpha \in H$,

$$\alpha(N) = N .$$

LEMMA 4. *Let* $A \in ZC_{2,p}$ *be* H*-invariant,* $B \in ZC_{2,p}$. *If* $AB \equiv 1 \pmod N$ *then* B *is* H*-invariant.*

Proof. If $AB = 1 + aN$, $a \in Z$, then

$$0 = \alpha(1+aN) - (1+aN) = A[\alpha(B)-B] , \quad \alpha \in H .$$

Since A is invertible mod N,

$$\alpha(B) = B + bN , \quad b \in Z ,$$

and, by induction, $\alpha^k(B) = B + kbN$. But for some $k \geq 1$ we have $\alpha^k = 1$, so $kbN = 0$ and $b = 0$.

LEMMA 5. *Let* $C \in ZC_{2,p}$ *and* C *be* H*-invariant. Then* $C = c_0 + c_1p(x) + c_2p(y) + c_3p(x)p(y)$ *where* $c_i \in Z$.

Proof. Let $C = \sum_{i,j=0}^{p-1} a_{ij}x^iy^j$. For each i, $1 \leq i \leq p-1$, we have an automorphism $\beta \in H$ such that $\beta(x) = x^i$, $\beta(y) = y$. Since C is invariant, $a_{ij} = a_{1j}$, $1 \leq i \leq p-1$. The same is true for j. This proves Lemma 5.

LEMMA 6. *If* $z \in \operatorname{Im} \varepsilon_{2,p}$, *then* $z^{2(p-1)^2} = 1$.

Proof. Since for all $A \in ZC_{2,p}$, $\alpha \in H$, $\varepsilon(\alpha(A)) = \varepsilon(A)$ and $|H| = (p-1)^2$, we need only prove that for H-invariant A, invertible mod N,

$$\varepsilon_{2,p}(A+(N)) = \pm 1 .$$

By Lemma 4, for H-invariant $B \in ZC_{2,p}$,

$$AB = 1 + aN , \quad a \in Z .$$

By Lemma 5 we can assume $A = a_0 + a_1p(x) + a_2p(y)$, $B = b_0 + b_1p(x) + b_2p(y)$. Note that $p(x)^2 = p\cdot p(x)$; so in $ZC_{2,p}$,

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$$AB = a_0b_0 + \left(a_0b_1+a_1b_0+pa_1b_1\right)p(x) + p(y)\left(a_0b_2+a_2b_0+pa_2b_2\right) + \\ + p(x)p(y)\left(a_1b_2+a_2b_1\right) = 1 + ap(x)p(y) ,$$

which implies $a_0b_1 + a_1b_0 + pa_1b_1 = a_0b_2 + a_2b_0 + pa_2b_2 = 0$, $a_0b_0 = 1$. Hence $a_0 = b_0 = \pm 1$, $b_1 = \pm a_1\left(\pm 1+pb_1\right)$, and

$$|b_1| \geq |pb_1\pm 1| \geq p|b_1| - 1 \geq 2|b_1| ,$$

since $p \geq 3$. Thus $b_1 = a_1 = 0$. Similarly $a_2 = b_2 = 0$. Hence, finally, $A = \pm 1$. This completes the proof of Theorem 1.

THEOREM 2. *Let the numbers* r, n *be as in Theorem* 1. *Then there exists a projective nonfree* $ZC_{r,n}$*-module* T *of rank* r *with epimorphism* $l : T \to m$ (*the augmentation ideal of* $ZC_{r,n}$) *such that*

(i) *for some projective ideal* $J \triangleleft ZC_{r,n}$ *the module* $M = J \oplus T$ *is free of rank* $r + 1$,

(ii) *if* l *is trivially extended to* M , *so that* $l(J) = 0$, *and if* ρ *is a projection of* M *onto* T , *then for some basis* $w_0, \ldots, w_r$ *of* M ,

$$\rho\left(w_0\right) \in mT , \qquad l\left(w_0\right) = 0 ,$$

$$\rho\left(w_i\right) \equiv w_i \ (\mathrm{mod}\ mT) , \quad l\left(w_i\right) = x_i - 1 , \quad 1 \leq i \leq r .$$

Proof. Let $k \in Z$ and $k + \left(n^r\right) \in \left(Z/n^r\right)^*\backslash\mathrm{Im}\ \varepsilon_{r,n}$. Then $(k, n) = 1$ and for some $k', m \in Z$,

$$kk' = 1 + mn^r . \tag{4}$$

By results from [6] the ideals $I = (k, N)$, $J = (k', N)$, where $N = N_{r,n}$, are projective nonfree $ZC_{r,n}$-modules of rank 1 . There exists an isomorphism

$$f : \left(ZC_{r,n}\right)^2 \to J \oplus I ,$$

$$f(1, 0) = (ku'-mv', u) , \quad f(0, 1) = \left(mn^ru'-mk'v', k'u-mv\right) ,$$

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where $u' = k'$, $v' = N \in J$, $u = k$, $v = N \in I$. Take in $(ZC_{r,n})^2$ a new basis $e_0 = k(1, 0) - (0, 1)$, $e_1 = (1, 0)$. Then by (4),

(4') $\quad f_0 = f(e_0) = (u', mv)$, $f_1 = f(e_1) = (ku'-mv', u)$.

In [6] Swan noticed that $I = (k, N)$ is given by generators u, v and relations

(4") $\quad Nu = kv$, $xv = v$, for all $x \in C_{r,n}$.

Thus, if $l : I \to m$ is any homomorphism, then $kl(v) = Nl(u) = 0$ by Proposition 1, since $l(u) = \sum a_i(x_i-1)$. So, by (4"), all homomorphisms $l : I \to m$ are uniquely determinded by $l(u)$, and since $ZC_{r,n}u \simeq ZC_{r,n}$ there is no restriction on $l(u)$.

Now take $T = I \oplus ZC_{r,n}f_2 \oplus \ldots \oplus ZC_{r,n}f_r$.

LEMMA 7. *T is a projective nonfree $ZC_{r,n}$-module of rank r.*

Proof. If $T \simeq (ZC_{r,n})^r$, then since Krull-dim $ZC_{r,n} = 1$ (see [1], p. 600) by the 'cancellation' theorem ([1], p. 184), $I \oplus ZC_{r,n} = (ZC_{r,n})^2$. But $GL(2, ZC_{r,n})$ acts transitively on the set of all unimodular vectors in $ZC_{r,n}$. So $I \simeq ZC_{r,n}$ (see [3], p. 286). This contradicts $I \not\simeq ZC_{r,n}$.

LEMMA 8. *Let*

$$A_0 = -mn^{r-1}n(x) + kk'x + (y-1),$$

$$A_1 = k'mn^{r-1}n(x) - k'mn^r x - k'(y-1),$$

$$A_3 = x - 1.$$

Then

(i) $A_0 \equiv 1 \pmod{m}$, $A_2, A_3 \in m$,

(ii) $\left(mn^r A_0 + kA_1\right)(x-1) + A_2(y-1) = 0$,

(iii) the ideal $L = (A_0, A_1, A_2) = ZC_{r,n}$.

Proof. Since $A_i \equiv A_i(1, 1) \pmod{m}$, (4) implies

$$A_0(1, 1) = -mn^r + kk' = 1 , \quad A_1(1, 1) = k'mn^r - k'mn^r = 0 , \quad A_2(1, 1) = 0 .$$

Thus *(i)* is proved.

Now $n(x)(x-1) = 0$, so

$$\left[mn^r A_0 + kA_1\right](x-1) + A_2(y-1) =$$

$$= \left[mn^r\left(kk'x+(y-1)\right)+k\left(-k'mn^r x-k'(y-1)\right)\right](x-1) + (x-1)(y-1) = 0 .$$

Since $L \ni x-1$ by *(i)*, $A_0 \equiv 1 + (y-1) \equiv Y \pmod{L}$ and $1 \in L$.

LEMMA 9. *There exists* $A \in \mathrm{GL}\left(3, ZC_{2,n}, m\right)$ *such that the first row of* A *is* A_0, A_1, A_2 .

Proof. Since Krull-dim $ZC_{r,n} = 1$ by results of Chapter 5 of [1], there exists $C \in \mathrm{GL}\left(3, ZC_{r,n}\right)$ with the first row $c_{00} = A_0$, $c_{01} = A_1$, $c_{02} = A_2$. But $A_0 \equiv 1 \pmod{m}$, so, applying elementary transformations to C , we can suppose $c_{10}, c_{20} \in m$. Now mod m ,

$$C \equiv C_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, Z) ,$$

and we can take A to be $A = C_0^{-1}C \in \mathrm{GL}\left(3, ZC_{r,n}, m\right)$.

Note that

$$\begin{vmatrix} k' & mn^r \\ 1 & k \end{vmatrix} = 1 ,$$

so

$$B = \begin{pmatrix} A & \vdots & 0 \\ \cdots\cdots & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} k' & mn^r & 0 & \vdots & \\ 1 & k & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & \\ \cdots & \cdots & \cdots & 1 & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in \mathrm{GL}(r+1, ZC_{r,n}) ,$$

where A is from Lemma 9. Put $W_i = B^{-1}f_i$, $i = 0, \ldots, r$, and define $l(W_0) = 0$, $l(W_i) = x_i - 1$, $i = 1, \ldots, r$. We need to prove that $l(J) = 0$ and $W_0, \ldots, W_r$ satisfy *(ii)* from Theorem 2.

LEMMA 10. $l(J) = 0$.

Proof. Since $l(J) = ZC_{r,n}l(u')$, we need to prove, by (4'), that $l(f_0) = 0$. But $f_0 = b_{00}W_0 + b_{01}W_1 + b_{02}W_2$, $l(f_0) = \left(A_0mn^r + A_1k\right)(x-1) + A_2(y-1) = 0$ by *(ii)* from Lemma 8.

LEMMA 11. *If* ρ *is a projection* $\rho : M \to T$, *then* $\rho(W_0) \in mT$, $\rho(W_i) \equiv W_i \pmod{mM}$, $i = 1, \ldots, r$.

Proof. Since $A \in \mathrm{GL}(3, ZC_{r,n}, m)$ and $W_i \equiv f_i = \rho(f_i) \pmod{mM}$, $2 \le i \le r$, without loss of generality we suppose $A = 1$, $r = 2$. Then

$$B^{-1} = \begin{pmatrix} k & -mn^r & \vdots & \\ & & \vdots & 0 \\ -1 & k' & \vdots & \\ \cdots & \cdots & 1 & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} ,$$

$W_0 = kf_0 - mn^rf_1$, $W_1 = -f_0 + k'f_1$. By (4), (4'),

$$\rho(W_0) = kmv - mn^ru = m\left(N-n^r\right)u \in mu ,$$

$$W_1 - \rho(W_1) = (-u', -mv) + k'(ku'-mv', u) + (0, mv) - k'(0, u) =$$
$$= m\left(\left(N-n^r\right)u', 0\right) \in mJ .$$

This completes the proof of Theorem 2.

3. Projective $\underline{\underline{AA}}_n$-groups

Let P be a projective $\underline{\underline{AA}}_n$-group. As we have already noticed in the introduction, rank P = rank P/P'.

THEOREM 3. *Let* $r, n \geq 2$ *except* $r = n = 2$. *Then there exist projective nonfree* $\underline{\underline{AA}}_n$*-groups of rank* r *with* $r+1$ *generators.*

Proof. Let $ZC_{r,n} = Z[x_1, \ldots, x_r]/\left(x_i^n-1,\ i = 1, \ldots, r\right)$ with augmentation ideal m, $ZC_{r+1,n} = Z[x_0, \ldots, x_r]/\left(x_i^n-1,\ i = 0, \ldots, r\right)$ with augmentation ideal m_0, and T, l, J, W_i from Theorem 2. Let

$$S = ZC_{r+1,n}W_0 \oplus \ldots \oplus ZC_{r+1,n}W_r .$$

Define $l_0 : S \to m_0$ by $l_0(W_i) = x_i - 1$.

In [5] it is shown that the group F of matrices

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix},\quad a \in C_{r+1,n},\quad b \in S,\quad a - 1 = l(b),$$

is a free $\underline{\underline{AA}}_n$-group with $r+1$ free generators

$$z_i = \begin{pmatrix} x_i & 0 \\ W_i & 1 \end{pmatrix}.$$

Note that for $t_i = \begin{pmatrix} a_i & 0 \\ b_i & 1 \end{pmatrix} \in F$,

$$t_1t_2 = \begin{pmatrix} a_1a_2 & 0 \\ a_2b_1+b_2 & 1 \end{pmatrix},\quad t_1^{-1} = \begin{pmatrix} a_1^{-1} & 0 \\ -a_1^{-1}b_1 & 1 \end{pmatrix},$$

$$t_1^{t_2} = \begin{pmatrix} a_1 & 0 \\ a_2b_1-(a_1-1)b_2 & 1 \end{pmatrix},\quad t_1^n = \begin{pmatrix} 1 & 0 \\ n(a_1)b_1 & 1 \end{pmatrix}. \tag{5}$$

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Let φ be an endomorphism of F,

$$\varphi(z_i) = \begin{pmatrix} a_i & 0 \\ \Phi W_i & 1 \end{pmatrix}, \quad \Phi = \Phi(x_0, \ldots, x_r) \in Ma^t(r+1, ZC_{r+1,n}) .$$

LEMMA 12. *If*

$$z = \begin{pmatrix} a(x_0, \ldots, x_r) & 0 \\ \sum b_i(x_0, \ldots, x_r)W_i & 1 \end{pmatrix},$$

then

(6) $$\varphi(z) = \begin{pmatrix} a(a_0, \ldots, a_r) & 0 \\ \sum b_i(a_0, \ldots, a_r)\Phi W_i & 1 \end{pmatrix}.$$

The proof is trivial, since (6) is valid for $z_0, \ldots, z_r$, and if (6) is valid for t_1, t_2, then, by (5), it is valid for t_1t_2, t_1^{-1}.

LEMMA 13. *Let* φ *be an endomorphism of* F *such that*

$$\varphi(z_0) = \begin{pmatrix} 1 & 0 \\ \Phi W_0 & 1 \end{pmatrix}, \quad \varphi(z_i) = \begin{pmatrix} x_i & 0 \\ \Phi W_i & 1 \end{pmatrix}, \quad i = 1, \ldots, r .$$

Then

$$U = \left\{ \begin{pmatrix} 1 & 0 \\ \sum b_i W_i & 1 \end{pmatrix} \in F \;\middle|\; b_i \in (x_0-1) \right\} \triangleleft F$$

and $U \subseteq \ker \varphi$.

Proof. By (5), $U \triangleleft F$ and by (6), $U \subseteq \ker \varphi$, since $a_0 = 1$, $b_i(1, x_1, \ldots, x_r) = 1$.

Now we can prove the theorem. We have $M \subset S$. Let ρ be the projection $\rho : M \to T$, $\rho = (\rho_{ij})$. By *(ii)* from Theorem 2, $\rho_{i0} = \sum_{j=1}^{r} \alpha_{ij}(x_j-1)$. Define

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(7) $$\Phi_{ij} = \begin{cases} \rho_{i0} , & j = 0 , \\ \rho_{ij}-\alpha_{ij}(x_0-1) , & j \geq 1 . \end{cases}$$

Then

(7') $$\Phi_{ij} \equiv \rho_{ij} \; (\mathrm{mod}(x_0-1)) , \quad \Phi = (\Phi_{ij}) .$$

Put

(7") $$\varphi(z_i) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \Phi W_0 & 1 \end{pmatrix} , & i = 0 , \\ \begin{pmatrix} x_i & 0 \\ \Phi W_i & 1 \end{pmatrix} , & i \geq 1 . \end{cases}$$

LEMMA 14. $\varphi(z_i) \in F$.

Proof.

$$l_0(\Phi W_0) = \sum_{j=0}^{r} \Phi_{0j}(x_j-1) = \sum_{j=1}^{r} \alpha_{0j}(x_j-1)(x_0-1) + \sum_{j=1}^{r} (\rho_{0j}-\alpha_{0j}(x_0-1))(x_j-1) =$$

$$= \sum_{j=1}^{r} \rho_{0j}(x_j-1) = l(\rho(W_0)) = l(W_0) = 0$$

since $l(\ker \rho) = l(J) = 0$. Similarly for $i \geq 1$,

$$l_0(\Phi W_i) = \sum_{j=0}^{r} \Phi_{ij}(x_j-1) = \sum_{j=1}^{r} \alpha_{ij}(x_j-1)(x_0-1) + \sum_{j=1}^{r} [\rho_{ij}-\alpha_{ij}(x_0-1)](x_j-1) =$$

$$= \sum_{j=1}^{r} \rho_{ij}(x_j-1) = l(\rho(W_i)) = l(W_i) = x_i - 1 .$$

Thus $\varphi \in \mathrm{end}\, F$.

LEMMA 15. *Let* $\pi = \varphi^2$ *; then* $\pi^2 = \pi$.

Proof. We have

$$\varphi(z_0) = \begin{pmatrix} 1 & 0 \\ \rho(W_0)+(x_0-1)y_0 & 1 \end{pmatrix}, \quad \varphi(z_i) = \begin{pmatrix} x_i & 0 \\ \rho(W_i)+(x_0-1)y_i & 1 \end{pmatrix},$$

where $y_i \in M \subset S$. Then by (6), (7'), (5), and $\rho^2 = \rho$ we have $\varphi^2(z_i) \equiv \varphi(z_i) \pmod{U}$ where U is a normal subgroup from Lemma 13. Hence $\pi(z_i) = \varphi^2(z_i) = \varphi(z_i)u_i$, $u_i \in U \subseteq \ker \varphi$ and

$$\pi^2(z_i) = \varphi^4(z_i) = \varphi^3(z_i) = \varphi^2(z_i u_i) = \varphi^2(z_i) = \pi(z_i) .$$

Put $P = \operatorname{Im} \pi$. Then P is a projective $\underline{\underline{AA}}_n$-group with $r + 1$ generators.

LEMMA 16. $\operatorname{rank} P = r$.

Proof. Let

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \in F \right\} .$$

Then, by (5) and (6), G is a verbal subgroup of F corresponding to the variety $\underline{\underline{A}}_n$. By (7"), $P/P\cap G$ is a free $\mathbb{Z}/(n)$-module of rank r. But $r = \operatorname{rank} P/P\cap G = \operatorname{rank} P/P' = \operatorname{rank} P$.

Suppose that P is a free $\underline{\underline{AA}}_n$-group. Since every automorphism of P/P' can be lifted to an automorphism of P (see [5]) we can choose in P free generators $t_1, \ldots, t_r$ such that in F,

$$t_i = \begin{pmatrix} x_i & 0 \\ g_i & 1 \end{pmatrix}, \quad g_i \in S .$$

Let $g_i = g_i' + (x_0-1)g_i''$, where $g_i' \in M$, $g_i'' \in S$. Since $\pi(t_i) = t_i$, (6), (7'), (7") imply $g_1', \ldots, g_r' \in T$, and, by (5), the submodule generated by $g_1', \ldots, g_r'$ coincides with T. So the projective $\mathbb{Z}C_{r,n}$-module T of rank r has r generators. Then T is free. This contradiction shows that T is not free.

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