SUMS OF RATIONAL NUMBERS

W. A. WEBB

1. Introduction. Previously, the problem of expressing rational numbers as finite sums of rational numbers of a given type has been concerned with the Egyptian, or unit, fractions. It has long been known that any rational number is the sum of distinct unit fractions. In response to a problem proposed by E. P. Starke **(4)**, R. Breusch **(1)** and B. M. Stewart **(5)** showed that every rational number with an odd denominator is a sum of distinct odd unit fractions. P. J. Van Albada and J. H. Van Lint **(6)** extended this result to show that any integer is a sum of unit fractions with denominators from an arithmetic progression. This result was extended further by R. L. Graham **(2)**, who showed that every positive rational number a/b can be expressed as a finite sum of reciprocals of distinct elements of the arithmetic progression $r + sx$, if and only if $(b/(b, (r, s)), r/(r, s)) = 1$. In **(3)** Graham also has shown that for every positive rational number α there is an integer n such that for every integer $m \geqslant n$ there is a partition of m whose reciprocals have the sum α.

In this paper we deal with some related problems not of the unit fraction type, such as expressing rational numbers as sums of other rational numbers where numerators, or both numerators and denominators, are restricted in some way.

2. Restricted numerators. In this case, where only the numerators are restricted, we have the following quite general result.

THEOREM 1. *Given any infinite set S of positive integers such that in S there are infinitely many disjoint pairs of elements which are relatively prime, then any rational number a/b may be written as a finite sum of reduced fractions whose numerators are distinct elements of S and whose denominators are distinct.*

Proof. Assume a/b is positive. If it is negative, simply make all of the denominators negative. We write $a/b = 1/b + 1/b + \ldots + 1/b$ (a summands). If $b = 1$, write $2a/2 = 1/2 + \ldots + 1/2$ ($2a$ summands). So we may assume that $b \neq 1$. We now express each $1/b$ as follows.

Let s_1 and s_2 be elements of S, $(s_1, s_2) = 1$, and let $b = PQ$ where $(s_1, P) = 1$ and $(s_2, Q) = 1$. That is, P contains all the primes in (s_2, b), and Q contains all the primes in (s_1, b). We arbitrarily put all other prime factors of b in P.

Received August 1, 1964. This research was supported by NSF Grant No. GE 1106.

Then,

$$\frac{1}{b} = \frac{s_1}{P(s_1 Q + s_2 P)} + \frac{s_2}{Q(s_1 Q + s_2 P)}.$$

The fractions are reduced since $(s_1, s_2) = 1$, $(s_1, P) = 1$ and $(s_2, Q) = 1$. The denominators are not equal since P and Q have different prime factors. Repeat this process for s_3 and s_4 in S, distinct from s_1 and s_2, such that $(s_3, s_4) = 1$ and $s_3 + s_4 > (s_1 + s_2)b^2$. This can be done since we have infinitely many disjoint pairs of relatively prime elements in S. There is obviously no repetition of numerators since they were picked distinct. The denominators are different since both old denominators are less than $(s_1 + s_2)b^2$, while both new denominators are greater than $(s_1 + s_2)b^2$. We repeat this process for each $1/b$ picking $s_{2k} + s_{2k-1}$ greater than $(s_{2k-2} + s_{2k-3})b^2$. This gives us the desired representation.

We note that such sets as the primes, the kth powers of the primes, any arithmetic progression $r + sx$, where $(r, s) = 1$, or even the Fibonacci numbers are suitable for S.

3. Restricted numerators and denominators. In this case we have the following result concerning arithmetic progressions.

THEOREM 2. *Any positive rational number a/b where b is odd, a/b reduced, can be written as a finite sum of proper, reduced fractions whose numerators are distinct elements of the arithmetic progression $r + sx$, and whose denominators are distinct elements of the arithmetic progression $u + vy$; provided $(u, v) = 1$, $(r, s) = 1$, $(v, b) = 1$, $(v, s) = 1$, and $(v, r) = 1$.*

Proof. We divide the proof into two parts. In the first part we reduce the problem to the case where $b \equiv 1 \pmod{v}$, and in the second part we prove the result for this special case.

If $v = 1$, the result is true by Theorem 1. Therefore assume $v > 1$.

Write $b = c + yv$, $0 < c < v$. Clearly $(v, c) = 1$ since $(v, b) = 1$. By the Chinese remainder theorem the system of congruences

$$\begin{aligned} k &\equiv cu^2 \pmod{v}, \\ vk &\equiv 1 \pmod{b} \end{aligned} \tag{1}$$

can be solved for k, where k has the property $0 < k < bv$; $(k, v) = 1$ and $(k, b) = 1$. Since $(v, k^2) = 1$, $(v, b) = 1$, and $(b, k^2) = 1$, the system of congruences

$$\begin{aligned} y_1 v + u &\equiv 0 \pmod{k^2}, \\ y_1 v + u &\equiv 1 \pmod{b} \end{aligned} \tag{2}$$

can be solved for $y_1 = Y + m_1 bk^2$, where Y is the particular solution such that $0 < Y < bk^2$.

https://doi.org/10.4153/CJM-1965-096-3 Published online by Cambridge University Press

Now, if $p^\alpha \| k$, where p is a prime and $\alpha \geqslant 1$, then $p^\alpha \| (y_1 + k)v + u$ since $p^{2\alpha} \mid y_1 v + u$ and $p^\alpha \| kv$. Also, $(y_1 v + u, (y_1 + k)v + u) = k$ since k divides both, and if p^α divides both, it divides their difference vk. If p divides v, then it would not divide $y_1 v + u$, since $(v, u) = 1$. Therefore $p^\alpha \mid k$, which proves the statement.

Let $(r, u + vy_1) = d_1$. Take $x_1 = (u + vy_1)/d_1^*$, where d_1^* is defined as the smallest number such that $((u + vy_1)/d_1^*, r) = 1$. That is, if

$$d_1 = p_1^{\alpha_1} \ldots p_h^{\alpha_h} \qquad \text{and} \qquad u + vy_1 = p_1^{\beta_1} \ldots p_h^{\beta_h} \ldots p_l^{\beta_l},$$

then $d_1^* = p_1^{\beta_1} \ldots p_h^{\beta_h}$. Now, $(r + x_1 s, u + vy_1) = 1$, since if a prime q divides both, then either $q \mid r$, in which case $q \nmid x_1 s$ since $(x_1, r) = 1$ and $(s, r) = 1$, so $q \nmid r + x_1 s$, or $q \nmid r$ and hence $q \nmid d_1^*$, in which case $q \mid x_1$ since $q \mid u + vy_1$ and $u + vy_1 = x_1 d_1^*$. Here too we find $q \nmid r + x_1 s$, which contradicts the assumption that q was to divide both numbers.

Let $(r, u + v(y_1 + k)) = d_2$. Then let $x_2 = (u + v(y_1 + k))/d_2^*$, where d_2^* is defined in the same way as d_1^*. We then have $(r + x_2 s, u + v(y_1 + k)) = 1$ just as above.

We now show that we may pick m_1 such that $x_1 \neq x_2$. If $r = p_1 \ldots p_h$, let $y_1 v + u = P_1 Q_1$ and $(y_1 + k)v + u = P_2 Q_2$, where P_1 and P_2 contain only the p_i as prime factors, $(Q_1 Q_2, r) = 1$. Then, by definition, $x_1 = Q_1$ and $x_2 = Q_2$. Now assume that $x_1 = x_2$ for all choices of m_1, i.e. $Q_1 = Q_2$. Then we have $(y_1 v + u, (y_1 + k)v + u) = P_3 Q_1$, where P_3 contains only the p_i as prime factors. But we have already shown that the greatest common divisor of these two numbers is precisely k, i.e. $P_3 Q_1 = k$. By (2), $k^2 | y_1 v + u$; hence $(P_3 Q_1)^2 \mid P_1 Q_1$. Therefore, since the P_i are relatively prime to the Q_i, we have $Q_1^2 \mid Q_1$, which is possible only if $Q_1 = 1$. This means that $y_1 v + u$ contains only the primes $p_1, p_2, \ldots, p_h$ as prime factors. But since m_1 may be any positive integer, we may pick $y_1 v + u$ to be any member of an arithmetic progression. It is well known and easily shown that the elements of any arithmetic progression have infinitely many prime factors, which contradicts what we have just shown. Therefore, x_1 cannot be equal to x_2 for all choices of m_1, so we may pick m_1 such that $x_1 \neq x_2$.

We now pick j_1 and j_2 such that $j_1 \geqslant j_2$ and

$$\begin{aligned} &r + x_1 s < (u + y_1 v)^{j_1 \phi(v) + 1} = U_1, \\ (3) \qquad &r + x_2 s < (u + (y_1 + k)v)^{j_2 \phi(v) + 1} = U_2, \\ &\frac{r + x_1 s}{U_1} + \frac{r + x_2 s}{U_2} < \frac{a}{b}. \end{aligned}$$

By $\phi(v)$ we mean Euler's ϕ-function. Clearly U_1 is relatively prime to $r + x_1 s$, since $u + y_1 v$ is, and U_2 is relatively prime to $r + x_2 s$, since $u + (y_1 + k)v$ is.

Consider the fraction

$$(4) \qquad \frac{(r + x_1 s)U_2 + (r + x_2 s)U_1}{U_1 U_2}.$$

https://doi.org/10.4153/CJM-1965-096-3 Published online by Cambridge University Press

Clearly $(U_1, U_2) = k^{j_2\phi(v)+1}$, since $(u + y_1 v, u + (y_1 + k)v) = k$ and $j_1 \geqslant j_2$. This implies that $k^{j_2\phi(v)+1}$ is the greatest common divisor of the numerator and denominator of (4). We can easily see this since any common divisor of the numerator and denominator must be a common divisor of U_1 and U_2. Therefore, after a factor of $k^{j_2\phi(v)+1}$ has been removed, the fraction (4) is in reduced form N/D.

Now,

$$Dk^{j_2\phi(v)+1} = U_1 U_2, \qquad U_1 \equiv u^{j_1\phi(v)+1} \equiv u \pmod{v},$$

and

$$U_2 \equiv u^{j_2\phi(v)+1} \equiv u \pmod{v}$$

since $(u, v) = 1$. This means that U_1 and U_2 are themselves elements of the arithmetic progression $u + yv$. Also, $Dk^{j_2\phi(v)+1} \equiv Dk \pmod{v}$ since $(k, v) = 1$. Therefore, $Dk \equiv U_1 U_2 \equiv u^2 \pmod{v}$, and we find that $D \equiv u^2k^{-1} \equiv C \pmod{v}$, where $cC \equiv 1 \pmod{v}$ by (1). We now have $D = C + Mv$ and can use (4) to write

$$\frac{a}{b} = \frac{r + x_1 s}{U_1} + \frac{r + x_2 s}{U_2} - \frac{N}{C + Mv} + \frac{a}{b}$$

or

$$\frac{a}{b} = \frac{r + x_1 s}{U_1} + \frac{r + x_2 s}{U_2} + \frac{a'}{b'},$$

where

$$\frac{a'}{b'} = \frac{a}{b} - \frac{N}{C + Mv} = \frac{a}{c + yv} - \frac{N}{C + Mv} = \frac{N_1}{cC + M_1 v} = \frac{N_1}{1 + M_2 v}.$$

This fraction is greater than zero by (3). We also have that $N_1/(1 + M_2 v)$ is reduced, since N/D and a/b are reduced and $(b, D) = 1$. To show this, we consider

$$u + y_1 v \equiv 1 \pmod{b}, \qquad u + (y_1 + k)v \equiv 2 \pmod{b}.$$

These relations are true by (1) and (2). So, since b is odd, it is relatively prime to both numbers, and hence relatively prime to $U_1 U_2$. Since D is a factor of $U_1 U_2$, we have $(b, D) = 1$.

We have thus expressed a/b as

$$\frac{a}{b} = \frac{r + x_1 s}{U_1} + \frac{r + x_2 s}{U_2} + \frac{a'}{b'}.$$

All of these fractions are reduced; the first two are of the desired type since U_1 and U_2 are elements of the arithmetic progression $u + yv$, and $b' \equiv 1 \pmod{v}$. Hence this completes the first part of the proof.

We may now assume that $b \equiv 1 \pmod{v}$, and write $a/b = 1/b + \ldots + 1/b$ (a summands). If we let $(b, r) = d_3$, then $(b, r + z(b/d_3^*)s) = 1$ if $(z, r) = 1$, where d_3^* is defined in the same way as d_1^* and d_2^*. The system of congruences

$$\begin{aligned} z &\equiv 1 \pmod{r}, \\ 2z(b/d_3^*)s &\equiv u - 2r - s \pmod{v} \end{aligned} \tag{5}$$

has a solution, since $(r, v) = 1$, and we can show that the second congruence has a solution. This is because if $2 \mid v$, then $2 \nmid u$ and $2 \nmid s$, so $2 \mid u - s$ and the 2 may be cancelled from the congruence. Now, $((b/d_3^*)s, v) = 1$ since $(b, v) = 1$ and $(s, v) = 1$. Hence, the coefficient of z is relatively prime to v (or $v/2$ if a 2 must be cancelled from the second congruence), so the system (5) has a solution $(\text{mod}\, rv)$. There are then infinitely many such z. Note that the first congruence assures that $(z, r) = 1$. Now, for any such z, set $z' = z(b/d_3^*)$, which assures that $(b, r + z's) = 1$, and consider the identity

$$\frac{1}{b} = \frac{r + z's}{b(r + z's + b(r + (z' + 1)s))} + \frac{r + (z' + 1)s}{r + z's + b(r + (z' + 1)s)}.$$

Note that $(r + z's, r + (z' + 1)s) = 1$, because if q, a prime, divides both, then it divides their difference s. But then $q \nmid r$ since $(r, s) = 1$, so $q \nmid r + z's$. Therefore the fractions are reduced and $1/b$ is now expressed in the desired form since the numerators are in the progression $r + xs$, and we can prove that the denominators are in the progression $u + yv$.

Let $V = r + z's + b(r + (z' + 1)s$. Then since $b \equiv 1 \pmod{v}$, we have $bV \equiv V \equiv 2r + 2z's + s \pmod{v}$. But $2z's = 2z(b/d_3^*)s$ and

$$2z(b/d_3^*)s \equiv u - 2r - s \pmod{v},$$

so $bV \equiv V \equiv u \pmod{v}$. This proves that both denominators are in the desired progression.

We need now only do this for each $1/b$ by picking $z_1, z_2, \ldots, z_a$ in such a way that

$$\begin{aligned} r + z_i' s &> r + (z'_{i-1} + 1)s; \\ r + z_i' s + b(r + (z_i' + 1)s) &> b(r + z'_{i-1} s + b(r + (z_i'_{-1} + 1)s)); \\ r + z_1' s &> \max(r + x_1 s, r + x_2 s); \\ V &> \max(U_1, U_2). \end{aligned} \tag{6}$$

This can be done since we have infinitely many z to pick from. The inequalities (6) assure distinctness of all numerators and denominators that we use.

This completes the second part of the proof, and we have shown that a/b can be represented in the desired form.

COROLLARY. *The above theorem holds for b odd or even if u is a primitive root of v.*

Proof. Let $(u, b) = d$, $y_i = k_i(b/d^*)$, where d^* has the same meaning as d_1^*, etc. in Theorem 2. Then $(y_i, d) = 1$ if $(k_i, d) = 1$. Hence

$$(b, u + y_i v) = 1.$$

For x_1 we may pick a k_1 (and hence a y_1) such that $(r + x_1 s, u + y_1 v) = 1$ by using a "d^* argument" as was used in Theorem 2.

We now have

$$\frac{a}{b} - \frac{r + x_1 s}{u + y_1 v} = \frac{N_1}{bu + Y_1 v}.$$

Repeat this for $x_1, y_1, \ldots, x_t, y_t$, where $bu^t \equiv 1 \pmod{v}$. At each stage we pick y_i such that $u + y_i v$ is relatively prime to both

$$r + x_i s \quad \text{and} \quad bu^{i-1} + Y_{i-1} v.$$

This can be done just as it was for y_1. That $bu^t \equiv 1 \pmod{v}$ has a solution follows from the fact that u is a primitive root of v. We also pick y_i large enough so that we have $N_i/(bu^i + Y_i v) > 0$. We then have

$$\frac{a}{b} = \frac{r + x_1 s}{u + y_1 v} + \ldots + \frac{r + x_t s}{u + y_t v} + \frac{N_t}{bu^t + Y_t v}.$$

But the denominator of this last fraction is congruent to 1 (mod v). Hence, we may proceed with the second part of Theorem 2.

The hypotheses $(r, s) = 1$ and $(v, b) = 1$ are necessary for Theorem 2 to hold in this generality. The condition $(u, v) = 1$ appears almost impossible to omit. The other conditions $(v, s) = 1$ and $(v, r) = 1$ may possibly be weakened. For instance, it can be shown using another "d^* argument" that $(v, r) = 1$ may be replaced by $(v, r, u - s) = 1$.

I should like to express my thanks to Dr. B. M. Stewart for many valuable suggestions.

References

1. R. Breusch, *A special case of Egyptian fractions*, Amer. Math. Monthly, *61* (1954), 200–201.
2. R. L. Graham, *On finite sums of unit fractions*, Proc. London Math. Soc. (3), *14* (1964), 193–207.
3. ——— *A theorem of partitions*, J. Austral. Math. Soc. III-*4* (1963), 435–441.
4. E. P. Starke, *Advanced problem* 4512, Amer. Math. Monthly, *59* (1952), 640.
5. B. M. Stewart, *Sums of distinct divisors*, Amer. J. Math., *76* (1954), 779–785.
6. P. J. Van Albada and J. H. Van Lint, *Reciprocal bases for the integers*, Amer. Math. Monthly, *70* (1963), 170–174.

Michigan State University