

## TYPE TRANSITION OF SIMPLE RANDOM WALKS ON RANDOMLY DIRECTED REGULAR LATTICES

MASSIMO CAMPANINO,\* *Università degli Studi di Bologna*

DIMITRI PETRITIS,\*\* *Université de Rennes 1*

### Abstract

Simple random walks on a partially directed version of  $\mathbb{Z}^2$  are considered. More precisely, vertical edges between neighbouring vertices of  $\mathbb{Z}^2$  can be traversed in both directions (they are undirected) while horizontal edges are one-way. The horizontal orientation is prescribed by a random perturbation of a periodic function; the perturbation probability decays according to a power law in the absolute value of the ordinate. We study the type of simple random walk that is recurrent or transient, and show that there exists a critical value of the decay power, above which it is almost surely recurrent and below which it is almost surely transient.

*Keywords:* Markov chain; random environment; recurrence criteria; random graph; directed graph

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### 1. Introduction

#### 1.1. Motivations

We study simple symmetric random walks (i.e. jumping with uniform probability to one of the available neighbours of a given vertex) on partially directed regular sublattices of  $\mathbb{Z}^2$  obtained from  $\mathbb{Z}^2$  by imposing horizontal lines to be unidirectional. Although random walks on partially directed lattices were introduced some time ago to study the hydrodynamic dispersion of a tracer particle in a porous medium [15], very little was known about them beyond some computer simulation heuristics [19]. Therefore, it came as a surprise to us that so little rigorous research had been undertaken when we first considered simple random walks on partially directed 2-dimensional lattices; see [3] and [4]. In these papers, the type of simple random walks on lattices was obtained from  $\mathbb{Z}^2$  by keeping vertical edges bi-directional while horizontal edges became one-way. Depending on how the allowed horizontal direction, to the left or the right, is determined we obtain dramatically different behaviour [3, Theorems 1.6, 1.7, and 1.8] (for completeness these results are reproduced as Theorem 1.1 in this paper).

This result triggered several developments by various authors. In [13] the orientation is chosen by means of a correlated sequence or by a dynamical system; in both cases, provided that some variance condition holds, almost sure transience is established and, in [12], a functional limit theorem is obtained. In [16] the case of orientations chosen according to a stationary sequence was examined. In [17] our results of [3] and [4] are used to study corner percolation

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\* Postal address: Dipartimento di Matematica, Università degli Studi di Bologna, Piazza di Porta San Donato 5, I-40126 Bologna, Italy. Email address: massimo.campanino@unibo.it

\*\* Postal address: Institut de Recherche Mathématique, Université de Rennes I, Campus de Beaulieu, F-35042 Rennes Cedex, France. Email address: dimitri.petritis@univ-rennes1.fr

on  $\mathbb{Z}^2$ . In [7] the Martin boundary of these walks has been studied for the models that are transient and proved to be trivial, i.e. the only positive harmonic functions for the Markov kernel of these walks are the constants. In [8] a model where the horizontal directions are chosen according to an arbitrary (deterministic or random) sequence, probability of performing a horizontal or vertical move is not determined by the degree but by a sequence of nondegenerate random variables and shown to be almost surely (a.s.) transient.

It is worth noting that all the previous directed lattices are regular in the sense that both the inward and the outward degrees are constant (and equal to 3) all over the lattice. Therefore, the dramatic change of type is due only to the nature of the directedness. However, the type result was always either recurrent or transient. This paper provides an example where the type of random walk is determined through the tuning of a parameter controlling the overall number of defects, thus, improving the insight we have on these nonreversible random walks. Let us also mention that beyond their theoretical interest (a short list of problems remaining open in the context of such random walks is given in the conclusion), directed random walks are much more natural models of propagation on large networks such as the Internet than reversible ones.

**1.2. Notation and definitions**

*Directed* versions of  $\mathbb{Z}^2$  are obtained as follows: let  $\mathbf{u} = (u_1, u_2)$  and  $\mathbf{v} = (v_1, v_2)$  be arbitrary elements of  $\mathbb{Z}^2$  and suppose that a sequence of  $\{-1, 1\}$ -valued variables  $\boldsymbol{\varepsilon} = (\varepsilon_y)_{y \in \mathbb{Z}}$  is given. The pair  $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^2 \times \mathbb{Z}^2$  is an allowed edge of the lattice if either  $[u_2 = u_1] \wedge [v_2 = v_1 \pm 1]$  or  $[v_2 = v_1] \wedge [u_2 = u_1 + \varepsilon_{v_1}]$ . The directed sublattice of  $\mathbb{Z}^2$  depends obviously on the choice of the sequence  $\boldsymbol{\varepsilon}$ ; we denote this partially directed lattice by  $\mathbb{Z}_{\boldsymbol{\varepsilon}}^2$ . The choice of  $\boldsymbol{\varepsilon}$  can be deterministic or random and will be specified later.

**Definition 1.1.** A *simple random walk* on  $\mathbb{Z}_{\boldsymbol{\varepsilon}}^2$  is a  $\mathbb{Z}^2$ -valued Markov chain  $(M_n)_{n \in \mathbb{N}}$  with transition probability matrix  $P$  having as matrix elements

$$P(\mathbf{u}, \mathbf{v}) = \mathbb{P}(M_{n+1} = \mathbf{v} \mid M_n = \mathbf{u}) = \begin{cases} \frac{1}{3} & \text{if } (\mathbf{u}, \mathbf{v}) \text{ is an allowed edge of } \mathbb{Z}_{\boldsymbol{\varepsilon}}^2, \\ 0 & \text{otherwise.} \end{cases}$$

**Remark 1.1.** The Markov chain  $(M_n)_{n \in \mathbb{N}}$  *cannot be reversible*. Therefore, all the powerful techniques based on the analogy with electrical circuits (see [9] and [20] for a modern exposition) or spectral properties of graph Laplacians (see [2], [5], and [6]) do not apply. Connections of graph Laplacians with electric circuits date back to the early 1920s reference [21]. (Connections with modern cohomology can be found in [14], [20], and [22], but again much older examples exist; see [10], [11], and references therein.)

Several  $\boldsymbol{\varepsilon}$ -horizontally directed lattices have been introduced in [3], where the following theorem has been established.

**Theorem 1.1.** ([3, Theorems 1.6, 1.7, and 1.8].) *Consider a  $\mathbb{Z}_{\boldsymbol{\varepsilon}}^2$  directed lattice.*

1. *If the lattice is alternatively directed, i.e.  $\varepsilon_y = (-1)^y$ , for  $y \in \mathbb{Z}$ , then the simple random walk on it is recurrent.*
2. *If the lattice has directed half-planes, i.e.  $\varepsilon_y = -\mathbf{1}_{]-\infty, 0[}(y) + \mathbf{1}_{]0, \infty[}(y)$ , then the simple random walk on it is transient.*
3. *If  $\boldsymbol{\varepsilon}$  is a sequence of  $\{-1, 1\}$ -valued random variables, independent and identically distributed with uniform probability, the simple random walk on it is transient for almost all possible choices of the horizontal directions.*

Note that the above simple random walks are defined on topologically nontrivial directed graphs in the sense that  $\lim_{N \rightarrow \infty} (1/N) \sum_{y=-N}^N \varepsilon_y = 0$ . For the first two cases this is shown by a simple calculation and for the third case this is an almost sure statement stemming from the independence of the sequence  $\varepsilon$ . The above condition guarantees that transience is not a trivial consequence of a nonzero drift but an intrinsic property of the walk in spite of its jumps being statistically symmetric.

### 1.3. Results

In this paper we consider again a  $\mathbb{Z}_e^2$  lattice, but the sequence  $\varepsilon$  is specified as follows.

**Definition 1.2.** Let  $f : \mathbb{Z} \rightarrow \{-1, 1\}$  be a  $Q$ -periodic function with some even integer  $Q \geq 2$  verifying  $\sum_{y=1}^Q f(y) = 0$  and  $\rho = (\rho_y)_{y \in \mathbb{Z}}$  a Rademacher sequence, i.e. a sequence of independent and identically distributed  $\{-1, 1\}$ -valued random variables having uniform distribution on  $\{-1, 1\}$ . Let  $\lambda = (\lambda_y)_{y \in \mathbb{Z}}$  be a  $\{0, 1\}$ -valued sequence of independent random variables, independent of  $\rho$ , and suppose there exists constants  $\beta$  (and  $c$ ) such that  $\mathbb{P}(\lambda_y = 1) = c/|y|^\beta$  for large  $|y|$ . We define the horizontal orientations  $\varepsilon = (\varepsilon_y)_{y \in \mathbb{Z}}$  through  $\varepsilon_y = (1 - \lambda_y)f(y) + \lambda_y \rho_y$ . Then the  $\mathbb{Z}_e^2$ -directed lattice defined above is termed a *randomly horizontally directed lattice with randomness decaying in power  $\beta$* .

**Theorem 1.2.** Consider the horizontally directed lattice  $\mathbb{Z}_e^2$  with randomness decaying in power  $\beta$ .

1. If  $\beta < 1$  then the simple random walk is transient for almost all realisations of the sequence  $(\lambda_y, \rho_y)$ .
2. If  $\beta > 1$  then the simple random walk is recurrent for almost all realisations of the sequence  $(\lambda_y, \rho_y)$ .

**Remark 1.2.** It is worth noting that the periodicity of the function  $f$  is required only to prove recurrence; for proving transience, any function  $f$  can be used.

**Remark 1.3.** In the previous model the levels  $y$ , where  $\lambda_y \neq 0$ , can be viewed as random defects perturbing a periodically directed model whose horizontal directions are determined by the periodic function  $f$ . Thus, it is natural to consider the random variable  $\|\lambda\| := \|\lambda\|_1 = \text{card}\{y \in \mathbb{Z} : \lambda_y = 1\}$  as the *strength* of the perturbation. When  $\beta > 1$ , by Borel–Cantelli lemma,  $\|\lambda\| < \infty$  a.s., meaning that there are a.s. finitely many levels  $y$  where the horizontal direction is randomly perturbed with respect to the direction determined by the periodic function; if  $\beta < 1$  then  $\|\lambda\| = \infty$  a.s.

An extreme choice of ‘random’ perturbation is when  $\lambda$  is a deterministic  $\{0, 1\}$ -valued sequence. We then have the following proposition.

**Proposition 1.1.** When  $\lambda$  is a deterministic  $\{0, 1\}$ -valued sequence with  $\|\lambda\| < \infty$ , then the simple random walk is recurrent.

Note however that the previous proposition does not provide us with a necessary condition for recurrence. We shall give in Section 5 the following counterexample.

**Example 1.1.** There are deterministic  $\{0, 1\}$ -sequences  $\lambda$ , with  $\|\lambda\| = \infty$  (infinitely many deterministic defects), leading nevertheless to recurrent random walks.

### 2. Technical preliminaries

Since the general framework developed in [3] is still useful, we need only recall the basic facts. It is always possible to choose a sufficiently large abstract probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  on which are defined all the sequences of random variables we shall use, namely  $(\rho_y), (\lambda_y)$ , etc. and in particular the Markov chain  $(M_n)_{n \in \mathbb{N}}$ . When the initial probability of the chain is  $\nu$ , then obviously  $\mathbb{P} := \mathbb{P}_\nu$ . When  $\nu = \delta_x$ , we write simply  $\mathbb{P}_x$  instead of  $\mathbb{P}_{\delta_x}$ .

The idea of the proof is to decompose the stochastic process  $(M_n)_{n \in \mathbb{N}}$  into a vertical skeleton—obtained by the vertical projection of  $(M_n)$  that is stripped out of the waiting times corresponding to the horizontal moves—and a horizontal component. More precisely, define  $T_0 := 0$  and for  $k \geq 1$  recursively:  $T_k = \inf\{n > T_{k-1} : \langle M_n - M_{n-1} | e_2 \rangle \neq 0\}$ . Then introduce the sequences  $\psi_k = \langle M_{T_k} - M_{T_{k-1}} | e_2 \rangle$  for  $k \geq 1$ , and  $Y_n = \sum_{k=1}^n \psi_k$ , for  $n \geq 0$  (with the convention  $Y_0 = 0$ ). The process  $(Y_n)$  is a simple random walk on the vertical axis, called the *vertical skeleton*; its occupation measure of level  $\{y\}$  is denoted  $\eta_n(y) = \sum_{k=0}^n \mathbf{1}_{\{y\}}(Y_k)$ . Similarly, we define the sequences of waiting times. For all  $y \in \mathbb{Z}$ , define  $S_0(y) := -1$  and recursively for  $k \geq 1$ :  $S_k(y) = \inf\{l > S_{k-1}(y) : Y_l = y\}$ . The random variables  $\xi_k^{(y)} = T_{S_k(y)+1} - T_{S_k(y)} - 1$  represent then the waiting time at level  $y$  during the  $k$ th visit at that level. Due to strong Markov property, the doubly infinite sequence  $(\xi_k^{(y)})_{y \in \mathbb{Z}, k \in \mathbb{N}^*}$  are independent  $\mathbb{N}$ -valued random variables with geometric distribution of parameter  $p = \frac{1}{3}$ ;  $q$  always stands for  $1 - p$  in the sequel.

**Definition 2.1.** Suppose the vertical skeleton and the environment of the orientations are given. Let  $(\xi_n^{(y)})_{n \in \mathbb{N}, y \in \mathbb{Z}}$  be the previously defined doubly infinite sequence of geometric random variables of parameter  $p = \frac{1}{3}$ , and let  $\eta_n(y)$  be the occupation measures of the vertical skeleton. We call the process  $(X_n)_{n \in \mathbb{N}}$  a *horizontally embedded* random walk with

$$X_n = \sum_{y \in \mathbb{Z}} \varepsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}, \quad n \in \mathbb{N}.$$

**Lemma 2.1.** (See [3, Lemma 2.7].) Let  $T_n = n + \sum_{y \in \mathbb{Z}} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)}$  be the instant just after the random walk  $(M_k)$  has performed its  $n$ th vertical move (with the convention that the sum  $\sum_i$  vanishes whenever  $\eta_{n-1}(y) = 0$ ). Then  $M_{T_n} = (X_n, Y_n)$ .

Define  $\sigma_0 = 0$  and recursively, for  $n = 1, 2, \dots$ ,  $\sigma_n = \inf\{k > \sigma_{n-1} : Y_k = 0\} > \sigma_{n-1}$ , the  $n$ th return to the origin for the vertical skeleton. Then obviously,  $M_{T_{\sigma_n}} = (X_{\sigma_n}, 0)$ . To study the recurrence or the transience of  $(M_k)$ , we must study how often  $M_k = (0, 0)$ . Now,  $M_{T_k} = (0, 0)$  if and only if  $X_k = 0$  and  $Y_k = 0$ . Since  $(Y_k)$  is a simple random walk, the event  $\{Y_k = 0\}$  is realised only at the instants  $\sigma_n, n = 0, 1, 2, \dots$ .

**Remark 2.1.** The significance of the random variable  $X_n$  is that it is the horizontal displacement after  $n$  vertical moves of the skeleton  $(Y_l)$ . Note that the random walk  $(X_n)$  has unbounded (although integrable) increments. As a matter of fact, they are signed integer-valued geometric random variables. Contrary to  $(X_n)$ , the increments of the process  $(X_{\sigma_n})_{n \in \mathbb{N}}$ , sampled at instants  $\sigma_n$ , are unbounded with heavy tails.

Recall that all random variables are defined on the same probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ ; introduce the following sub- $\sigma$ -algebras:  $\mathcal{H} = \sigma(\xi_i^{(y)}; i \in \mathbb{N}^*, y \in \mathbb{Z})$ ,  $\mathcal{G} = \sigma(\rho_y, \lambda_y; y \in \mathbb{Z})$ , and  $\mathcal{F}_n = \sigma(\psi_i; i = 1, \dots, n)$ , with  $\mathcal{F} \equiv \mathcal{F}_\infty$ .

**Lemma 2.2.** (See [3, Lemma 2.8].) *It holds that*

$$\sum_{l=0}^{\infty} \mathbb{P}(\mathbf{M}_l = (0, 0) \mid \mathcal{F} \vee \mathcal{G}) = \sum_{n=0}^{\infty} \mathbb{P}(I(X_{\sigma_n}, \varepsilon_0 \xi_0^0) \ni 0 \mid \mathcal{F} \vee \mathcal{G}),$$

where  $\xi_0^0$  has the same law as  $\xi_1^{(0)}$  and, for  $x \in \mathbb{Z}$ ,  $z \in \mathbb{N}$ , and  $\varepsilon = \pm 1$ ,  $I(x, \varepsilon z) = \{x, \dots, x+z\}$  if  $\varepsilon = +1$  and  $\{x-z, \dots, x\}$  if  $\varepsilon = -1$ .

**Lemma 2.3.** (See [3, Lemma 2.9].)

1. If  $\sum_{n=0}^{\infty} \mathbb{P}_0(X_{\sigma_n} = 0 \mid \mathcal{F} \vee \mathcal{G}) = \infty$  then  $\sum_{l=0}^{\infty} \mathbb{P}(\mathbf{M}_l = (0, 0) \mid \mathcal{F} \vee \mathcal{G}) = \infty$ .
2. If  $(X_{\sigma_n})_{n \in \mathbb{N}}$  is transient then  $(M_n)_{n \in \mathbb{N}}$  is also transient.

Let  $\xi$  be a geometric random variable equidistributed with  $\xi_i^{(y)}$ . Denote

$$\chi(\theta) = \mathbb{E} \exp(i\theta \xi) = \frac{q}{1 - p \exp(i\theta)} = r(\theta) \exp(i\alpha(\theta)), \quad \theta \in [-\pi, \pi]$$

as its characteristic function, where

$$r(\theta) = |\chi(\theta)| = \frac{q}{\sqrt{q^2 + 2p(1 - \cos \theta)}} = r(-\theta);$$

$$\alpha(\theta) = \arctan \frac{p \sin \theta}{1 - p \cos \theta} = -\alpha(-\theta).$$

Note that  $r(\theta) < 1$  for  $\theta \in [-\pi, \pi] \setminus \{0\}$ . Then

$$\begin{aligned} \mathbb{E} \exp(i\theta X_n) &= \mathbb{E}(\mathbb{E}(\exp(i\theta X_n) \mid \mathcal{F} \vee \mathcal{G})) \\ &= \mathbb{E} \left( \mathbb{E} \left( \exp \left( i\theta \sum_{y \in \mathbb{Z}} \varepsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)} \right) \mid \mathcal{F} \vee \mathcal{G} \right) \right) \\ &= \mathbb{E} \left( \prod_{y \in \mathbb{Z}} \chi(\theta \varepsilon_y)^{\eta_{n-1}(y)} \right). \end{aligned}$$

### 3. Proof of transience

Adopting the notation of [3], we introduce the constants  $\delta_i > 0$  for  $i = 1, 2, 3$ , and for  $n \in \mathbb{N}$  the sequence of events  $A_n = A_{n,1} \cap A_{n,2}$  and  $B_n$  that are defined by

$$A_{n,1} = \left\{ \omega \in \Omega : \max_{0 \leq k \leq 2n} |Y_k| < n^{1/2+\delta_1} \right\},$$

$$A_{n,2} = \left\{ \omega \in \Omega : \max_{y \in \mathbb{Z}} \eta_{2n-1}(y) < n^{1/2+\delta_2} \right\},$$

$$B_n = \left\{ \omega \in A_n : \left| \sum_{y \in \mathbb{Z}} \varepsilon_y \eta_{2n-1}(y) \right| > n^{1/2+\delta_3} \right\},$$

the range of possible values for  $\delta_i$ ,  $i = 1, 2, 3$ , will be chosen later (see the end of the proof of Proposition 3.2). Obviously  $A_{n,1}$ ,  $A_{n,2}$ , and, hence,  $A_n$  belong to  $\mathcal{F}_{2n}$ ; moreover,  $B_n \subseteq A_n$  and  $B_n \in \mathcal{F}_{2n} \vee \mathcal{G}$ . We denote in the sequel generically  $d_{n,i} = n^{1/2+\delta_i}$  for  $i = 1, 2, 3$ .

Since  $B_n \subseteq A_n$  and both sets are  $\mathcal{F}_{2n} \vee \mathcal{G}$ -measurable, decomposing the unity as  $1 = \mathbf{1}_{B_n} + \mathbf{1}_{A_n \setminus B_n} + \mathbf{1}_{A_n^c}$ , we obtain  $p_n = p_{n,1} + p_{n,2} + p_{n,3}$ , where  $p_n = \mathbb{P}(X_{2n} = 0; Y_{2n} = 0)$ ,  $p_{n,1} = \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; B_n)$ ,  $p_{n,2} = \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n)$ , and  $p_{n,3} = \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; A_n^c)$ . By repeating verbatim the reasoning of Propositions 4.1 and 4.3 of [3], we obtain the following proposition.

**Proposition 3.1.** *For large  $n$ , there exists  $\delta > 0$ ,  $\delta' > 0$ ,  $c > 0$ , and  $c' > 0$  such that*

$$p_{n,1} = \mathcal{O}(\exp(-cn^\delta)) \quad \text{and} \quad p_{n,3} = \mathcal{O}(\exp(-c'n^{\delta'})).$$

Consequently,  $\sum_{n \in \mathbb{N}} (p_{n,1} + p_{n,3}) < \infty$ . The proof will be complete if we can show that  $\sum_{n \in \mathbb{N}} p_{n,2} < \infty$ .

**Lemma 3.1.** *On the set  $A_n \setminus B_n$ , we have, uniformly on  $\mathcal{F} \vee \mathcal{G}$ ,*

$$\mathbb{P}(X_{2n} = 0 \mid \mathcal{F} \vee \mathcal{G}) = \mathcal{O}\left(\sqrt{\frac{\ln n}{n}}\right).$$

*Proof.* Use the  $\mathcal{F} \vee \mathcal{G}$ -measurability of the variables  $(\varepsilon_y)_{y \in \mathbb{Z}}$  and  $(\eta_n(y))_{y \in \mathbb{Z}, n \in \mathbb{N}}$  to express the conditional characteristic function of the variable  $X_{2n}$  as follows:

$$\chi_1(\theta) = \mathbb{E}(\exp(i\theta X_{2n}) \mid \mathcal{F} \vee \mathcal{G}) = \prod_{y \in \mathbb{Z}} \chi(\theta \varepsilon_y)^{\eta_{2n-1}(y)}.$$

Hence,  $\mathbb{P}(X_{2n} = 0 \mid \mathcal{F} \vee \mathcal{G}) = (1/2\pi) \int_{-\pi}^{\pi} \chi_1(\theta) d\theta$ . Now use the decomposition of  $\chi$  into the modulus part,  $r(\theta)$  (that is an even function of  $\theta$ ) and the angular part of  $\alpha(\theta)$  and the fact that there is a constant  $K < 1$  such that for  $\theta \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$  we can bound  $r(\theta) < K$  to majorise

$$\mathbb{P}(X_{2n} = 0 \mid \mathcal{F} \vee \mathcal{G}) \leq \frac{1}{\pi} \int_0^{\pi/2} r(\theta)^{2n} d\theta + \mathcal{O}(K^n).$$

Fix  $a_n = \sqrt{(\ln n)/n}$  and split the integration integral  $[0, \pi/2]$  into  $[0, a_n] \cup [a_n, \pi/2]$ . For the first part, we majorise the integrand by 1, so that  $\int_0^{a_n} r(\theta)^{2n} d\theta \leq a_n$ .

For the second part, we use the fact that  $r(\theta)$  is decreasing for  $\theta \in [0, \pi/2]$ . Hence,  $(1/\pi) \int_{a_n}^{\pi/2} r(\theta)^{2n} d\theta \leq \frac{1}{2} r(a_n)^{2n}$ . Now,  $\lim_{n \rightarrow \infty} a_n = 0$ , hence, for large  $n$  it is enough to study the behaviour of  $r$  near 0, namely  $r(\theta) \asymp 1 - \frac{3}{8}\theta^2 + \mathcal{O}(\theta^4)$ . It follows that  $r(a_n)^{2n} \asymp (1 - \frac{3}{4}((\ln n)/2n))^{2n} \asymp \exp(-\frac{3}{4} \ln n) = n^{-3/4}$ . Since the estimate of the first part dominates, the result follows. This completes the proof.

**Lemma 3.2.** *Let  $d$  be a positive integer,  $Z$  an integer-valued random variable with law  $\mu_Z$ , and  $G$  a centred Gaussian random variable with variance  $d^2$ , but otherwise independent of  $Z$ . Then, there exists a constant  $C > 0$  (independent of  $d$  and of the law of  $Z$ ) such that*

$$\mathbb{P}\left(|Z| \leq \frac{d}{2}\right) \leq C\mathbb{P}(|Z + G| \leq d).$$

*Proof.* Denote by  $\gamma(g) = (1/\sqrt{2\pi d^2}) \exp(-g^2/2d^2)$  the density of the Gaussian random variable  $G$  and observe that on  $[-d/2, d/2]$  the density is minorised by  $\gamma(g) \geq 2C^{-1}/d$  with  $C = \sqrt{2\pi e}$ . Then

$$\mathbb{P}(|Z + G| \leq d) \geq \int_{-d/2}^{d/2} \mu_Z([-d - g, \dots, d - g]) \gamma(g) dg \geq 2C^{-1} \mu_Z\left(\left[-\frac{d}{2}, \dots, \frac{d}{2}\right]\right).$$

This completes the proof.

**Proposition 3.2.** *For all  $\beta < 1$ , there exists a  $\delta_\beta > 0$  such that—uniformly in  $\mathcal{F}$ —for all large  $n$*

$$\mathbb{P}(A_n \setminus B_n \mid \mathcal{F}) = \mathcal{O}(n^{-\delta_\beta}).$$

*Proof.* The required probability is an estimate, on the event  $A_n$ , of the conditional probability  $\mathbb{P}(|\sum_{y \in \mathbb{Z}} \zeta_{y,n}| \leq d_{n,3} \mid \mathcal{F})$ , where we denote  $\zeta_{y,n} = \varepsilon_y \eta_{2n-1}(y)$ . Extend the probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  to carry an auxiliary variable  $G$  assumed to be a centred Gaussian with variance  $d_{n,3}^2$ , (conditionally on  $\mathcal{F}$ ) independent of the  $\zeta_{y,n}$ s. Since  $G$  is a symmetric random variable and  $[-d_{n,3}, d_{n,3}]$  is a symmetric set around 0, then by Lemma 3.2, there exists a positive constant  $c := \sqrt{2\pi}e$  (hence, independent of  $n$ ) such that

$$\mathbb{P}\left(\left|\sum_{y \in \mathbb{Z}} \zeta_{y,n}\right| \leq d_{n,3} \mid \mathcal{F}\right) \leq c\mathbb{P}\left(\left|\sum_{y \in \mathbb{Z}} \zeta_{y,n} + G\right| \leq d_{n,3} \mid \mathcal{F}\right).$$

Let  $\chi_2(t) = \mathbb{E}(\exp(it \sum_y \zeta_{y,n}) \mid \mathcal{F}) = \prod_y A_{y,n}(t)$ , where  $A_{y,n}(t) = \mathbb{E}(\exp(it \zeta_{y,n}) \mid \mathcal{F})$  and  $\chi_3(t) = \mathbb{E}(\exp(itG) \mid \mathcal{F}) = \exp(-t^2 d_{n,3}^2 / 2)$ . Therefore,  $\mathbb{E}(\exp(it(\sum_y \zeta_{y,n} + G)) \mid \mathcal{F}) = \chi_2(t)\chi_3(t)$ , and using the Plancherel’s formula,

$$\mathbb{P}\left(\left|\sum_{y \in \mathbb{Z}} \zeta_{y,n} + G\right| \leq d_{n,3} \mid \mathcal{F}\right) = \frac{d_{n,3}}{\pi} \int_{\mathbb{R}} \frac{\sin(td_{n,3})}{td_{n,3}} \chi_2(t)\chi_3(t) dt \leq C d_{n,3} I,$$

where  $I = \int_{\mathbb{R}} |\chi_2(t)| \exp(-t^2 d_{n,3}^2 / 2) dt$ . Fix  $b_n = n^{\delta_4} / d_{n,3}$ , for some  $\delta_4 > 0$  and split the integral defining  $I$  into  $I_1 + I_2$ , the first part being for  $|t| \leq b_n$  and the second for  $|t| > b_n$ .

We have

$$I_2 \leq C \int_{|t| > b_n} \exp\left(-\frac{t^2 d_{n,3}^2}{2}\right) \frac{dt}{2\pi} = \frac{C}{d_{n,3}} \int_{|s| > n^{\delta_4}} \exp\left(-\frac{s^2}{2}\right) \frac{ds}{2\pi} \leq 2 \frac{C}{d_{n,3}} \frac{1}{n^{\delta_4}} \frac{\exp(-n^{2\delta_4} / 2)}{2\pi}$$

because the probability that a centred normal random variable of variance 1, whose density is denoted by  $\phi$ , exceeds a threshold  $x > 0$  is majorised by  $\phi(x)/x$ .

For  $I_1$  we obtain  $I_1 \leq \int_{|t| \leq b_n} \prod_y |A_{y,n}(t)| dt$ .

Assume for the moment that the inequality  $|t\eta_{2n-1}(y)| \leq 1$  holds. Use the fact that  $\cos(x) \leq 1 - x^2/4$ , valid for  $|x| \leq 1$ , to write

$$\begin{aligned} |A_{y,n}(t)| &= |\mathbb{E}_0(\exp(it\varepsilon_y \eta_{2n-1}(y)) \mid \mathcal{F})| \\ &= \left| \left(1 - \frac{c}{|y|^\beta}\right) \exp(it\eta_{2n-1}(y)f(y)) + \frac{c}{|y|^\beta} \cos(t\eta_{2n-1}(y)) \right| \\ &\leq 1 - \frac{c}{|y|^\beta} + \frac{c}{|y|^\beta} \left(1 - \frac{t^2 \eta_{2n-1}^2(y)}{4}\right) \\ &= 1 - \frac{ct^2 \eta_{2n-1}^2(y)}{4|y|^\beta} \\ &\leq \exp\left(-\frac{ct^2 \eta_{2n-1}^2(y)}{4|y|^\beta}\right). \end{aligned}$$

The assumed inequality  $|t\eta_{2n-1}(y)| \leq 1$  is verified whenever the constants  $\delta_2, \delta_3$ , and  $\delta_4$  are chosen so that  $\delta_2 + \delta_4 - \delta_3 < 0$  holds, because  $|t| \leq b_n$  and  $b_n = n^{d_4} / n^{1/2 + \delta_3}$ , whereas  $\eta_{2n-1}(y) \leq n^{1/2 + \delta_2}$ . Therefore,

$$|\chi_2(t)| \leq \prod_y \exp\left(-\frac{t^2}{4} \eta_{2n-1}^2(y) \frac{c}{|y|^\beta}\right).$$

Now, define  $\pi_n(y) = \eta_{2n-1}(y)/2n$ ; obviously  $\sum_y \pi_n(y) = 1$ , establishing that  $(\pi_n(y))_y$  is a probability measure on  $\mathbb{Z}$ . Therefore, applying Hölder’s inequality we obtain  $I_1 \leq \prod'_y J_n(y)^{\pi_n(y)}$ , where  $\prod'_y$  means that the product runs over those  $y$  such that  $\eta_{2n-1}(y) \neq 0$  and

$$\begin{aligned} J_n(y) &= \int_{-b_n}^{b_n} \exp\left(-\frac{t^2}{4} \eta_{2n-1}^2(y) \frac{c}{|y|^\beta} \frac{1}{\pi_n(y)}\right) dt \\ &= \sqrt{\frac{2\pi |y|^\beta}{cn \eta_{2n-1}(y)}} \int_{-b_n \sqrt{cn \eta_{2n-1}(y)/|y|^\beta}}^{b_n \sqrt{cn \eta_{2n-1}(y)/|y|^\beta}} \exp\left(-\frac{v^2}{2} \frac{dv}{\sqrt{2\pi}}\right) \\ &\leq \sqrt{\frac{4\pi}{c}} \exp\left(-\log 2n - \frac{1}{2} \log \pi_n(y) + \frac{\beta}{2} \log |y|\right). \end{aligned}$$

We conclude that

$$I_1 \leq \prod'_y J_n(y)^{\pi_n(y)} \leq \sqrt{\frac{2\pi}{c}} \exp\left(-\log 2n + \frac{1}{2} H(\pi_n) + \frac{\beta}{2} \sum_y \pi_n(y) \log |y|\right)$$

and  $H(\pi_n)$  is the entropy of the probability measure  $\pi_n$ , reading (with the convention that  $0 \log 0 = 0$ )

$$H(\pi_n) := - \sum_y \pi_n(y) \log \pi_n(y) \leq \log \text{card } C_n,$$

where  $C_n := \text{supp } \pi_n$  and, on  $A_n$ ,  $\text{card } C_n \leq 2n^{1/2+\delta_1}$ . We conclude that we can always chose the parameters  $\delta_1$  and  $\delta_3$  such that, for every  $\beta < 1$  there exists a parameter  $\delta_\beta > 0$  such that  $d_{n,3} I_1 \leq Cn^{-\delta_\beta}$ . This completes the proof.

**Corollary 3.1.** *We have*

$$\sum_{n \in \mathbb{N}} p_{n,2} < \infty.$$

*Proof.* Recall that for the standard random walk  $\mathbb{P}(Y_{2n} = 0) = \mathcal{O}(n^{-1/2})$ ; combining with the estimates obtained in (3.1) and (3.2), we have

$$\begin{aligned} p_{n,2} &= \mathbb{P}(X_{2n} = 0; Y_{2n} = 0; A_n \setminus B_n) \\ &= \mathbb{E}[\mathbb{E}(\mathbf{1}_{Y_{2n}=0} [\mathbb{E}(\mathbf{1}_{A_n \setminus B_n} \mathbb{P}(X_{2n} = 0 \mid \mathcal{F} \vee \mathcal{G}) \mid \mathcal{F})])] \\ &= \mathcal{O}\left(n^{-1/2} n^{-\delta_\beta} \sqrt{\frac{\ln n}{n}}\right) \\ &= \mathcal{O}(n^{-(1+\delta_\beta)} \sqrt{\ln n}), \end{aligned}$$

proving thus the summability of  $p_{n,2}$ .

*Proof of the statement on transience of Theorem 1.2.* The function  $p_n = p_{n,1} + p_{n,2} + p_{n,3}$  is summable because all the partial probabilities  $p_{n,i}$  for  $i = 1, 2, 3$  are all shown to be summable. This completes the proof.

### 4. Proof of recurrence

We define the following sequence of random times:

$$\tau_0 \equiv 0 \quad \text{and} \quad \tau_{n+1} = \inf\{k : k > \tau_n, |Y_k - Y_{\tau_n}| = Q\} \quad \text{for } n \geq 0.$$



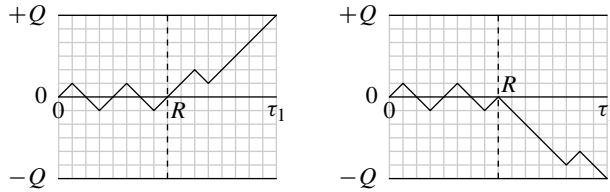


FIGURE 1: Illustration of the modified reflection principle. The left-hand plot depicts a detail of the upcrossing excursion, occurring between times 0 and  $\tau_1$ . The right-hand plot depicts the details of a new admissible path bijectively obtained from  $Y$  by defining it as identical to  $Y$  for the times  $0 \leq t \leq R$  and then by reverting the flow of time and displacing the remaining portion of the path by  $-Q$ , as explained in the text.

The random variables  $(\tau_{n+1} - \tau_n)_{n \geq 0}$  are independent and for all  $n$  the variable  $\tau_{n+1} - \tau_n$  has the same distribution (under  $\mathbb{P}_0$ ) as  $\tau_1$ . It is straightforward to show (see, e.g. [1, Proposition 1.13.4]) that these random variables have exponential moments, i.e.  $\mathbb{E}_0(\exp(\alpha \tau_1)) < \infty$  for sufficiently small  $|\alpha|$ .

Let  $\mathbb{Z}_Q = \mathbb{Z}/Q\mathbb{Z} = \{0, 1, \dots, Q - 1\}$  with integer addition replaced by addition modulo  $Q$  and for any  $y \in \mathbb{Z}$  denote  $\bar{y} = y \bmod Q \in \mathbb{Z}_Q$ . Consistently, we define  $\bar{Y}_n = Y_n \bmod Q$ .

**Lemma 4.1.** Define for  $n \geq 1$  and  $\bar{y} \in \mathbb{Z}_Q$ ,

$$N_n(\bar{y}) := \bar{\eta}_{\tau_{n-1}, \tau_n-1}(\bar{y}) = \sum_{k=\tau_{n-1}}^{\tau_n-1} \mathbf{1}_{\bar{y}}(\bar{Y}_k).$$

Then, for every  $\bar{y} \in \mathbb{Z}_Q$ ,

1. the conditional laws of  $N_1(\bar{y})$  with respect to the events  $\{Y_{\tau_1} = Q\}$  and  $\{Y_{\tau_1} = -Q\}$  are the same; and
2. the following equalities hold:

$$\mathbb{E}_0 N_1(\bar{y}) = \mathbb{E}_0(N_1(\bar{y}) \mid Y_{\tau_1} = Q) = \mathbb{E}_0(N_1(\bar{y}) \mid Y_{\tau_1} = -Q) = \frac{\mathbb{E}_0 \tau_1}{Q}.$$

*Proof.* 1. Denote by  $U := \{Y_{\tau_1} = Q\}$  and  $D := \{Y_{\tau_1} = -Q\}$  the sets of conditioning. Assume that  $Y$  is a trajectory in  $U$  and define  $R := \max\{t : 0 \leq t < \tau_1, Y_t = 0\}$ . Now, between times 0 and  $R$ , the path  $Y$  wanders around level 0. For times  $t$  such that  $R < t < \tau_1$ , the path remains strictly confined within the (interior of the) strip.

For any trajectory  $Y$  in  $U$ , we shall define a new trajectory  $V$  (bijectively determined from  $Y$ ) belonging to  $D$  as follows:

$$V_t = \begin{cases} Y_t & \text{for } 0 \leq t \leq R, \\ Y_{\tau_1-(t-R)} - Q & \text{for } R \leq t \leq \tau_1. \end{cases}$$

Obviously, the above construction is a bijection. Hence, for trajectories not in  $U$  (i.e. trajectories in  $D$ ) the modified trajectory is defined by inverting the previous transformation. Figure 1 illustrates the construction (modified reflection principle).

Denoting  $\eta$  to be the occupation measure of the process  $Y$  and  $\kappa$  to be the occupation measure of  $V$ , we have  $\bar{\kappa}_{\tau_1-1}(\bar{y}) := \sum_{i=0}^{\tau_1-1} \mathbf{1}_{\bar{y}}(\bar{V}_i) = \sum_{i=0}^{\tau_1-1} \mathbf{1}_{\bar{y}}(\bar{Y}_i) =: \bar{\eta}_{\tau_1-1}(\bar{y})$  by construction of the path  $V$ . This remark implies that  $\bar{\eta}_{\tau_1-1}(\cdot)$  and  $\bar{\kappa}_{\tau_1-1}(\cdot)$  have the same law.

2. Since the random walk  $(Y_n)$  is symmetric, the probability of exiting the strip of width  $Q$  by upcrossing is the same as for a downcrossing. Hence,  $\mathbb{E}_0 N_1(\bar{y}) = \frac{1}{2} \mathbb{E}_0(N_1(\bar{y}) \mid Y_{\tau_1} = Q) + \frac{1}{2} \mathbb{E}_0(N_1(\bar{y}) \mid Y_{\tau_1} = -Q)$ . This remark, combined with the equality of the conditional laws proved in 1 establishes the left-most and the central equalities of the statement.

To prove the right-most equality, let  $g : \mathbb{Z} \rightarrow \mathbb{R}$  be a bounded function and denote by  $S_n[g] = \sum_{k=0}^{n-1} g(Y_k)$ . On defining  $W_n[g] = \sum_{k=\tau_n}^{\tau_{n+1}-1} g(Y_k)$  and  $R_n = \max\{k : \tau_k \leq n\}$ , we have the decomposition

$$S_n[g] = \sum_{k=0}^{R_n} W_k[g] - \sum_{k=n}^{\tau_{R_n+1}-1} g(Y_k).$$

Since  $\tau_{R_n+1} - n \leq \tau_{R_n+1} - \tau_{R_n}$ , we have, thanks to the boundedness of  $g$ , that

$$\frac{1}{n} \left| \sum_{k=n}^{\tau_{R_n+1}-1} g(Y_k) \right| \leq \frac{\tau_{R_n+1} - \tau_{R_n}}{n} \sup_{y \in \mathbb{Z}} |g(y)|.$$

Since

$$\mathbb{P}(\tau_{R_n+1} - \tau_{R_n} = l) \leq \sum_{k=0}^n \mathbb{P}(\tau_{k+1} - \tau_k = l; R_n = k) \leq \sum_{k=0}^n \mathbb{P}(\tau_{k+1} - \tau_k = l),$$

it follows that for all  $\varepsilon > 0$ , we have  $\sum_{k=1}^n \mathbb{P}(\tau_{k+1} - \tau_k \geq \varepsilon n) \leq n \mathbb{P}(\tau_1 \geq \varepsilon n)$  which tends to 0, when  $n \rightarrow \infty$ , thanks to Markov inequality and the existence of exponential moments for  $\tau_1$ .

It remains to estimate  $S_n[g]/n$  by  $(R_n/n) \sum_{k=0}^{R_n} W_k[g]$ . Obviously,  $R_n \rightarrow \infty$  a.s. as  $n \rightarrow \infty$ , and, by the renewal theorem (see [1, p. 221] for instance),  $R_n/n \rightarrow 1/\mathbb{E}_0 \tau_1$  a.s. Fix any  $\bar{y} \in \mathbb{Z}_Q$  and choose  $g(z) := \mathbf{1}_{\{\bar{y}\}}(z \bmod Q)$ . For  $g$ , we have  $S_n[g] = \bar{\eta}_n(\bar{y})$ , where  $\bar{\eta}_n(\bar{y}) = \sum_{k=0}^{n-1} \mathbf{1}_{\{\bar{y}\}}(\bar{Y}_k)$ . But  $(\bar{Y}_k)$  is a simple random walk on the finite set  $\mathbb{Z}_Q$ , therefore, admits a unique invariant probability  $\bar{\pi}(\bar{y}) = 1/Q$ . By the ergodic theorem for Markov chains, we have  $S_n[g]/n \rightarrow 1/Q$  a.s.

Additionally, for this choice of  $g$ , the sequence  $(W_k[g])_{k \in \mathbb{N}}$  are independent random variables, identically distributed as  $N_1(\bar{y})$ . We conclude by applying the law of large numbers to the ratio  $(1/R_n) \sum_{k=1}^{R_n} W_k[g]$ .

To prove an almost sure recurrence, it is enough to show  $\sum_{k \in \mathbb{N}} \mathbb{P}_0(X_{\sigma_k} = 0, Y_{\sigma_k} = 0 \mid \mathcal{G}) = \infty$ . If  $\beta > 1$  then  $\sum_y \mathbb{P}(\lambda_y = 1) < \infty$ ; hence, by the Borel–Cantelli lemma, there is almost surely a finite number of  $y$ s such that  $\lambda_y = 1$ , i.e. the  $\mathcal{G}$ -measurable random variable  $l(\omega) = \max\{|y| : \lambda_y = 1\}/Q$  is almost surely finite. Fixing an integer  $L(\omega) \geq l(\omega) + 1$ , and introducing the  $\mathcal{F} \vee \mathcal{G}$ -measurable random sets, we obtain

$$F_{L,2n}(\omega) = \{k : 0 \leq k \leq 2n - 1; |Y_{\tau_k(\omega)}(\omega)| \leq L(\omega)Q; |Y_{\tau_{k+1}(\omega)}(\omega)| \leq L(\omega)Q\},$$

$$G_{L,2n}(\omega) = \{k : 0 \leq k \leq 2n - 1; |Y_{\tau_k(\omega)}(\omega)| \geq L(\omega)Q; |Y_{\tau_{k+1}(\omega)}(\omega)| \geq L(\omega)Q\}.$$

To simplify the notation, we drop the explicit reference to the  $\omega$  dependence of those sets.

We denote  $\text{Adm}(2n)$  to be the set of *admissible paths*  $\mathbf{z} = (z_0, z_1, \dots, z_{2n-1}, z_{2n}) \in \mathbb{Z}^{2n+1}$  satisfying  $|z_{i+1} - z_i| = 1$  for  $i = 0, \dots, 2n - 1$  and  $z_0 = 0$ . For any  $\mathbf{z} \in \text{Adm}(2n)$ , we denote  $C[\mathbf{z}]$  to be the cylinder set

$$C[\mathbf{z}] = \{\omega \in \Omega : Y_0(\omega) = Qz_0 = 0, Y_{\tau_1(\omega)}(\omega) = Qz_1, \dots, Y_{\tau_{2n}(\omega)}(\omega) = Qz_{2n}\} \in \mathcal{F}.$$

Denoting  $\theta_k = X_{\tau_{k+1}} - X_{\tau_k}$ , for  $k \in \{0, \dots, 2n - 1\}$ , and observing that

$$X_{\tau_{2n}} = \sum_{k=0}^{2n-1} \theta_k = \sum_{k \in F_{L,2n}} \theta_k + \sum_{k \in G_{L,2n}} \theta_k,$$

the summations appearing in the above decomposition refer to disjoint excursions. This completes the proof.

**Proposition 4.1.** *For every  $z \in \text{Adm}(2n)$  and every  $k \in G_{L,2n}(\omega)$ , with  $\omega \in C[z]$ ,*

$$a_k := \mathbb{E}_0(\theta_k \mid C[z]; \mathcal{G}) = 0.$$

*Proof.* Let  $z$  be an arbitrary admissible path and suppose that  $k$  corresponds, say, to an upcrossing (i.e.  $z_{k+1} - z_k = 1$ ) and abbreviate  $z := z_k$  and  $z + 1 = z_{k+1}$  in order to simplify the notation. Since  $z \in \text{Adm}(2n)$ , then for all  $\omega \in C[z]$ , the random times  $\tau_1, \dots, \tau_{2n}$  are compatible with  $z$ , meaning, in particular, that

$$Y_{\tau_k} = Qz \quad \text{and} \quad Y_{\tau_{k+1}} = Qz + Q.$$

The horizontal increments  $(\theta_k)_{k \in G_{L,2n}}$ , conditionally on  $C[z]$  and  $\mathcal{G}$ , are independent. To simplify the notation, we introduce the symbol  $T := \tau_{k+1} - \tau_k$ ; obviously,  $T \stackrel{D}{=} \tau_1$  conditionally on the starting point; more precisely  $\mathbb{P}_{Qz_k}(T = t) = \mathbb{P}_0(\tau_1 = t)$ , for all  $t \in \mathbb{N}$ .

We now complete the proof of the proposition.

$$\begin{aligned} a_k &= \mathbb{E}_0 \left( \sum_{y \in \mathbb{Z}} f(y) \sum_{i=0}^{\eta_{\tau_k, \tau_{k+1}-1}(y)} \xi_i^y \mid C[z]; \mathcal{G} \right) \\ &= \mathbb{E}(\xi_0^0) \sum_{y=Qz-Q+1}^{Qz+Q-1} f(\bar{y}) \mathbb{E}_{Qz}(\eta_{T-1}(y) \mid Y_T = Qz + Q; C[z]; \mathcal{G}) \\ &\quad \times \mathbb{P}_{Qz}(Y_T = Qz + Q \mid C[z]; \mathcal{G}) \\ &= \mathbb{E}(\xi_0^0) \sum_{\bar{y} \in \mathbb{Z}_Q} f(\bar{y}) \mathbb{E}_0(N_1(\bar{y}) \mid Y_{\tau_1} = Q) \\ &= \sum_{\bar{y} \in \mathbb{Z}_Q} f(\bar{y}) \frac{\mathbb{E}_0(\tau_1)}{Q} \mathbb{E}(\xi_0^0) \\ &= 0, \end{aligned}$$

where we use the strong Markov property from Lemma 4.1, and the centring condition  $\sum_{\bar{y} \in \mathbb{Z}_Q} f(\bar{y}) = 0$  to conclude. This completes the proof.

The sampled process  $Z_k = Y_{\tau_k}/Q \in \mathbb{Z}$  is a standard simple symmetric nearest neighbour random walk on  $\mathbb{Z}$ . For  $z \in \mathbb{Z}$ , we define the occupation measure  $\varpi_n(z) := \varpi(\{z\}) = \sum_{k=1}^n \mathbf{1}_{\{z\}}(Z_k)$ .

**Lemma 4.2.** *Fix  $K > 0$ . For every  $\delta > 0$  there exists a constant  $c > 0$  such that, for all sufficiently large  $n$ ,*

$$\mathfrak{P}_n = \mathbb{P}_0 \left( \max_{z: |z| \leq K} \varpi_{2n}(z) > c\sqrt{n} \mid Z_{2n} = 0 \right) < \delta.$$

**Remark 4.1.** Lemma 4.2 will be used in the course of the proof by fixing a  $\mathcal{G}$ -measurable almost surely finite  $K$ , while  $n$  will tend to  $\infty$ . Of course,  $c = c(K, \delta)$  depends on the choice of  $K$  and  $\delta$ .

*Proof.* On denoting  $m_n = \lfloor c\sqrt{n} \rfloor$ , the conditional Markov inequality implies that,

$$\mathfrak{P}_n \leq \sum_{z=-K}^K \mathbb{P}_0(\varpi_{2n}(z) > m_n \mid Z_{2n} = 0) \leq \sum_{z=-K}^K \frac{\mathbb{E}_0(\varpi_{2n}(z) \mid Z_{2n} = 0)}{m_n}.$$

Now

$$\mathbb{E}_0(\varpi_{2n}(z) \mid Z_{2n} = 0) = \sum_{k=1}^{2n} \mathbb{P}_0(Z_k = z \mid Z_{2n} = 0) = \sum_{k=1}^{2n} \frac{P^k(0, z)P^{2n-k}(z, 0)}{P^{2n}(0, 0)},$$

where  $P^l(0, z) = \mathbb{P}_0(Z_l = z)$ . For all  $z \geq 0$  and all  $l \geq z$ , then  $P^l(0, z) = 2^{-l} \binom{l}{l+z/2}$  if  $l + z$  is even, and 0 otherwise. We majorise  $\binom{l}{l+z/2} \leq \binom{l}{l/2}$  when  $l$  is even, and  $\binom{l}{l+z/2} \leq \binom{l}{l-1/2}$  when  $l$  is odd. Using Stirling’s formula, we see that for all sufficiently large  $l$ , the probability  $P^l(0, z)$  is majorised independently of the parity of  $l$  by a term equivalent (for large  $l$ ) to  $1/\sqrt{l}$ . Consequently by choosing an appropriate constant  $C$ , the same majorisation holds for the remaining finite set of values of  $l$ . By approximating the sum by an integral, we obtain

$$\mathbb{E}_0(\varpi(z) \mid Z_{2n} = 0) \leq C \int_0^{2n} \sqrt{\frac{2n}{t(2n-t)}} dt \leq e\sqrt{n}.$$

We conclude that  $\mathfrak{P}_n \leq \delta$  provided that  $c > (2K + 1)/C$ .

*Proof of the recurrence statement of Theorem 1.2.* We shall now fix  $K = L$ . For  $\delta \in (0, 1)$ , let  $c = c(K, \delta)$  as in Lemma 4.2. Also from Lemma 4.2, we have  $\mathbb{P}_0(\text{card } F_{L,2n} \leq c\sqrt{n}) \geq 1 - \delta$  on the set  $\{Z_{2n} = 0\}$ . Fix some constant  $d$  and define

$$\begin{aligned} &\text{ConsAdm}(L, 2n, d) \\ &= \{z \in \text{Adm}(2n) : z_{2n} = 0; |\{k : 0 \leq k < 2n, |z_k| \leq L; |z_{k+1}| \leq L\}| \leq d\sqrt{n}\} \end{aligned}$$

the set of *constrained admissible paths*. (Here and in the sequel, we use the symbols  $|A|$  or  $\text{card } A$  indistinguishably to denote the cardinality of the discrete set  $A$ ). On the set  $\{Z_{2n} = 0\}$ , obviously the equality  $\{\text{card } F_{L,2n} \leq d\sqrt{n}\} = \bigcup_{z \in \text{ConsAdm}(L,2n,d)} C[z]$  holds.

$$\begin{aligned} &\mathbb{P}_0(X_{\tau_{2n}} = 0; Y_{\tau_{2n}} = 0 \mid \mathcal{G}) \\ &= \mathbb{P}_0(X_{\tau_{2n}} = 0; Z_{2n} = 0 \mid \mathcal{G}) \\ &\geq \mathbb{P}_0(X_{\tau_{2n}} = 0; Y_{\tau_{2n}} = 0; |F_{L,2n}| \leq d\sqrt{n} \mid \{Z_{2n} = 0\}; \mathcal{G})\mathbb{P}_0(Z_{2n} = 0) \\ &= \sum_{z \in \text{ConsAdm}(L,2n,d)} \mathbb{P}_0(\{X_{\tau_{2n}} = 0\} \cap C[z] \mid \{Z_{2n} = 0\}; \mathcal{G})\mathbb{P}_0(Z_{2n} = 0) \\ &= \sum_{z \in \text{ConsAdm}(L,2n,d)} \mathbb{P}_0(X_{\tau_{2n}} = 0 \mid C[z]; \mathcal{G})\mathbb{P}_0(C[z] \mid \mathcal{G}). \end{aligned}$$

Now, for any  $z \in \text{ConsAdm}(L, 2n, d)$ ,

$$\begin{aligned} \mathbb{P}_0(X_{\tau_{2n}} = 0 \mid \mathcal{G}, C[z]) &\geq \sum_{|m| \leq d\sqrt{n}} \mathbb{P}_0\left(\sum_{k \in F_{L,2n}} \theta_k = m; \sum_{k \in G_{L,2n}} \theta_k = -m \mid \mathcal{G}, C[z]\right) \\ &= \sum_{|m| \leq d\sqrt{n}} \mathbb{P}_0\left(\sum_{k \in F_{L,2n}} \theta_k = m \mid \mathcal{G}, C[z]\right) \\ &\quad \times \mathbb{P}_0\left(\sum_{k \in G_{L,2n}} \theta_k = -m \mid \mathcal{G}, C[z]\right). \end{aligned}$$

The joint probability factors,  $\mathbb{P}_0$ , that appear in the last line of this equation are due to the  $\mathcal{G}$ -measurable set-valued random variables  $G_{L,2n}$  and  $F_{L,2n}$  taking disjoint values, hence, the terms in  $F_{L,2n}$  and  $G_{L,2n}$  refer to different excursions of the random walk  $Y$ . Independence follows as a consequence of the strong Markov property.

By Proposition 4.1, we have  $\mathbb{E}(\theta_k \mid C[z], \mathcal{G}) = 0$ . The variables  $(\theta_k)_{k \in G_{L,2n}}$  are independent and identically distributed conditionally to  $\mathcal{G}$  and  $C[z]$ ; their common variance,  $\sigma^2$ , is finite because

$$\begin{aligned} \sigma^2 &= \mathbb{E}_0(\theta_k^2 \mid \mathcal{G}, C[z]) \\ &= \mathbb{E}_{Q^{z_k}}\left(\left[\sum_y \varepsilon_y \sum_{i=0}^{\eta_{\tau_k, \tau_{k+1}-1}(y)} \xi_i^y\right]^2 \mid \mathcal{G}\right) \\ &\leq \mathbb{E}_0(\tau_1)\mathbb{E}((\xi_0^0)^2) + \mathbb{E}_0(\tau_1^2)[\mathbb{E}(\xi_0^0)]^2 + [\mathbb{E}_0(\tau_1)]^2[\mathbb{E}(\xi_0^0)]^2 \\ &< \infty, \end{aligned}$$

where we have used the strong Markov property to bound the last term of the first line in the previous equation by the second line.

For  $z \in \text{ConsAdm}(L, 2n, d)$ , we additionally have (on  $C[z]$ ) that  $2n - d\sqrt{n} \leq |G_{L,2n}| \leq 2n$ . Hence, for  $|m| \leq d\sqrt{n}$ , we can apply the local limit theorem (see, for instance, [18, Proposition 52.12, p. 706]), thus,

$$\mathbb{P}_0\left(\sum_{k \in G_{L,2n}} \theta_k = -m \mid \mathcal{G}, C[z]\right) \geq \frac{c_1}{\sqrt{|G_{L,2n}|\sigma^2}} \exp\left(-\frac{m^2}{2|G_{L,2n}|\sigma^2}\right),$$

to obtain  $\mathbb{P}_0(\sum_{k \in G_{L,2n}} \theta_k = -m \mid \mathcal{G}, C[z]) \geq c_2/\sqrt{n}$ , uniformly in  $z$ . We can summarise the estimate obtained so far

$$\begin{aligned} \mathbb{P}_0(X_{\tau_{2n}} = 0, Y_{\tau_{2n}} = 0 \mid \mathcal{G}) &\geq \frac{c_3}{\sqrt{n}} \sum_{z \in \text{ConsAdm}(L, 2n, d)} \mathbb{P}_0(C[z] \mid \mathcal{G}) \mathbb{P}_0\left(\left|\sum_{k \in F_{L,2n}} \theta_k\right| \leq d\sqrt{n} \mid C[z]; \mathcal{G}\right). \end{aligned}$$

Now,

$$\left\{\left|\sum_{k \in F_{L,2n}} \theta_k\right| \leq d\sqrt{n}\right\} \supseteq \left\{\sum_{k \in F_{L,2n}} |\theta_k| \leq d\sqrt{n}\right\} \supseteq \left\{\sum_{k \in F_{L,2n}} \Theta_k \leq d\sqrt{n}\right\},$$

where  $\Theta_k = \sum_y \sum_{i=\tau_k(y)}^{\tau_{k+1}-1(y)} \xi_i^y$  are independent and identically distributed conditionally on  $C[z]$ , with finite mean  $0 \leq \mu = \mathbb{E}\Theta_k = \mathbb{E}(\xi_0^0)\mathbb{E}(T_k) < \infty$  and variance  $0 \leq \sigma^2 = \text{var } \Theta_k < \infty$ , where  $T_k = \tau_{k+1} - \tau_k$  is the time needed for the vertical random walk to cross the strip bounded by  $z_k$  and  $z_{k+1}$ .

Additionally,  $\lim_{n \rightarrow \infty} |F_{L,2n}| = \infty$  a.s., due to the recurrence of the simple symmetric vertical random walk  $(Y_k)$ . From the weak law of large numbers, it follows that for all  $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}_0 \left( \left| \frac{\sum_{k \in F_{L,2n}} \Theta_k}{|F_{L,2n}|} - \mu \right| \leq \varepsilon \right) = 1,$$

hence, for all  $\alpha \in (0, 1)$  and sufficiently large  $n$ ,  $\mathbb{P}_0(|\sum_{k \in F_{L,2n}} \Theta_k / |F_{L,2n}| - \mu| \leq \varepsilon) \geq \alpha$ . Since, for  $z \in \text{ConsAdm}(L, 2n, d)$  we have  $|F_{L,2n}| \leq d\sqrt{n}$ , we conclude that, for all sufficiently large  $n$ ,  $\mathbb{P}_0(|\sum_{k \in F_{L,2n}} \theta_k| \leq d'\sqrt{n} \mid \mathcal{G}) > \alpha$ , with any  $d' > \mu d$ . Finally, for sufficiently large  $n$ ,

$$\begin{aligned} \sum_{z \in \text{ConsAdm}(L, 2n, d)} \mathbb{P}_0(C[z] \mid \mathcal{G}) &= \mathbb{P}_0 \left( \sum_{z: |z| \leq L+1} \varpi(z) \leq d\sqrt{n} \mid Z_{2n} = 0 \right) \mathbb{P}_0(Z_{2n} = 0) \\ &\geq (1 - \delta) \frac{c_3}{\sqrt{n}} \end{aligned}$$

from Lemma 4.2, where  $Z_k = Y_{\tau_k}$ . This concludes the proof of the recurrence.

*Proof of Proposition 1.1.* Since  $\|\lambda\| < \infty$ , it follows that there is a positive integer  $l$  such that  $\lambda_y = 0$  for all  $y$  with  $|y| > l$ . Choosing then an integer  $L > [l/Q] + 1$  in the proof of the recurrence part, we immediately conclude the proof.

### 5. Conclusion, open problems, and further developments

As was apparent during the course of the proof of the recurrence, the condition  $\beta > 1$  is used only to show that there are almost surely finitely as many lines where the periodicity imposed by  $f$  is perturbed by a random defect. Therefore, this condition can be improved. As a matter of fact, the walk is recurrent provided that there exists an arbitrarily large integer  $l$  such that the decay is of the form  $c(|y| \ln |y| \cdots \ln_{l-1} |y| \ln_l^{\beta_l} |y|)^{-1}$  for some  $\beta_l > 1$  (arbitrarily close to 1), where  $\ln_l$  is the  $l$ -times iterated logarithm. Nevertheless, our methods do not allow the treatment of the really critical case of  $\beta_0 = 1$ .

It is easy to build up examples in which the random walk is recurrent although there are infinitely many defects, provided they are sparse. The following deterministic construction illustrates this fact. Let  $(a_n)_{n \in \mathbb{N}}$  be an increasing sequence of positive numbers such that  $a_n \rightarrow \infty$ . For an arbitrary  $\{0, 1\}$ -valued sequence  $\lambda = (\lambda_y)_{y \in \mathbb{Z}}$  we denote  $\lambda \upharpoonright_k$  to be its restriction to  $\{-k, \dots, k\}$  (meaning that  $(\lambda \upharpoonright_k)_y = \lambda_y$  if  $|y| \leq k$  and vanishes otherwise) for  $k \in \mathbb{N}$ ; define also  $\lambda \upharpoonright_\infty \equiv \lambda$ . Obviously  $\|\lambda \upharpoonright_k\| = \text{card}\{y \in \mathbb{Z}: |y| \leq k, \lambda_y = 1\}$ . For a given sequence  $\lambda$ , we write  $\mathbb{P}[\lambda]_0(\cdot)$  for the probability measure corresponding to the environment  $\lambda$ . We shall construct iteratively an infinite deterministic sequence  $\lambda = (\lambda_y)_{y \in \mathbb{Z}}$  of defects perturbing the periodic sequence determined by  $f$  so that the corresponding random walk is recurrent. The first defect is inserted at level 0, i.e. we initialise the sequence to  $\lambda_y^{(1)} = \delta_{y,0}$  so that  $\|\lambda^{(1)}\| = 1$ . Now the random walk with only one defect in the whole vertical axis is recurrent, meaning that

$$\sum_{n \in \mathbb{N}} \mathbb{P}[\lambda^{(1)}]_0(M_n = (0, 0)) = \infty.$$

Therefore, there exists a positive integer  $L_1 > 0$  such that

$$\sum_{n=1}^{L_1} \mathbb{P}[\lambda^{(1)}]_0(\mathbf{M}_n = (0, 0)) \geq a_1.$$

But since in time  $L_1$ , the vertical random walk cannot be further than  $L_1$  from 0, nothing changes if, instead of choosing the sequence  $\lambda^{(1)}$  as above with  $\|\lambda^{(1)}\| = 1$ , we chose *any other* sequence  $\lambda'$  with  $\lambda'_0 = \lambda_0^{(1)} = 1$  and  $\|\lambda' \upharpoonright_{L_1}\| = 1$  in the above equation. The second defect is inserted at level  $L_1 + 1$ , i.e. we modify the sequence into  $\lambda^{(2)}$  verifying  $\lambda_y^{(2)} = \delta_{y,0} + \delta_{y,L_1+1}$ ,  $\lambda^{(2)} \upharpoonright_{L_1} = \lambda^{(1)}$ , and  $\|\lambda^{(2)}\| = 2$ . Again, we can choose a positive integer  $L_2 > L_1$  such that  $\sum_{n=1}^{L_2} \mathbb{P}[\lambda^{(2)}]_0(\mathbf{M}_n = (0, 0)) \geq a_2$ , and so on. In this way, we construct a deterministic sequence

$$\lambda = \lim_{k \rightarrow \infty} \lambda^{(k)} \quad \text{such that} \quad \lambda_y = \delta_{y,0} + \sum_{k=1}^{\infty} \delta_{y,L_k+1} \quad \text{for } y \in \mathbb{Z},$$

verifying (by construction)  $\|\lambda\| = \infty$  and  $\|\lambda \upharpoonright_{L_k}\| = k$  for all  $k$ . For every  $k \in \mathbb{N}$ , during its first  $L_k$  steps, the random walk can reach no more than the first  $k$  defects, and for  $n \leq L_k$ , we therefore have  $\mathbb{P}[\lambda]_0(\mathbf{M}_n = (0, 0)) = \mathbb{P}[\lambda \upharpoonright_{L_k}]_0(\mathbf{M}_n = (0, 0))$ . Hence,

$$\sum_{n=0}^{\infty} \mathbb{P}[\lambda]_0(\mathbf{M}_n = (0, 0)) \geq \sum_{n=0}^{L_k} \mathbb{P}[\lambda]_0(\mathbf{M}_n = (0, 0)) = \sum_{n=0}^{L_k} \mathbb{P}[\lambda \upharpoonright_{L_k}]_0(\mathbf{M}_n = (0, 0)) \geq a_k.$$

Since  $k$  is arbitrary, this implies recurrence.

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