

Invariant family of leaf measures and the Ledrappier–Young property for hyperbolic equilibrium states

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Abstract. The manifold M is a Riemannian, boundaryless, and compact manifold with $\dim M \geq 2$, and f is a $C^{1+\beta}$ ($\beta > 0$) diffeomorphism of M . φ is a Hölder continuous potential on M . We construct an invariant and absolutely continuous family of measures (with transformation relations defined by φ), which sit on local unstable leaves. We present two main applications. First, given an ergodic homoclinic class $H_\chi(p)$, we prove that φ admits a local equilibrium state on $H_\chi(p)$ if and only if φ is ‘recurrent on $H_\chi(p)$ ’ (a condition tested by counting periodic points), and one of the leaf measures gives a positive measure to a set of positively recurrent hyperbolic points; and if an equilibrium measure exists, the said invariant and absolutely continuous family of measures constitute as its conditional measures. An immediate corollary is the local product structure of hyperbolic equilibrium states. Second, we prove a Ledrappier–Young property for hyperbolic equilibrium states: if φ admits a conformal family of leaf measures and a hyperbolic local equilibrium state, then the leaf measures of the invariant family (relative to φ) are equivalent to the conformal measures (on a full measure set). This extends the celebrated result by Ledrappier and Young for hyperbolic Sinai–Ruelle–Bowen measures, which states that a hyperbolic equilibrium state of the geometric potential (with pressure zero) has conditional measures on local unstable leaves which are absolutely continuous with respect to the Riemannian volume of these leaves.

Key words: thermodynamics formalism, smooth dynamics, hyperbolic dynamics

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1. Introduction

When studying a complex system of many possible configurations and its dynamics, one approach is to find a measure on the space of all possible configurations of the system which is invariant under its dynamics, to represent a distribution of possible events when the system is in equilibrium. The collection of all possible such invariant measures could be very big, and the field of thermodynamic formalism offers canonical ways to single out specific invariant measures of importance.

One way to single out measures of importance, is by the notion of equilibrium states: probability measures which maximize a dynamic quantity with respect to some potential (function) on the system. More explicitly, let M be the space of states of the system (a compact boundaryless Riemannian manifold), let $f \in \text{Diff}^{1+}(M)$ be the dynamics of the system, and let $\varphi : M \rightarrow \mathbb{R}$ be a bounded measurable potential on M ; then an equilibrium state of φ maximizes $\mu \mapsto h_\mu(f) + \int_M \varphi d\mu$, where $h_\mu(f)$ is the metric entropy of μ with respect to f .

Two important instances of this notion are measures of maximal entropy (when $\varphi \equiv 0$), and Sinai–Ruelle–Bowen (SRB) measures (when φ is the geometric potential, that is, minus the logarithm of the Jacobian of f when restricted to the dynamics on unstable leaves). The fact that equilibrium states of the geometric potential (when $\sup\{h_\mu(f) + \int \varphi d\mu\} = 0$) are SRB measures is highly non-trivial, and it is due to Ledrappier and Young [LY85a]. It is an instance of their more general formula to compute the entropy of general measures via local dimension of conditional measures on unstable leaves (see also [LY85b]). When f preserves a smooth measure, the entropy formula is due to Pesin [Pes77].

SRB measures are measures which have conditional measures which are absolutely continuous with respect to the induced Riemannian volume on unstable leaves, when disintegrated by a measurable partition subordinated to the foliation of unstable leaves (see Rokhlin’s disintegration theorem). Hyperbolic and ergodic SRB measures are of physical importance, as the set of points which satisfy the forward-time Birkhoff’s ergodic theorem are of positive Riemannian volume (and thus, by convention, are observable in an experiment or a simulation). This property is due to Pesin’s absolute continuity theorem (see [KSLP86, Pes77, PS89]). Tsujii showed a converse, a condition via Lyapunov exponents on orbits such that a physical measure must be an ergodic hyperbolic SRB, see [Tsu91].

While SRB measures are important, it is not clear what generality systems admit them. Sinai had shown their existence in Anosov systems, and Bowen and Ruelle in Axiom A systems. The dual characterization through conditional leaf measures, and as an equilibrium state, by Ledrappier and Young, offered a new way to test whether a system admits an SRB measure.

Singling out measures of importance through their conditional leaf measures is another important tool in thermodynamic formalism, in addition to the notion of equilibrium states. Some important examples of this are the Margulis construction for the measure of maximal entropy through its leaf measures [Mar70] and the extension of the horocyclic flow to high dimension variable negative curvature manifolds (see [BM77, Mar75]). Unstable leaves can be viewed as the equivalence class of an equivalence relation on orbits. The notion of defining measures by conditional measures on unstable leaves has been further generalized and studied as an equivalence class of orbits on manifolds (see [Ser80]) and symbolic spaces (see [Ser80, PS97]).

Given a Hölder continuous potential, the idea of an equilibrium state is well defined, but it is not clear how to generally define the leaf measures corresponding to a potential. The Margulis leaf measures are conformal, that is, when pushed forward, they remain invariant up to a factor of $e^{-h_{\text{top}}(f)}$, where $h_{\text{top}}(f) = \sup\{h_{\mu}(f) : \mu \text{ is an invariant probability measure}\}$. Similarly, the conditional leaf measures of SRB measures are the induced Riemannian volume, which gains a factor of $\exp(\varphi)$ (geometric potential). We therefore are looking for a family of conditional leaf measures which are conformal with respect to our potential. More explicitly, if φ is a bounded measurable potential and V^u is a local unstable leaf, we wish to find measures which satisfy $m_{V^u} \circ f^{-1} = e^{\varphi - P(\varphi)} \cdot m_{f[V^u]}$, where $P(\varphi)$ is the pressure of φ (the quantity which an equilibrium state maximizes), which acts as a calibrating factor. In [CPZ19, CPZ20], Climenhaga, Pesin, and Zelerowicz give a construction using Carathéodory dimension. Their construction works for a general Hölder continuous potential in the partially hyperbolic setup with some restrictions on the central stable foliation and on the transitivity of the system (see the remark after Definition 2.15 for more details).

The result of Ledrappier and Young connects the two approaches of singling out measures of importance as equilibrium states and through conditional measures on unstable leaves. This naturally raises the question of extending their result for more potentials than just the geometric potential. Given a Grassmann–Hölder continuous potential, we give a construction of an invariant and absolutely continuous family of leaf measures

which give a necessary and sufficient condition for the existence of a hyperbolic local equilibrium state via a leaf condition, and which act as the conditional leaf measures of the equilibrium state when it exists. We prove a Ledrappier–Young property, such that if the potential admits a hyperbolic local equilibrium state and a conformal system of measures, then the conditional measures of the equilibrium state are equivalent to the conformal measures on a set of full measure, and, in particular, the conformal measures give a positive measure to the hyperbolic points. For the definitions of a hyperbolic local equilibrium state, the leaf condition, and a Grassmann–Hölder continuous potential, see §2. In particular, Grassmann–Hölder continuity applies to the family of potentials $\{t \cdot \varphi\}_{t \in [0,1]}$, where φ is the geometric potential, and this family varies continuously between the 0 potential and the geometric potential.

2. Basic definitions and main results

Throughout this paper, M is a Riemannian, boundaryless, and compact manifold of dimension $d \geq 2$, and $f \in \text{Diff}^{1+\beta}(M)$, $\beta > 0$. Fix $\chi > 0$. See Appendix A for the special notation used within this paper.

2.1. Basic definitions

Definition 2.1. (χ -summable points) A point $x \in M$ is called χ -summable if there is a unique splitting $T_x M = H^s(x) \oplus H^u(x)$ such that

$$\sup_{\xi_s \in H^s(x), |\xi_s|=1} \sum_{m=0}^{\infty} |d_x f^m \xi_s|^2 e^{2\chi m} < \infty, \quad \sup_{\xi_u \in H^u(x), |\xi_u|=1} \sum_{m=0}^{\infty} |d_x f^{-m} \xi_u|^2 e^{2\chi m} < \infty.$$

Let χ -summ define the set of χ -summable points. An f -invariant probability measure carried by χ -summ is called χ -hyperbolic.

For each $x \in \chi$ -summ, write $s(x) := \dim(H^s(x))$, $u(x) := \dim(H^u(x))$. Note that χ -summ is f -invariant and $H^s(\cdot)$, $H^u(\cdot)$ are invariant under $d \cdot f$.

THEOREM 2.2. (Pesin–Oseledec reduction theorem) *For each point $x \in \chi$ -summ, there exists an invertible linear map $C_\chi(x) : \mathbb{R}^d \rightarrow T_x M$, which depends measurably on x , such that $C_\chi(x)[\mathbb{R}^s(x) \times \{0\}] = H^s(x)$, $C_\chi(x)[\{0\} \times \mathbb{R}^u(x)] = H^u(x)$. Here, $C_\chi(\cdot)$ are chosen measurably on χ -summ, and the choice is unique up to a composition with an orthogonal mapping of the ‘stable’ and of the ‘unstable’ subspaces of the tangent space. In addition,*

$$C_\chi^{-1}(f(x)) \circ d_x f \circ C_\chi(x) = \begin{pmatrix} D_s(x) & \\ & D_u(x) \end{pmatrix},$$

where $D_s(x)$, $D_u(x)$ are square matrices of dimensions $s(x)$, $u(x)$, respectively, and $\|D_s(x)\|$, $\|D_u^{-1}(x)\| \leq e^{-\chi}$, $\|D_s^{-1}(x)\|$, $\|D_u(x)\| \leq \kappa$ for some constant $\kappa = \kappa(f, \chi) > 1$.

The Pesin–Oseledec reduction theorem has many different versions, which are suitable for different setups. We use the version which appears, with proof, in [BO18, Theorem 2.4].

It is possible to show that $\|C_\chi^{-1}(x)\|^2 = \sup_{\substack{|\xi_s + \xi_u|=1, \\ \xi_s \in H^s(x), \xi_u \in H^u(x)}} \{2 \sum_{m \geq 0} |d_x f^m \xi_s|^2 e^{2\chi m} + 2 \sum_{m \geq 0} |d_x f^{-m} \xi_u|^2 e^{2\chi m}\}$. In addition, $\|C_\chi(\cdot)\| \leq 1$. See [BO18, Theorem 2.4,

Lemma 2.9] for details. Here, $\|C_\chi^{-1}(x)\|$ serves as a measurement of the hyperbolicity of x : a greater norm means a worse hyperbolicity (that is, slow contraction/expansion on stable/unstable spaces, or small angle between the stable and unstable spaces).

Definition 2.3. Let $\epsilon > 0$ and let $x \in \chi$ -summ, then

$$Q_\epsilon(x) := \max \left\{ Q \in \{e^{-\ell\epsilon/3}\}_{\ell \in \mathbb{N}} : Q \leq \frac{1}{36/\beta} \epsilon^{90/\beta} \|C_\chi^{-1}(x)\|^{-48/\beta} \right\}.$$

Definition 2.4. (Recurrent ϵ -weak temperability) Let $\epsilon > 0$. A point $x \in \chi$ -summ is called ϵ -weakly temperable if there exists $q : \{f^n(x)\}_{n \in \mathbb{Z}} \rightarrow \{e^{-\ell\epsilon/3}\}_{\ell \in \mathbb{N}}$ such that:

- (1) $q \circ f/q = e^{\pm\epsilon}$;
- (2) for all $n \in \mathbb{Z}$, $q(f^n(x)) \leq Q_\epsilon(f^n(x))$.

If, in addition to items (1) and (2), $q : \{f^n(x)\}_{n \in \mathbb{Z}} \rightarrow \{e^{-\ell\epsilon/3}\}_{\ell \in \mathbb{N}}$ can be chosen to also satisfy:

- (3) $\limsup_{n \rightarrow \pm\infty} q(f^n(x)) > 0$;

then we say that x is recurrently ϵ -weakly temperable. A function $q : \{f^n(x)\}_{n \in \mathbb{Z}} \rightarrow \{e^{-\ell\epsilon/3}\}_{\ell \in \mathbb{N}}$ which satisfies items (1) and (2), is called an ϵ -weakly tempered kernel of x . Define $WT_\chi^\epsilon := \{x \in \chi\text{-summ} : x \text{ is } \epsilon\text{-weakly temperable}\}$, and $RWT_\chi^\epsilon := \{x \in \chi\text{-summ} : x \text{ is recurrently } \epsilon\text{-weakly temperable}\}$.

Remark. Every point $x \in WT_\chi^\epsilon$ (for sufficiently small $\epsilon > 0$) admits a global unstable manifold $W^u(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \xrightarrow{n \rightarrow \infty} 0\}$, and a global stable manifold $W^s(x) = \{y \in M : d(f^n(x), f^n(y)) \xrightarrow{n \rightarrow \infty} 0\}$, which are immersed submanifolds tangent to $H^u(x)$ and $H^s(x)$, respectively. See [BS02, §5.6] for more details. In Definition 3.12, we give an alternative (yet equivalent) notion of global stable/unstable manifolds.

Claim 2.5. For all $\epsilon > 0$ and $\epsilon' \in (0, \frac{2}{3}\epsilon]$, $RWT_\chi^{\epsilon'} \subseteq RWT_\chi^\epsilon$, and $WT_\chi^{\epsilon'} \subseteq WT_\chi^\epsilon$.

For proof, see [BO20, Claim 2.6].

LEMMA 2.6. *There exists $\epsilon_\chi > 0$ which depends on M, f, β and χ such that for all $\epsilon \in (0, \epsilon_\chi]$, RWT_χ^ϵ has a Markov partition and a finite-to-one coding with a Hölder continuous factor map, where the induced coding space is locally compact.*

See [BO20] for full details regarding the terms ‘Markov partition’ and ‘factor map’. See also Theorems 3.2–3.7 for more details regarding the notion of ‘a finite-to-one coding’.

Definition 2.7

$$RWT_\chi := \{x \in \chi\text{-summ} : x \text{ is recurrently } \epsilon_\chi\text{-weakly temperable}\}.$$

Remark. RWT_χ carries all χ -hyperbolic f -invariant probability measures; and its definition depends only on the quality of hyperbolicity of the orbits of its elements. In the following parts of this paper, when $\chi > 0$ is fixed, the subscript of ϵ_χ would be omitted to ease notation.

Definition 2.8. (Pesin set) For all $N \geq 1$, the set

$$\Lambda_N := \left\{ x \in \text{WT}_\chi^\epsilon : \text{there exists } \epsilon\text{-weakly tempered kernel of } x \text{ such that } q(x) \geq \frac{1}{N} \right\}$$

is called the *Pesin set of level N* (also sometimes called *level set* or *Pesin level set*).

Definition 2.9. (Positively recurrent points) For all $N \geq 1$, denote by Λ_N the Pesin set of level N . Let

$$\text{RWT}_\chi^{\text{PR}} := \left\{ x \in \text{RWT}_\chi : \text{there exists } N \in \mathbb{N} \text{ such that } \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{\Lambda_N}(f^k(x)) > 0 \right\}.$$

For the next definition, we assume without loss of generalization that there exists $s \in \{1, \dots, d - 1\}$ such that for all points $x \in \text{WT}_\chi^\epsilon$, $\dim H^s(x) = s$. This is possible, since we may simply split WT_χ^ϵ into $d - 1$ disjoint sets, and all results apply to each subset.

Definition 2.10. (Grassmann–Hölder continuity) Let TM be the tangent bundle of M , and let $x, y \in \text{WT}_\chi^\epsilon$. Let $d_s(H^s(x), H^s(y))$ be the Grassmannian distance (of subspaces of dimension s , over TM) between the stable space of x and stable space of y . Similarly, let $d_u(H^u(x), H^u(y))$ be the Grassmannian (of subspaces of dimension $d - s$, over TM) distance between the unstable space of x and stable space of y . The *Lyapunov distance* on WT_χ^ϵ is $d_{GH}(x, y) := d(x, y) + d_s(H^s(x), H^s(y)) + d_u(H^u(x), H^u(y))$. A function $\varphi : \text{WT}_\chi^\epsilon \rightarrow \mathbb{R}$ is called *Grassmann–Hölder continuous* if it is Hölder continuous with respect to $d_{GH}(\cdot, \cdot)$.

Let $E \supseteq \text{WT}_\chi^\epsilon$ be a measurable subset of $\bigcup \{W^u(x) : x \in \text{WT}_\chi^\epsilon\}$. A function $\varphi : E \rightarrow \mathbb{R}$ is called *unstable-Grassmann–Hölder continuous on E* if it is Hölder continuous with respect to $d(\cdot, \cdot) + d_u(\cdot, \cdot)$ on E , where for $x \in \bigcup \{W^u(x) : x \in \text{WT}_\chi^\epsilon\}$, $H^u(x)$ is defined as $T_x W^u(x)$ (which is well defined).

As $f \in \text{Diff}^{1+\beta}(M)$, for all $t > 0$, the scaled geometric potential $\varphi_t(x) = -t \cdot \log \text{Jac}(d_x f|_{H^u(x)})$ is unstable-Grassmann–Hölder continuous. Every Hölder continuous potential is unstable-Grassmann–Hölder continuous, and every unstable-Grassmann–Hölder continuous potential on WT_χ^ϵ is Grassmann–Hölder continuous.

Note also that a Grassmann–Hölder continuous potential must be bounded, since M has a finite diameter.

Definition 2.11. (Ergodic homoclinic class) The *ergodic homoclinic class* of a hyperbolic periodic point p (with respect to χ) is

$$H_\chi(p) := \{x \in \text{RWT}_\chi : W^u(x) \pitchfork W^s(o(p)) \neq \emptyset, W^s(x) \pitchfork W^u(o(p)) \neq \emptyset\}.$$

Here, $o(p) = \{f^k(p)\}$, \pitchfork denotes transverse intersections of full codimension, and $W^{s/u}(\cdot)$ are the global stable and unstable manifolds of the point, respectively.

See the remark after Definition 2.4 for the definition of global stable and unstable leaves. Definition 2.11 was introduced in [RHRHTU11], with a set of Lyapunov regular points replacing RWT_χ . By a well-known argument, using the inclination lemma (see [BS02, Theorem 5.7.2]), every two ergodic homoclinic classes are either disjoint modulo

all conservative measures, or coincide. (Conservative measures are measures which satisfy Poincaré’s recurrence theorem: for any set of positive measure, almost every point in it returns to it infinitely often.) We caution the reader that $H_\chi(p)$ is not closed, and that this definition is different from the definition in [New72].

Every ergodic χ -hyperbolic probability is carried by a unique ergodic homoclinic class (see Claim 3.15).

Definition 2.12. (Local equilibrium state) Let $\varphi : \text{WT}_\chi^\epsilon \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Let μ be a χ -hyperbolic f -invariant ergodic probability measure. Let $H_\chi(p)$ be the unique ergodic homoclinic class which carries μ . Here, μ is called a hyperbolic local equilibrium state of φ on $H_\chi(p)$ if

$$h_\mu(f) + \int \varphi \, d\mu = \sup \left\{ h_\nu(f) + \int \varphi \, d\nu : \nu \text{ is an } f\text{-invariant probability measure on } H_\chi(p) \right\}.$$

Note that $\int \varphi \, d\mu < \infty$ since φ is bounded since it is Grassmann–Hölder continuous.

Definition 2.13. (Local pressure) Let p be a periodic χ -hyperbolic point, and let $H_\chi(p)$ be its ergodic homoclinic class. The local pressure of φ on $H_\chi(p)$ is

$$P_{H_\chi(p)}(\varphi) := \sup \left\{ h_\nu(f) + \int \varphi \, d\nu : \nu \text{ is an } f\text{-invariant probability measure on } H_\chi(p) \right\}.$$

The boundedness of φ implies $P_{H_\chi(p)}(\varphi) < \infty$. Let E be as in Definition 2.10.

Definition 2.14. Given a sequence of C^1 embedded submanifolds, V_n^u , we write $V_n^u \xrightarrow[n \rightarrow \infty]{C^1} V^u$, where V^u is an embedded submanifold, if $\sup_{x \in V_n^u} \inf_{y \in V^u} \{d(x, y) + d_{TM}(T_x V_n^u, T_y V^u)\} \xrightarrow[n \rightarrow \infty]{} 0$.

Definition 2.15. (φ -conformal system of measures) Let $\varphi : E \rightarrow \mathbb{R}$ be an unstable-Grassmann–Hölder continuous potential. Let $H_\chi(p)$ be the ergodic homoclinic class of a χ -hyperbolic and periodic point p . A φ -conformal system of measures on $H_\chi(p)$ is a collection of non-zero (finite) Radon measures $\{m_{V^u}\}$ indexed by the collection of local (‘local’ here means diffeomorphic to a ball, and bounded, in the induced metric) unstable leaves of points in $H_\chi(p)$ such that m_{V^u} is supported on V^u , and:

- (1) $V_n^u \xrightarrow[n \rightarrow \infty]{C^1} V^u$ implies $m_{V_n^u}^\varphi(1) \xrightarrow[n \rightarrow \infty]{} m_{V^u}^\varphi(1)$;
- (2) for every local unstable leaf of a point in $H_\chi(p)$, V^u ,

$$m_{V^u}^\varphi \circ f^{-1} = e^{\varphi - P_{H_\chi(p)}(\varphi)} \cdot m_{f[V^u]}^\varphi,$$

where $m_{V^u}^\varphi, m_{f[V^u]}^\varphi$ are the φ -conformal measures on the local unstable leaves $V^u, f[V^u]$, respectively.

An example of a conformal family of measures is the Riemannian volume on unstable leaves. It is φ -conformal with φ being the geometric potential $-\log \text{Jac}(df|_{\text{unstable leaves}})$, when the pressure is 0 (e.g. when an SRB exists).

It is not clear *a priori* when do conformal families exist. In [CPZ20], Climenhaga, Pesin, and Zelerowicz give a construction of such families in the partially hyperbolic setup using the Carathéodory dimension structure (which extends their result in the uniformly hyperbolic setup [CPZ19]), under the assumption that φ has the Bowen property (that is, uniformly summable variations on stable and unstable leaves, under a certain regularity assumption). Searching for an object which exists always as an alternative to conformal measures is the motivation for the next definition of invariant families.

Definition 2.16. (φ -invariant family of leaf measures) Let $\varphi : \text{WT}_\chi^e \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Let $H_\chi(p)$ be the ergodic homoclinic class of a periodic χ -hyperbolic point p such that $P_{H_\chi(p)}(\varphi) < \infty$. A φ -invariant family of leaf measures on $H_\chi(p)$ is a family \mathcal{F} of non-zero and finite measures such that:

- (1) for all $\mu \in \mathcal{F}$, μ is carried by WT_χ^e , and its support is contained in a local unstable leaf of a point $x \in H_\chi(p)$;
- (2) there exists a map $A : \cup\{\text{supp}(\mu')\}_{\mu' \in \mathcal{F}} \rightarrow \mathbb{R}$ which is measurable when restricted to measurable sets, such that for all $\mu \in \mathcal{F}$, $\mu \circ f^{-1} = e^{-P_{H_\chi(p)}(\varphi)} \sum_{i=1}^{N(\mu)} e^{\phi(\mu^{(i)})} \mu^{(i)}$, where $\mu^{(i)} \in \mathcal{F}$ for all $1 \leq i \leq N(\mu) < \infty$, and $\phi(\mu^{(i)})$ is a constant such that $\phi(\mu^{(i)}) = \varphi(x) + A(x) - A(f^{-1}(x))$ for $\mu^{(i)}$ -almost every (a.e.) $x \in M$ where φ is defined;
- (3) there exists a σ -compact metric space X such that $\mathcal{F} = \{\mu_x\}_{x \in X}$ and $x_n \rightarrow y \Rightarrow \mu_{x_n} \xrightarrow{\text{weak-*}} \mu_y$.

The measures in \mathcal{F} are not required to be mutually singular.

One cannot expect uniqueness of such a family, as the Radon–Nikodym derivative of the push-forwards can always be changed by a coboundary.

2.2. Main results

THEOREM A. *Let $\varphi : \text{WT}_\chi^e \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Let $H_\chi(p)$ be the ergodic homoclinic class of a periodic χ -hyperbolic point p . Then $H_\chi(p)$ admits a φ -invariant family of leaf measures, $\mathcal{F}_{H_\chi(p)}(\varphi)$.*

(See Theorem 5.7.) While conformal families of measures are generally a simpler object for calculations, their general existence is a hard problem which is only solved in a few setups (we mention a few in §1). The main advantage of invariant families is that they exist with (almost) no preceding assumptions. Furthermore, when a conformal family does exist, and so does an equilibrium state for the respective potential, the invariant family of measures must coincide in measure-class with the conformal family. See Theorem E.

In the following statement, $\mathcal{F}_{H_\chi(p)}(\varphi)$ is as constructed in Theorem A. That is, in Theorem A, we construct an invariant family, and the leaf condition applies to any invariant family as constructed by that theorem. We remind the reader that the invariant family need not be unique, as the leaf measures can always be changed by a coboundary factor. In addition, we use the notation $\varphi_n(q) := \sum_{k=0}^{n-1} \varphi(f^{-k}(q))$ in the following statement.

Definition 2.17. (φ -leaf condition) Let $\varphi : \text{WT}_\chi^\epsilon \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Let $H_\chi(p)$ be an ergodic homoclinic class of a χ -hyperbolic periodic point p . We say that the φ -leaf condition is satisfied for $H_\chi(p)$, if there exists $\mu \in \mathcal{F}_{H_\chi(p)}(\varphi)$ such that $\mu(\text{RWT}_\chi^{\text{PR}}) > 0$.

THEOREM B. Let $\varphi : \text{WT}_\chi^\epsilon \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Let $H_\chi(p)$ be an ergodic homoclinic class of a χ -hyperbolic periodic point p such that p belongs to a Pesin level set Λ_l . Then, φ admits a χ -hyperbolic local equilibrium state on $H_\chi(p)$ if and only if $\sum_{n \geq 1} \sum_{f^n(q)=q, q \in H_\chi(p) \cap \Lambda_l} e^{\varphi_n(q) - n \cdot P_{H_\chi(p)}(\varphi)} = \infty$ and the φ -leaf condition is satisfied for $H_\chi(p)$. If the local equilibrium state exists, then it is unique and the φ -invariant family is its conditional measures (in the explicit sense given by equation (11)).

(See Theorem 6.1.) Note that if the assumption above holds for one l , then it holds for all $l' \geq l$, since $\Lambda_{l'} \supseteq \Lambda_l$. Theorems 6.1 and 5.7 yield the following corollary.

COROLLARY C. Local equilibrium states of Grassmann–Hölder continuous potentials have local product structure.

Local product structure for a non-uniformly hyperbolic measure means that every Pesin block can be covered by finitely many sets with a local product structure, on which the measure can be approximated by the product of a measure on a local stable leaf and a measure on a local unstable leaf (see Corollary 6.2). This follows from the absolute continuity property of the φ -invariant family we construct. This extends previous results for the Axiom A setup (see [Hay94, Lep00]).

COROLLARY D. (M, f) admits a χ -hyperbolic equilibrium state for φ if and only if there exists an ergodic homoclinic class $H_\chi(p)$ which satisfies the assumptions of Theorem B, and in addition, $P_{H_\chi(p)}(\varphi) = P_{\text{top}}(\varphi)$ ($= \sup\{h_\nu(f) + \int \varphi d\nu : \nu \text{ is an } f\text{-inv. probability on } M\}$).

THEOREM E. Let $\varphi : \text{WT}_\chi^\epsilon \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Let $H_\chi(p)$ be an ergodic homoclinic class of a χ -hyperbolic periodic point p which admits a (unique) χ -hyperbolic equilibrium state for φ , ν_φ . Assume that $H_\chi(p)$ admits a φ -conformal system of measures. Let $\mathcal{F}_{H_\chi(p)}(\varphi)$ be any φ -invariant family of conditional leaf measures as constructed in Theorem A, then there is a sub-family \mathcal{F}' which disintegrates ν_φ (as in Theorem B), and for which the following holds. For any $\mu \in \mathcal{F}'$, given a local unstable leaf V_{loc}^u which carries μ , and a corresponding φ -conformal measure on V_{loc}^u , $m_{V_{\text{loc}}^u}^\varphi$, there exists a measurable set of full μ -measure, E_μ , such that $\mu \sim m_{V_{\text{loc}}^u}^\varphi \upharpoonright_{E_\mu}$. In particular, $m_{V_{\text{loc}}^u}^\varphi(\text{RWT}_\chi^{\text{PR}}) > 0$.

(See Theorem 6.3.) The last property holds in particular when φ is unstable-Grassmann–Hölder continuous and is defined on unstable leaves, and thus we *a priori* have no reason to know that its conformal measures ‘see’ the hyperbolic points.

If φ is the geometric potential, then the φ -conformal measures exist and are the induced Riemannian volume of the unstable leaves, and this theorem proves that every

χ -hyperbolic equilibrium state of the geometric potential with pressure 0 has absolutely continuous conditional measures (and thus is an SRB measure). This is a new proof to the result by Ledrappier and Young in [LY85a], when restricting to hyperbolic measures; and extends it to additional potentials.

Remark. In several of the main results, we use the assumption $\sum_{n \geq 1} \sum_{f^n(q)=q, q \in H_\chi(p) \cap \Lambda_l} e^{\varphi_n(q) - n \cdot P_{H_\chi(p)}(\varphi)} = \infty$ for some hyperbolic periodic point p which belongs to a level set Λ_l (where $\varphi : \text{WT}_\chi^\epsilon \rightarrow \mathbb{R}$ is a Grassmann–Hölder continuous potential). This assumption is used to show that φ is recurrent (see §4.2) when lifted to an irreducible component which codes $H_\chi(p)$ (see §3.3). This assumption is always used together with the leaf condition (see Definition 2.17). We do not know if the first assumption can be omitted (that is, does the leaf condition imply recurrence of the lifted potential). We were not able to prove that using the methods of this paper. The difference between this paper and [BO21], where the leaf condition is sufficient, is the overlapping property of leaf measures (that is, the Lebesgue measures of two unstable leaves which intersect coincide on the intersection). Here, we use a coding to construct our φ -invariant family of leaf measures and, to gain a bound on the overlapping of measures, we must use the bound on the multiplicity of the coding map $\widehat{\pi}$; which in turn is only bounded on the recurrent part of the symbolic space in our construction; and hence the need for recurrence. There are other constructions of codings with better multiplicity bounds for the coding map (e.g. [Buz20]), though those multiplicity improvements come at the expense of other properties which are necessary for this paper.

3. Preliminary constructions

3.1. *Symbolic dynamics.* Sinai [Sin68] and Adler and Weiss [AW70] constructed Markov partitions and for Anosov diffeomorphisms and linear toral automorphisms, respectively. In [Sar13], Sarig constructed a Markov partition for non-uniformly hyperbolic surface diffeomorphisms. We extended his results to manifolds of any dimension greater or equal to 2 in [BO18]. In [BO20], we modify the codings from [BO18, Sar13] and introduce the set RWT_χ . Here, RWT_χ is defined canonically, and still consists of all recurrently codable points (see Proposition 3.8) in the modified symbolic dynamics. In the following section, we review those results.

As M is compact, there exists $r = r(M) > 0$, $\rho = \rho(M) > 0$ such that the exponential map

$$\exp_x : \{v \in T_x M : |v| \leq r\} \rightarrow B_\rho(x) = \{y \in M : d(x, y) < \rho\}$$

is injective.

Definition 3.1. (Pesin charts) When $\epsilon \leq r$, the following is well defined since $C_\chi(\cdot)$ is a contraction (see [BO18, Lemma 2.9]):

- (1) $\psi_x^\eta := \exp_x \circ C_\chi(x) : \{v \in T_x M : |v| \leq \eta\} \rightarrow B_\rho(x)$, $\eta \in (0, Q_\epsilon(x)]$ is called a *Pesin chart*;
- (2) a *double Pesin chart* is an ordered couple $\psi_x^{p^s, p^u} := (\psi_x^{p^s}, \psi_x^{p^u})$, where $\psi_x^{p^s}$ and $\psi_x^{p^u}$ are Pesin charts.

THEOREM 3.2. *For all $\chi > 0$ such that there exists a χ -hyperbolic periodic point p , there exists a countable and locally finite (finite out-going and in-going degree at each vertex) directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ which induces a topological Markov shift $\Sigma := \{\underline{u} \in \mathcal{V}^{\mathbb{Z}} : (u_i, u_{i+1}) \in \mathcal{E}, \text{ for all } i \in \mathbb{Z}\}$ such that Σ admits a factor map $\pi : \Sigma \rightarrow M$ with the following properties:*

- (1) $\sigma : \Sigma \rightarrow \Sigma$ is defined by $(\sigma \underline{u})_i := u_{i+1}, i \in \mathbb{Z}$ (the left-shift) and satisfies $\pi \circ \sigma = f \circ \pi$;
- (2) π is a Hölder continuous map with respect to the metric $d(\underline{u}, \underline{v}) := \exp(-\min\{i \geq 0 : u_i \neq v_i \text{ or } u_{-i} \neq v_{-i}\})$;
- (3) let $\Sigma^\# := \{\underline{u} \in \Sigma : \text{there exists } n_k, m_k \uparrow \infty \text{ such that } u_{n_k} = u_{n_0}, u_{-m_k} = u_{-m_0}, \text{ for all } k \geq 0\}$, then $\pi[\Sigma^\#]$ carries all f -invariant, χ -hyperbolic probability measures.

This theorem is the content of [BO18, Theorem 3.13] (and, similarly, the content of [Sar13, Theorem 4.16] when $d = 2$). Here, \mathcal{V} is a collection of double Pesin charts (see Definition 3.1), which is discrete.

Definition 3.3

- (1) For all $u \in \mathcal{V}, Z(u) := \pi[\{\underline{u} \in \Sigma^\# : u_0 = u\}], \mathcal{Z} := \{Z(u) : u \in \mathcal{V}\}$.
- (2) \mathcal{R} is defined to be a countable partition of $\bigcup_{v \in \mathcal{V}} Z(v) = \pi[\Sigma^\#]$ such that:
 - (a) \mathcal{R} is a refinement of \mathcal{Z} : for all $Z \in \mathcal{Z}, R \in \mathcal{R}, R \cap Z \neq \emptyset \Rightarrow R \subseteq Z$;
 - (b) for all $v \in \mathcal{V}, \#\{R \in \mathcal{R} : R \subseteq Z(v)\} < \infty$ [Sar13, §11];
 - (c) the rectangles property: for all $R \in \mathcal{R}$, for all $x, y \in R$ there exists $!z := [x, y]_R \in R$, such that for all $i \geq 0, R(f^i(z)) = R(f^i(y)), R(f^{-i}(z)) = R(f^{-i}(x))$, where $R(t) :=$ the unique partition member of \mathcal{R} which contains t , for $t \in \pi[\Sigma^\#]$.
- (3) For all $R, S \in \mathcal{R}$, we say $R \rightarrow S$ if $R \cap f^{-1}[S] \neq \emptyset$, and let $\widehat{\mathcal{E}} := \{(R, S) \in \mathcal{R}^2 \text{ such that } R \rightarrow S\}$.
- (4) $\widehat{\Sigma} := \{\underline{R} \in \mathcal{R}^{\mathbb{Z}} : R_i \rightarrow R_{i+1}, \text{ for all } i \in \mathbb{Z}\}$.

Remark. Given \mathcal{Z} , the definition of \mathcal{R} is proper, since such a refining partition as \mathcal{R} exists by the Bowen–Sinai refinement, see [Sar13, §11.1]. By property (2)(b), and since Σ is locally compact (see Theorem 3.2, local-finiteness of \mathcal{G} implies local-compactness of Σ), $\widehat{\Sigma}$ is also locally compact.

Definition 3.4

- (1) $\widehat{\Sigma}^\# := \{\underline{R} \in \widehat{\Sigma} : \text{there exists } n_k, m_k \uparrow \infty \text{ such that } R_{n_k} = R_{n_0}, R_{-m_k} = R_{-m_0}, \text{ for all } k \geq 0\}$.
- (2) Two partition members $R, S \in \mathcal{R}$ are said to be *affiliated* if there exists $u, v \in \mathcal{V}$ such that $R \subseteq Z(u), S \subseteq Z(v)$ and $Z(u) \cap Z(v) \neq \emptyset$ (this terminology is due to O. Sarig [Sar13, §12.3]).

Claim 3.5. (Local finiteness of the cover \mathcal{Z}) For all $Z \in \mathcal{Z}, \#\{Z' \in \mathcal{Z} : Z' \cap Z \neq \emptyset\} < \infty$.

This claim is the content of [BO18, Theorem 5.2] (and similarly [Sar13, Theorem 10.2] when $d = 2$).

Remark. By Claim 3.5 and Definition 3.3(2)(b), it follows that every partition member of \mathcal{R} has only a finite number of partition members affiliated to it.

Definition 3.6. Let $R \in \mathcal{R}$, $N(R) := \#\{S \in \mathcal{R} : S \text{ is affiliated to } R\}$.

THEOREM 3.7. Given $\widehat{\Sigma}$ from Definition 3.3, there exists a factor map $\widehat{\pi} : \widehat{\Sigma} \rightarrow M$ such that:

- (1) $\widehat{\pi}$ is Hölder continuous with respect to the metric $d(\underline{R}, \underline{S}) = \exp(-\min\{i \geq 0 : R_i \neq S_i \text{ or } R_{-i} \neq S_{-i}\})$;
- (2) $f \circ \widehat{\pi} = \widehat{\pi} \circ \sigma$, where σ denotes the left-shift on $\widehat{\Sigma}$;
- (3) $\widehat{\pi}|_{\widehat{\Sigma}^\#}$ is finite-to-one;
- (4) for all $\underline{R} \in \widehat{\Sigma}$, $\widehat{\pi}(\underline{R}) \in \overline{R_0}$;
- (5) $\widehat{\pi}[\widehat{\Sigma}^\#]$ carries all χ -hyperbolic invariant probability measures.

This theorem is the content of the main theorem of [BO18, Theorem 1.1] (and similarly the content of [Sar13, Theorem 1.3] when $d = 2$). Note that the fibers of $\widehat{\pi}$ being finite in $\widehat{\Sigma}^\#$ does not imply being uniformly bounded.

PROPOSITION 3.8. We have $\widehat{\pi}[\widehat{\Sigma}^\#] = \pi[\Sigma^\#] = \bigcup \mathcal{R} = \text{RWT}_\chi$.

This is the content of [BO20, Proposition 4.11, Corollary 4.12].

3.2. Maximal dimension unstable leaves

Definition 3.9. An unstable leaf (of f) in M , V^u , is a $C^{1+\beta/3}$ embedded Riemannian submanifold of M , such that for all $x, y \in V^u$, $\limsup_{n \rightarrow \infty} 1/n \log d(f^{-n}(x), f^{-n}(y)) < 0$. Similarly, a stable leaf is an unstable leaf of f^{-1} .

Definition 3.10. An unstable leaf is called an unstable leaf of maximal dimension if it is not contained in any unstable leaf of a greater dimension.

Note that if $x \in \text{RWT}_\chi$ belongs to an unstable leaf of maximal dimension V^u , then $\dim H^u(x) = \dim V^u$. This can be seen using the following claim.

Claim 3.11. For all $\underline{u} \in \Sigma$, there exists a maximal dimension unstable leaf $V^u(\underline{u})$, which depends only on $(u_i)_{i \leq 0}$, and a maximal dimension stable leaf $V^s(\underline{u})$, which depends only on $(u_i)_{i \geq 0}$, such that $\{\pi(\underline{u})\} = V^u(\underline{u}) \cap V^s(\underline{u})$, and $f[V^s(\underline{u})] \subset V^s(\sigma \underline{u})$ and $f^{-1}[V^u(\underline{u})] \subset V^u(\sigma^{-1} \underline{u})$.

This is the content of [BO18, Propositions 3.12, 4.4, Theorem 3.13] (and similarly [Sar13, Propositions 4.15, 6.3, Theorem 4.16] when $d = 2$). By construction, $V^s(\underline{u})$, $V^u(\underline{u})$ are local, in the sense that they have finite (intrinsic) diameter.

3.3. Ergodic homoclinic classes and maximal irreducible components

Definition 3.12. Let $x \in \text{RWT}_\chi$ and let $\underline{u} \in \Sigma^\#$ such that $\pi(\underline{u}) = x$. The global stable (respectively unstable) manifold of x is $W^s(x) := \bigcup_{n \geq 0} f^{-n}[V^s(\sigma^n \underline{u})]$ (respectively $W^u(x) := \bigcup_{n \geq 0} f^n[V^u(\sigma^{-n} \underline{u})]$).

This definition is proper and is independent of the choice of u by the construction of $V^u(u)$. For more details, see [BO18, Definitions 2.23 and 3.2]. Recall the remark after Definition 2.4.

Let p be a periodic point in χ -summ, that is, a hyperbolic periodic point. Since p is periodic, $\|C_\chi^{-1}(\cdot)\|$ is bounded along the orbit of p , and therefore $p \in \text{RWT}_\chi$.

Definition 3.13. Let Σ' be a topological Markov shift. A *cylinder* $[R_0, \dots, R_n]$, $n \geq 0$, is the set $\{\underline{R}' \in \Sigma' : R'_i = R_i \text{ for all } i = 0, \dots, n\}$. An element of a topological Markov shift is called a *chain*.

Definition 3.14. Consider the Markov partition \mathcal{R} from Definition 3.3.

- (1) For any $a, b \in \mathcal{R}$, $n \in \mathbb{N}$, we write $a \xrightarrow{n} b$ if there exists a non-empty cylinder $[W_1, \dots, W_n]$ such that $W_1 = a$ and $W_n = b$.
- (2) Define $\sim \subseteq \mathcal{R} \times \mathcal{R}$ by $R \sim S$ if and only if there exists $n_{RS}, n_{SR} \in \mathbb{N}$ such that $R \xrightarrow{n_{RS}} S$, $S \xrightarrow{n_{SR}} R$. The relation \sim is transitive and symmetric. When restricted to $\{R \in \mathcal{R} : R \sim R\}$, it is also reflexive, and thus an equivalence relation. Denote the corresponding equivalence class of some representative $R \in \mathcal{R}$, $R \sim R$, by $\langle R \rangle$.
- (3) A *maximal irreducible component* in $\widehat{\Sigma}$, corresponding to $R \in \mathcal{R}$ such that $R \sim R$, is $\{\underline{R} \in \widehat{\Sigma} : \underline{R} \in \langle R \rangle^{\mathbb{Z}}\}$.
- (4) If a topological Markov shift is a maximal irreducible component of itself, we call it *irreducible*.

If $\widetilde{\Sigma}$ is a maximal irreducible component of $\widehat{\Sigma}$, then $\widehat{\pi}[\widetilde{\Sigma}^\#]$ is a subset of an ergodic homoclinic class, where $\widetilde{\Sigma}^\# := \widehat{\Sigma}^\# \cap \widetilde{\Sigma}$.

Recall Definition 2.11.

Claim 3.15. Let μ be an f -invariant ergodic probability measure which is carried by RWT_χ . Then there exists a unique ergodic homoclinic class $H_\chi(p)$, where p is a χ -hyperbolic periodic point, which carries μ .

For proof, see [RHRHTU11].

PROPOSITION 3.16. *Let p be a periodic χ -hyperbolic point. Then, there exists a maximal irreducible component, $\widetilde{\Sigma} \subseteq \widehat{\Sigma}$, such that $\widehat{\pi}[\widetilde{\Sigma}^\#] \supseteq H_\chi(p) \cap \widehat{\pi}[\widehat{\Sigma}^{\#\#}]$, where*

$$\widehat{\Sigma}^{\#\#} = \{\underline{R} \in \widehat{\Sigma} : \text{there exists } R \text{ such that } \#\{i \geq 0 : R_i = R\} = \#\{i \leq 0 : R_i = R\} = \infty\}.$$

In particular, $\widehat{\pi}[\widetilde{\Sigma}^\#] = H_\chi(p)$ modulo all conservative measures.

This is the content of [BO20, Theorem 5.9], and is based on the result of Buzzi, Crovisier, and Sarig in [BCS22] for homoclinic classes of the type of Newhouse [New72], and the f -invariant, χ -hyperbolic, probability measures which they carry.

3.4. The canonical part of the symbolic space

Definition 3.17

$$\begin{aligned} \widehat{\Sigma}_L &:= \{(R_i)_{i \leq 0} \in \mathcal{R}^{-\mathbb{N}} : \text{for all } i \leq 0, R_{i-1} \rightarrow R_i\}, \sigma_R : \widehat{\Sigma}_L \rightarrow \widehat{\Sigma}_L, \sigma_R((R_i)_{i \leq 0}) \\ &= (R_{i-1})_{i \leq 0}. \end{aligned}$$

Note that σ_R is the right-shift, not the left-shift. To prevent any confusion, we will always notate σ_R with a subscript R (for ‘right’), when considering the right-shift.

Definition 3.18. (The canonical coding $\underline{R}(\cdot)$) Recall that $R(x)$ is the unique \mathcal{R} -element which contains x . Given

$$x \in \pi[\Sigma^\#] = \bigcup \mathcal{R}, \text{ let } (\underline{R}(x))_i := R(f^i(x)), i \in \mathbb{Z}.$$

One should note that $\widehat{\pi}(\underline{R}(x)) = x, \underline{R}(x) \in \widehat{\Sigma}^\circ$ (see Definition 3.21 for $\widehat{\Sigma}^\circ$).

Definition 3.19

$$\text{For all } \underline{R} \in \widehat{\Sigma}_L, \quad W^u(\underline{R}) := \bigcap_{j=0}^\infty f^j[R_{-j}].$$

Here, $W^u(\underline{R})$ might be empty, and is not expected to be an immersed submanifold. It is a subset of RWT_λ . We have the following important property.

COROLLARY 3.20. For all $\underline{R} \in \widehat{\Sigma}_L$,

$$f[W^u(\underline{R})] = \bigcup_{\sigma_R \underline{S} = \underline{R}} W^u(\underline{S}).$$

See [BO21, Corollary 3.19] for proof.

Definition 3.21. We have

$$\begin{aligned} \widehat{\Sigma}^\circ &:= \{\underline{R} \in \widehat{\Sigma} : \text{for all } n \in \mathbb{Z}, f^n(\widehat{\pi}(\underline{R})) \in R_n\}, \\ \widehat{\Sigma}_L^\circ &:= \{\underline{R} \in \widehat{\Sigma}_L^\# : W^u(\underline{R}) \neq \emptyset\}, \end{aligned}$$

where $\widehat{\Sigma}_L^\# := \{(R_i)_{i \leq 0} : (R_i)_{i \in \mathbb{Z}} \in \widehat{\Sigma}^\#\}$. We call $\widehat{\Sigma}^\circ, \widehat{\Sigma}_L^\circ$ the canonical parts of the respective symbolic spaces.

Note that $\underline{R}(\cdot)$ is the inverse of $\widehat{\pi}|_{\widehat{\Sigma}^\circ}$, and that $\widehat{\Sigma}_L^\circ = (\widehat{\Sigma}^\circ)_L := \{(R_i)_{i \leq 0} : (R_i)_{i \in \mathbb{Z}} \in \widehat{\Sigma}^\circ\}$.

Claim 3.22. We claim that

$$\widehat{\pi}[\widehat{\Sigma}^\circ] = \widehat{\pi}[\widehat{\Sigma}^\#] = \pi[\Sigma^\#] = \bigcup \mathcal{R}.$$

See [BO21, Corollary 3.21] for proof.

LEMMA 3.23. If $R_0 \rightarrow R_1$, where $R_0, R_1 \in \mathcal{R}$, and $R_0 \subseteq Z(u_0), u_0 \in \mathcal{V}$, then there exists $u_1 \in \mathcal{V}$ such that $u_0 \rightarrow u_1$ and $R_1 \subseteq Z(u_1)$. Analogously, if $R_0 \rightarrow R_1$ and $R_1 \subseteq Z(v_1)$, then there exists v_0 such that $v_0 \rightarrow v_1$ and $R_0 \subseteq Z(v_0)$.

See [BO21, Lemma 4.1] for proof.

Definition 3.24. Given a chain $\underline{R} \in \widehat{\Sigma}_L$, we say that a chain $\underline{u} \in \mathcal{V}^{-\mathbb{N}}$ covers the chain \underline{R} if \underline{u} is admissible and $R_i \subseteq Z(u_i)$ for all $i \leq 0$. We write $\underline{u} \curvearrowright \underline{R}$.

By Lemma 3.23, for every chain $\underline{R} \in \widehat{\Sigma}_L$, the collection of chains which cover \underline{R} is not empty.

Definition 3.25. We have $\Sigma_L := \{(u_i)_{i \leq 0} : \underline{u} \in \Sigma\}$.

PROPOSITION 3.26. For all $\underline{R} \in \widehat{\Sigma}_L$, there exists a local unstable leaf of maximal dimension $V^u(\underline{R})$ such that $f^{-1}[V^u(\underline{R})] \subseteq V^u(\sigma_R \underline{R})$, and $V^u(\underline{R})$ is an open submanifold of M , with finite and positive induced Riemannian volume. In addition, $V^u(\underline{R}) \subseteq \psi_x[\{v : |v|_\infty \leq p^u\}]$ for all $u = \psi_x^{p^s, p^u} \in \mathcal{V}$ such that $Z(u) \supseteq R_0$, and $\underline{R} \mapsto V^u(\underline{R})$ is continuous in C^1 -norm in any local choice of coordinates. Moreover, $V^u(\underline{R}) \subseteq V^u(\underline{u})$ for all $\underline{u} \in \Sigma_L$ such that $\underline{u} \curvearrowright \underline{R}$.

See [BO21, §4.1] for proof. (Note that $W^u(\underline{R}) \subseteq V^u(\underline{R}) \cap \text{RWT}_\chi$.)

4. A Markovian and absolutely continuous family of measures on the symbolic space

4.1. The Ruelle operator and Sinai’s theorem.

Definition 4.1. (Ruelle operator) Let $\widetilde{\Sigma}_L$ be a one-sided irreducible locally compact topological Markov shift (of negative chains), and let $\phi : \widetilde{\Sigma}_L \rightarrow \mathbb{R}$ be a Hölder continuous potential. The associated Ruelle operator $L_\phi : C(\widetilde{\Sigma}_L) \rightarrow C(\widetilde{\Sigma}_L)$ is defined by

$$(L_\phi h)(\underline{R}) := \sum_{\widetilde{\sigma}_R \underline{S} = \underline{R}} e^{\phi(\underline{S})} h(\underline{S}),$$

where $h \in C(\widetilde{\Sigma}_L)$, and $\widetilde{\sigma}_R : \widetilde{\Sigma}_L \rightarrow \widetilde{\Sigma}_L$ is the right-shift.

Definition 4.2. Let X be a topological Markov shift (either one-sided or two-sided). A function $\widehat{\varphi} : X \rightarrow \mathbb{R}$ is called *weakly Hölder continuous* if there exists $C > 0, \alpha > 0$, such that for all $\underline{x}, \underline{y} \in X$ such that $d(\underline{x}, \underline{y}) \leq e^{-1}$, we have $|\widehat{\varphi}(\underline{x}) - \widehat{\varphi}(\underline{y})| \leq C \cdot d(\underline{x}, \underline{y})^\alpha$.

This notion is introduced in [Sar09], with the weaker assumption of $d(\underline{x}, \underline{y}) \leq e^{-2}$.

Recall Definition 2.10.

Claim 4.3. Let $\varphi : \text{WT}_\chi^\epsilon \rightarrow \mathbb{R}$ be a potential which is Grassmann–Hölder continuous. Let $\widehat{\varphi} : \widehat{\Sigma} \rightarrow \mathbb{R}, \widehat{\varphi} := \varphi \circ \widehat{\pi}$, be the lift of φ . Then $\widehat{\varphi}$ is weakly Hölder continuous on $\widehat{\Sigma}$.

This follows from [BO18, Proposition 6.1].

THEOREM 4.4. (Sinai) Let $\varphi : \text{RWT}_\chi \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Then there exist two functions $\varphi^*, A : \bigcup \mathcal{R} \rightarrow \mathbb{R}$ such that $\varphi^*(x) = \varphi^*((R(f^{-i}(x)))_{i \geq 0})$ depends only on the past of the itinerary of x , $\varphi^* = \varphi + A - A \circ f^{-1}$, A is bounded, and $\widehat{A} := A \circ \widehat{\pi}, \phi := \varphi^* \circ \widehat{\pi}$ are weakly Hölder continuous.

This theorem appears in Sinai’s fundamental paper [Sin72]. He states it in the context of a topological Markov shift with a finite alphabet (for the countable case, see [Dao13]). We give the proof in the case when a countable Markov partition exists.

Proof. First, for all $a \in \mathcal{R}$, fix a point $y_a \in a$. For all $x \in \bigcup \mathcal{R}$, define $x^* := [x, y_{R(x)}]_{R(x)} \in R(x)$, where $R(x)$ is unique partition element of \mathcal{R} which contains x , and $[\cdot, \cdot]_{R(x)}$

denotes the Smale bracket of two points in $R(x)$ (the right-hand term yields its future, and the left-hand terms yields its past). Define $A : \bigcup \mathcal{R} \rightarrow \mathbb{R}$,

$$A(x) := \sum_{n \geq 0} \varphi(f^{-n}(x^*)) - \varphi(f^{-n}(x)).$$

By Claim 4.3, $\widehat{\varphi} := \varphi \circ \widehat{\pi}$ is weakly Hölder continuous. In addition, $d(\underline{R}(f^{-n}(x)), \underline{R}(f^{-n}(x^*))) \leq e^{-n}$. Let $C > 0, \theta \in (0, 1)$ be the weak Hölder constants of $\widehat{\varphi}$ such that $d(\underline{R}^{(1)}, \underline{R}^{(2)}) \leq e^{-n} \Rightarrow |\widehat{\varphi}(\underline{R}^{(1)}) - \widehat{\varphi}(\underline{R}^{(2)})| \leq C \cdot \theta^n$. Then, \widehat{A} is a uniformly convergent series with uniformly bounded and equicontinuous summands, and so \widehat{A} is bounded and continuous. In particular, A is also bounded.

Let $\varphi^* := \varphi + A - A \circ f^{-1}$ and $\phi := \varphi^* \circ \widehat{\pi}$. The weak Hölder continuity of $\widehat{A} = A \circ \widehat{\pi}$ and of ϕ follows from the weak Hölder continuity of $\widehat{\varphi}$ and the proof of the finite alphabet case in [Sin72], whose details are given in [Bow75, Lemma 1.6].

It remains to show that $\varphi(x) = \varphi((R(f^{-i}(x))))_{i \geq 0}$.

$$\begin{aligned} \varphi^*(x) &= \varphi(x) + A(x) - A(f^{-1}(x)) \\ &= \varphi(x) + \sum_{n \geq 0} \varphi(f^{-n}(x^*)) - \varphi(f^{-n}(x)) - \sum_{n \geq 0} \varphi(f^{-n}((f^{-1}(x))^*)) - \varphi(f^{-n-1}(x)) \\ &= \varphi(x^*) + \sum_{n \geq 0} \varphi(f^{-n}(f^{-1}(x^*))) - \varphi(f^{-n}((f^{-1}(x))^*)). \end{aligned}$$

The last expression depends only on $(R(f^{-i}(x)))_{i \geq 0}$. □

Remark. Although $\widehat{\varphi}, \widehat{A}, \phi$ are defined on $\widehat{\Sigma}^\circ$, they extend continuously to $\widehat{\Sigma}$ because of their uniform moduli of continuity and the fact that $\widehat{\Sigma}^\circ$ is dense in $\widehat{\Sigma}$. Thus, $\phi : \widehat{\Sigma}_L \rightarrow \mathbb{R}$ is well defined and weakly Hölder continuous.

4.2. *Recurrence, harmonic functions, and conformal measures.* As before, $\widetilde{\Sigma}_L$ is a maximal irreducible component of $\widehat{\Sigma}_L$, and $\phi : \widetilde{\Sigma}_L \rightarrow \mathbb{R}$ is a weakly Hölder continuous potential.

In this section, we use the Gurevich pressure, and we begin by recalling its definition. For a weakly Hölder potential $\zeta : \widehat{\Sigma}_L \rightarrow \mathbb{R}$, the *local partition functions* are for $n \geq 1$,

$$Z_n(\zeta, R) := \sum_{\underline{S} \in \widehat{\Sigma}_L \cap [R], \sigma_R^n \underline{S} = \underline{S}} e^{\sum_{k=0}^{n-1} \zeta(\sigma_R^k \underline{S})},$$

where the sum has finitely many terms since $\widehat{\Sigma}_L$ is locally compact. The *Gurevich pressure* of the potential ζ is

$$P_G(\zeta) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\zeta, R) \in (-\infty, \infty].$$

When $\widehat{\Sigma}_L$ is irreducible, the limit is independent of the choice of the symbol R (see [Sar09, Proposition 3.2]). We say that the potential ζ is *recurrent* if for some symbol R ,

$$\sum_{n \geq 1} e^{-nP_G(\zeta)} Z_n(\zeta, R) = \infty.$$

In this case, the sum diverges for all R , see [Sar09, Corollary 3.1]. We say that the potential ζ is *positive recurrent* if it is recurrent, and for some symbol R ,

$$\sum_{n \geq 1} n \cdot e^{-nP_G(\zeta)} \sum_{\substack{\underline{S} \in \widehat{\Sigma}_L \cap [R], \text{ such that } \sigma_R^n \underline{S} = \underline{S}, \\ \text{and for all } 0 < l < n, S_{-l} \neq R}} e^{\sum_{k=0}^{n-1} \zeta(\sigma_R^k \underline{S})} < \infty.$$

Again, this property turns out to be independent of R . For more details, see [Sar09, §3.1.3]. For a detailed review of the properties of recurrent potentials, see [Sar09].

Definition 4.5. Let $\phi : \widetilde{\Sigma}_L \rightarrow \mathbb{R}$ be a weakly Hölder continuous potential such that $P_G(\phi) < \infty$, where $\widetilde{\Sigma}_L$ is a maximal irreducible component of $\widehat{\Sigma}_L$. A positive and continuous function $\psi : \widetilde{\Sigma}_L \rightarrow \mathbb{R}^+$ is called ϕ -harmonic if $L_\phi \psi = e^{P_G(\phi)} \psi$. A non-zero Radon measure p on $\widetilde{\Sigma}_L$ is called ϕ -conformal if $L_\phi^* p = e^{P_G(\phi)} p$, where L_ϕ^* is the dual operator of L_ϕ .

Note that $\widetilde{\Sigma}_L$ may be non-compact, and thus p may be infinite.

THEOREM 4.6. *Let $\phi : \widetilde{\Sigma}_L \rightarrow \mathbb{R}$ be a weakly Hölder continuous potential, where $\widetilde{\Sigma}_L$ is an irreducible locally compact one-sided topological Markov shift. If $P_G(\phi) < \infty$, then there exist a ϕ -harmonic function ψ and a ϕ -conformal measure p .*

This theorem is due to work of Sarig [Sar99, Sar01], Cyr [Cyr10], and Shwartz [Shw19]. Sarig had proven that when ϕ is recurrent, ψ and p exist and are unique up to scaling. In the transient case, Cyr showed the existence of a ϕ -conformal measure, and Shwartz proved the existence of a ϕ -harmonic function with a weakly Hölder logarithm. In the transient case, the ϕ -harmonic functions and the ϕ -conformal measures are not always unique up to scaling. For the structure of the set of these objects, see [Shw19].

LEMMA 4.7. *Let $\phi : \widetilde{\Sigma}_L \rightarrow \mathbb{R}$ be a weakly Hölder continuous potential, where $\widetilde{\Sigma}_L$ is an irreducible locally compact one-sided topological Markov shift. Let $\psi : \widetilde{\Sigma}_L \rightarrow \mathbb{R}^+$ be a ϕ -harmonic function. Then $\log \psi$ is weakly Hölder continuous.*

In the recurrent case, Sarig proves it in [Sar09, Proposition 3.4]. In the transient case, it follows from the proof of Theorem 5.7 in [Shw19]. Shwartz shows in the proof that $\log \psi$ is uniformly continuous. A close inspection of his argument tells that, in fact, if ϕ is weakly Hölder continuous, then so is $\log \psi$. (It follows from the estimate on $o_n(1)$, which depends on the regularity of the coboundary function ψ , where one can see that if ϕ is weakly Hölder continuous, then so is ψ ([Shw19, Proposition 5.4], see also Theorem 4.4 of the present paper). In particular, the error term $o_n(1)$ is exponentially small in n .

4.3. Recurrence and periodic orbits

LEMMA 4.8. *Let $\varphi : \text{RWT}_X \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential, and let $\phi : \widehat{\Sigma}_L \rightarrow \mathbb{R}$ be a corresponding weakly Hölder continuous one-sided potential given by*

Theorem 4.4 and the remark following it. Let $H_\chi(p)$ be an ergodic homoclinic class of a periodic and χ -hyperbolic point p , and let $\tilde{\Sigma}$ be a maximal irreducible component of $\tilde{\Sigma}$ such that $\hat{\pi}[\tilde{\Sigma}^\#] = H_\chi(p)$ modulo conservative measures (see Proposition 3.16). Then, $\phi : \tilde{\Sigma}_L \rightarrow \mathbb{R}$ is recurrent if and only if

$$\sum_{n \geq 1} \sum_{\substack{f^n(q)=q, \\ q \in H_\chi(p) \cap \Lambda_l}} e^{\varphi_n(q) - n \cdot P_{H_\chi(p)}(\varphi)} = \infty,$$

where Λ_l is a level set such that $p \in \Lambda_{l, e^{-2} \cdot \frac{3}{2}\epsilon}$ and $\varphi_n(q) = \sum_{k=0}^{n-1} \varphi(f^{-k}(q))$.

Proof. We start with the ‘if’ part. Let $q \in H_\chi(p) \cap \Lambda_l$ be a periodic χ -hyperbolic point of period n . By Proposition 3.16, q has a (periodic) coding in $\tilde{\Sigma}$. (Let $\underline{R} \in \tilde{\Sigma}^\# \cap \hat{\pi}^{-1}[\{q\}]$. Then for any $m \geq 0$, $\hat{\pi}(\sigma^{m \cdot n} \underline{R}) = q$, whence by the finiteness of $\tilde{\Sigma}^\# \cap \hat{\pi}^{-1}[\{q\}]$, there exists $m_1, m_2 \in \mathbb{N}$ such that $\sigma^{m_1 \cdot n} \underline{R} = \sigma^{m_2 \cdot n} \underline{R}$, and so $\sigma^{m \cdot n} \underline{R} = \underline{R}$ with $m = \max\{m_1, m_2\} - \min\{m_1, m_2\}$.) Let \underline{S}^q be such a periodic coding.

By [BO21, Claim 7.6], there is a finite collection of symbols $\mathcal{R}_l \subseteq \mathcal{R}$, which depends only on l , and which contains S_0^q .

By [BO18, Theorem 1.3], there exists $C_l = \max\{N(R)^2 : R \in \mathcal{R}_l\} \in \mathbb{N}$ such that $|\hat{\pi}^{-1}[\{q\}] \cap \tilde{\Sigma}^\#| \leq C_l$ (see Definition 3.6 for the definition of $N(R)$). In addition, by Theorem 4.4, there exists $C := 2\|A\|_\infty < \infty$ such that

$$\phi_n(\underline{S}^q) = \pm C + \varphi_n(q), \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Since φ is bounded (recall remark after Definition 2.10), so is ϕ . By the spectral decomposition (see [Sar15, Theorem 2.5]), we may assume without loss of generality that $(\tilde{\Sigma}, \sigma)$ is topologically mixing (otherwise apply the proof to each mixing component of a suitable period of σ_R). By the variational principle (see [Sar09, Theorem 4.4] and [IJT15] when the potential is not bounded), the Gurevich pressure of ϕ on $\tilde{\Sigma}_L$, $P_G(\phi)$, satisfies

$$P_G(\phi) = \sup\{h_\nu(\tilde{\sigma}_R) + \int_{\tilde{\Sigma}_L} \phi \, d\nu : \nu \text{ is an invariant probability measure on } \tilde{\Sigma}_L\}.$$

Every invariant probability measure on $\tilde{\Sigma}_L$, ν , extends uniquely to an invariant probability measure on $\tilde{\Sigma}$, ν' , such that $\nu' \circ \tau^{-1} = \nu$ and $h_\nu(\tilde{\sigma}_R) = h_{\nu'}(\sigma)$, where $\tau : \tilde{\Sigma} \rightarrow \tilde{\Sigma}_L$ is the projection onto the non-positive coordinates. Thus,

$$P_G(\phi) = \sup\{h_{\nu'}(\sigma) + \int_{\tilde{\Sigma}} \phi \, d\nu' : \nu' \text{ is an invariant probability measure on } \tilde{\Sigma}\}.$$

In addition, every invariant probability measure on $H_\chi(p)$, μ , lifts to a an invariant probability measure on $\tilde{\Sigma}$, μ' , such that $\mu' \circ \hat{\pi}^{-1} = \mu$ and $h_\mu(f) = h_{\mu'}(\sigma)$. Then, we get

$$P_G(\phi) = P_{H_\chi(p)}(\varphi). \tag{2}$$

By the Ruelle inequality, the topological entropy of f is finite, and since φ is bounded, $P_{H_\chi(p)}(\varphi) < \infty$.

For every $n \in \mathbb{N}$, there exists $a_n, b_n \in \mathcal{R}_l$ such that

$$\sum_{\substack{f^n(q)=q, q \in H_\chi(p) \cap \Lambda_l \\ \underline{S}^q \in \widehat{\pi}^{-1}[\{q\}] \cap \widetilde{\Sigma}^\#, S_0^q = a_n, S_n^q = b_n}} e^{\phi_n(\underline{S}^q)} \geq \frac{1}{|\mathcal{R}_l|^2} \cdot \sum_{\substack{f^n(q)=q, q \in H_\chi(p) \cap \Lambda_l \\ \underline{S}^q \in \widehat{\pi}^{-1}[\{q\}] \cap \widetilde{\Sigma}^\#}} e^{\phi_n(\underline{S}^q)}.$$

For every $n \geq 1$, for all $q \in H_\chi(p) \cap \Lambda_l$ such that $f^n(q) = q$, $|\widehat{\pi}^{-1}[\{q\}] \cap \widetilde{\Sigma}^\#| \leq C_l$. By assumption, $\infty = \sum_{n \geq 1} \sum_{f^n(q)=q, q \in H_\chi(p) \cap \Lambda_l} e^{\phi_n(q) - n \cdot P_{H_\chi(p)}(\varphi)}$. Recall equation (1), then by the pigeonhole principle, there exists $a, b \in \mathcal{R}_l$ and $n_k \uparrow \infty$ such that $a_{n_k} = a$ and $b_{n_k} = b$, and so that the following sum is infinite:

$$\infty = \sum_{k \geq 0} \sum_{\substack{f^{n_k}(q)=q, q \in H_\chi(p) \cap \Lambda_l \\ \underline{S}^q \in \widehat{\pi}^{-1}[\{q\}] \cap \widetilde{\Sigma}^\#, S_0^q = a, S_{n_k}^q = b}} e^{\phi_{n_k}(\underline{S}^q) - n_k \cdot P_G(\phi)}.$$

Let $C'_H > 0, \gamma \in (0, 1)$ be the Hölder constant and Hölder exponent of ϕ , respectively. Let $C_H := C'_H \cdot \sum_{n \geq 0} \gamma^n < \infty$.

Let \underline{W} be an admissible word such that $W_0 = b$ and $W_{m-1} = a$, where $m := |\underline{W}|$. For every finite word \underline{W}' such that $W'_0 = W'_{|\underline{W}'|-1}$, let $\underline{S}^{\underline{W}'}$ denote the periodic extension of \underline{W}' to a bi-infinite chain (in $\widetilde{\Sigma}^\#$).

For every $n \geq 0$, for every finite word \underline{W}' such that $|\underline{W}'| = n$, every extension of \underline{W}' to $\widetilde{\Sigma}^\#$ which codes a periodic point in $H_\chi(p) \cap \Lambda_l$ of period n must code the same point, by the shadowing lemma (see [BO18, Proposition 3.12(4)]). Therefore,

$$\begin{aligned} \infty &= \sum_{k \geq 0} \sum_{\substack{f^{n_k}(q)=q, q \in H_\chi(p) \cap \Lambda_l \\ \underline{S}^q \in \widehat{\pi}^{-1}[\{q\}] \cap \widetilde{\Sigma}^\#, S_0^q = a, S_{n_k}^q = b}} e^{\phi_{n_k}(\underline{S}^q) - n_k \cdot P_G(\phi)} \\ &\leq C_l e^{C_H \sum_{j=1}^{n_k} \gamma^j} \sum_{k \geq 0} \sum_{\substack{\widetilde{W}: |\widetilde{W}|=n_k, \widetilde{W}_0=a, \widetilde{W}_{n_k-1}=b}} e^{\phi_{n_k}(\underline{S}^{\widetilde{W}}) - n_k \cdot P_G(\phi)} \\ &\leq C_l e^{m \cdot (\|\varphi\|_\infty + P_G(\phi)) + C_H} \cdot \sum_{k \geq 0} \sum_{\substack{\underline{S} \in \widetilde{\Sigma}: \sigma^{n_k+m} \underline{S} = \underline{S}, S_0 = a}} e^{\phi_{n_k+m}(\underline{S}) - (n_k+m) \cdot P_G(\phi)} \\ &\leq C_l e^{m \cdot (\|\varphi\|_\infty + P_G(\phi)) + C_H} \cdot \sum_{n \geq 1} \sum_{\substack{\underline{S} \in \widetilde{\Sigma}: \sigma^n \underline{S} = \underline{S}, S_0 = a}} e^{\phi_n(\underline{S}) - n \cdot P_G(\phi)}. \end{aligned}$$

Therefore, ϕ is recurrent on $\widetilde{\Sigma}$.

It remains to show the ‘only if’ part. Assume that ϕ is recurrent on $\widetilde{\Sigma}$, then for all $a \in \mathcal{R}_l$,

$$\sum_{n \geq 1} \sum_{\substack{\underline{S} \in \widetilde{\Sigma}: \\ \sigma^n \underline{S} = \underline{S}, S_0 = a}} e^{\phi_n(\underline{S}) - n \cdot P_G(\phi)} = \infty.$$

Choose $a \in \mathcal{R}_l$ such that $[a] \cap \hat{\pi}^{-1}[\{p\}] \cap \tilde{\Sigma}^\# \neq \emptyset$. For all $n \geq 1$, every periodic chain $\underline{S} \in \tilde{\Sigma} \cap [a]$ of period n codes a periodic point of period n in $H_\chi(p) \cap \Lambda_l$ (see [BO18, Theorem 4.13]). There can be no more than C_l chains which code the same point, and thus

$$\begin{aligned} \infty &= \sum_{n \geq 1} \sum_{\underline{S} \in \tilde{\Sigma} : \sigma^n \underline{S} = \underline{S}, S_0 = a} e^{\phi_n(\underline{S}) - n \cdot P_G(\phi)} \\ &\leq C_l e^C e^{C_H} \sum_{n \geq 1} \sum_{f^n(q) = q, q \in H_\chi(p) \cap \Lambda_l} e^{\varphi_n(q) - n \cdot P_{H_\chi(p)}(\varphi)}. \end{aligned} \quad \square$$

5. Absolutely continuous invariant family of leaf measures

The following theorem is a version of the Ledrappier theorem ([Led74], [LLS16, Theorem 3.3]). We give a new proof here which is good for our needs, while the previous proofs provide a formula for the measures of cylinders.

THEOREM 5.1. *Let $\phi : \tilde{\Sigma}_L \rightarrow \mathbb{R}$ be a weakly Hölder continuous potential such that $P_G(\phi) < \infty$, where $\tilde{\Sigma}_L$ is an irreducible locally compact one-sided topological Markov shift. Let ψ be a ϕ -harmonic function on $\tilde{\Sigma}_L$. Then, there exists a family of probability measures on $\tilde{\Sigma}$, $\{\hat{p}_{\underline{R}}\}_{\underline{R} \in \tilde{\Sigma}_L}$, such that for all $\underline{R} \in \tilde{\Sigma}_L$, $\hat{p}_{\underline{R}}$ is carried by $\{\underline{v} \in \tilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}$ and $\hat{p}_{\underline{R}} \circ \sigma^{-1} = \sum_{\tilde{\sigma}_R \underline{S} = \underline{R}} e^{\phi(\underline{S}) + \log \psi(\underline{S}) - \log \psi \circ \tilde{\sigma}_R(\underline{S}) - P_G(\phi)} \hat{p}_{\underline{S}}$, where $\tilde{\sigma}_R := \sigma_R|_{\tilde{\Sigma}_L}$.*

Proof. Fix $[R] \subseteq \tilde{\Sigma}_L$, and let $\underline{R} \in \tilde{\Sigma}_L \cap [R]$. Let $\tau : \hat{\Sigma} \rightarrow \hat{\Sigma}_L$ be the projection to the non-positive coordinates. Write $\tilde{\phi} := \phi + \log \psi - \log \psi \circ \tilde{\sigma}_R - P_G(\phi)$, and $\tilde{\phi}_n := \sum_{k=0}^{n-1} \tilde{\phi} \circ \tilde{\sigma}_R^k$. It follows that $L_{\tilde{\phi}} 1 = 1$. Define

$$\hat{p}_{\underline{R}}^{(n)} := \sum_{\tilde{\sigma}_R^n \underline{S} = \underline{R}} e^{\tilde{\phi}_n(\underline{S})} \delta_{\sigma^{-n} \underline{S}^\pm}, \tag{3}$$

where $\delta_{\underline{S}^\pm}$ is a Dirac measure with mass at the chain \underline{S}^\pm , and \underline{S}^\pm is some (any) choice of a chain in $\tau^{-1}[\{\underline{S}\}] \cap \tilde{\Sigma}$. We will show that the limit $\hat{p}_{\underline{R}}^{(n)} \xrightarrow[n \rightarrow \infty]{\text{weak-}^*} \hat{p}_{\underline{R}}$ exists, and is independent of the choice of the representatives $\underline{S}^\pm \in \tau^{-1}[\{\underline{S}\}] \cap \tilde{\Sigma}$.

Step 1. For all $n \geq 0$, $\hat{p}_{\underline{R}}^{(n)}(1) = (L_{\tilde{\phi}}^n 1)(\underline{R}) = 1$. Here, $\{\tilde{\underline{R}} \in \tilde{\Sigma} \cap [R] : \tilde{R}_i = R_i \text{ for all } i \leq 0\}$ is a compact set which carries $\hat{p}_{\underline{R}}^{(n)}$, for all $n \geq 1$. Thus, there exists $n_k \uparrow \infty$ and a probability $\hat{p}_{\underline{R}}$ on $\{\tilde{\underline{R}} \in \tilde{\Sigma} \cap [R] : \tilde{R}_i = R_i, \text{ for all } i \leq 0\}$ such that $\hat{p}_{\underline{R}}^{(n_k)} \xrightarrow[k \rightarrow \infty]{} \hat{p}_{\underline{R}}$.

Step 2. Let $[w]_a := \sigma^a[w_a, w_{a+1}, \dots, w_{b-1}, w_b]$, where $a, b \in \mathbb{Z}$ and $a \leq b$. Let $n, m \geq 0$. Observe the following conditions.

- (1) If $b \leq 0$, $\hat{p}_{\underline{R}}^{(n)}(\mathbb{1}_{[w]_a}) = 1$ if $\underline{R} \in [w]_a$, and $\hat{p}_{\underline{R}}^{(n)}(\mathbb{1}_{[w]_a}) = 0$ otherwise.
- (2) If $a \leq 0 \leq b$, $\hat{p}_{\underline{R}}^{(n)}(\mathbb{1}_{[w]_a}) = \mathbb{1}_{\sigma^a[w_a, \dots, w_0]}(\underline{R}) \cdot \hat{p}_{\underline{R}}^{(n)}(\mathbb{1}_{[w_0, \dots, w_b]})$ (and in particular, this equals 0 if $w_0 \neq R_0$).

(3) Assume $\underline{w} = R, w_1, \dots, w_b$, then $[\underline{w}]_a = [\underline{w}]$. If $n > b$,

$$\begin{aligned} \widehat{P}_R^{(n+m)}(\mathbb{1}_{[\underline{w}]}) &= \sum_{\widetilde{\sigma}_R^n \underline{S}=\underline{R}} e^{\widetilde{\phi}_m(\underline{S})} \left(\sum_{\widetilde{\sigma}_R^n \underline{Q}=\underline{S}} e^{\widetilde{\phi}_n(\underline{Q})} \delta_{\sigma^{-n-m} \underline{Q}^\pm}(\mathbb{1}_{[\underline{w}]}) \right) \\ &= \sum_{\widetilde{\sigma}_R^n \underline{S}=\underline{R}} e^{\widetilde{\phi}_m(\underline{S})} \left(\sum_{\widetilde{\sigma}_R^n \underline{Q}=\underline{S}} e^{\widetilde{\phi}_n(\underline{Q})} \delta_{\sigma^{-n} \underline{Q}^\pm}(\mathbb{1}_{[\underline{w}] \circ \sigma^{-m}}) \right) \\ &= \sum_{\widetilde{\sigma}_R^m \underline{S}=\underline{R}} e^{\widetilde{\phi}_m(\underline{S})} \widehat{P}_S^{(n)}(\mathbb{1}_{[\underline{w}] \circ \sigma^{-m}}) = \sum_{\widetilde{\sigma}_R^m \underline{S}=\underline{R}} e^{\widetilde{\phi}_m(\underline{S})} (\widehat{P}_S^{(n)} \circ \sigma^m)(\mathbb{1}_{[\underline{w}]}) . \end{aligned} \tag{4}$$

Note that equation (4) holds regardless of the choice of $\underline{S}^\pm, \underline{Q}^\pm$.

Step 3. Let $\underline{w} = (R, w_1, \dots, w_{b-1}, w_b), b \geq 1, l \geq 1$. For all $n = b + l > b$,

$$\widehat{P}_R^{(n)}(\mathbb{1}_{[\underline{w}]}) = \sum_{\widetilde{\sigma}_R^b \underline{S}=\underline{R}} e^{\widetilde{\phi}_b(\underline{S})} (\widehat{P}_S^{(l)} \circ \sigma^b)(\mathbb{1}_{[\underline{w}]}) = \sum_{\widetilde{\sigma}_R^b \underline{S}=\underline{R}} e^{\widetilde{\phi}_b(\underline{S})} \mathbb{1}_{[\underline{w}]}(\underline{S}) . \tag{5}$$

This is a constant sequence for all $n > b$. (Having a consistent explicit formula for the limit measure of shifted cylinders allows us to use it as a definition and apply the Carathéodory extension theorem; however, we choose to present the argument which uses weak-* limits for future computations.) Therefore, since the cylinders (and their shifts) generate the Borel sigma algebra, and $\widehat{P}_R^{(nk)} \xrightarrow[k \rightarrow \infty]{\text{weak-*}} \widehat{P}_R$, we get that $\widehat{P}_R^{(n)} \xrightarrow[n \rightarrow \infty]{\text{weak-*}} \widehat{P}_R$; and the limit is independent of the choice of $\underline{S}^\pm \in \tau^{-1}[\{\underline{S}\}] \cap \widetilde{\Sigma}$ for a chain $\underline{S} \in \widetilde{\Sigma}_L$ in equation (3). In addition, it follows that for all $n \geq 0$,

$$\widehat{P}_R \circ \sigma^{-n} = \sum_{\widetilde{\sigma}_R^n \underline{S}=\underline{R}} e^{\widetilde{\phi}_n(\underline{S})} \widehat{P}_S . \tag{6}$$

□

This construction could *a priori* depend on the choice of ψ ; and in the case that ϕ is transient, ψ may not be unique. We later show applications in setups which imply the recurrence of ϕ , and thus remove the need to choose ψ .

COROLLARY 5.2. (Absolute continuity) *Let $\phi : \widetilde{\Sigma}_L \rightarrow \mathbb{R}$ be a weakly Hölder continuous potential such that $P_G(\phi) < \infty$, where $\widetilde{\Sigma}_L$ is an irreducible locally compact one-sided topological Markov shift. Let ψ be a ϕ -harmonic function on $\widetilde{\Sigma}_L$. Let $\{\widehat{P}_R\}_{R \in \widetilde{\Sigma}_L}$ be the measures constructed in Theorem 5.1. Let $[R] \subseteq \widetilde{\Sigma}_L$, let $\underline{R}, \widetilde{R} \in \widetilde{\Sigma}_L \cap [R]$, and let*

$$\widehat{\Gamma}_{\underline{R} \widetilde{R}} : \{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\} \rightarrow \{\underline{v} \in \widetilde{\Sigma} : v_i = \widetilde{R}_i, \text{ for all } i \leq 0\}$$

be the holonomy map $\underline{v} \mapsto (\widetilde{R}_i)_{i \leq 0} \cdot (v_i)_{i \geq 0}$, where \cdot denotes an admissible concatenation. Then, $\widehat{\Gamma}_{\underline{R} \widetilde{R}}$ is a continuous and invertible map such that $\widehat{P}_R \circ \widehat{\Gamma}_{\underline{R} \widetilde{R}}^{-1} \sim \widehat{P}_{\widetilde{R}}$.

Proof. Let ψ be the ϕ -harmonic function which is used in Theorem 5.1, and write $\widetilde{\phi} := \phi + \log \psi - \log \psi \circ \widetilde{\sigma}_R - P_G(\phi)$ and $\widetilde{\phi}_n := \sum_{k=0}^{n-1} \widetilde{\phi} \circ \widetilde{\sigma}_R^k$. By Lemma 4.7 by the local compactness of $\widetilde{\Sigma}_L$, the weak Hölder continuity of ϕ , implies that $\widetilde{\phi}$ is weakly Hölder continuous.

Let $[w]_a := \sigma^a[w_a, w_{a+1}, \dots, w_{b-1}, w_b]$, where $a, b \in \mathbb{Z}$ and $a \leq b$, and write $g := \mathbb{1}_{[w]_a}$. Let $j \geq 0$, and fix an extension $\underline{S}^{\pm, (j)} \in \tau^{-1}[\{\underline{S}\}]$ for each $\underline{S} \in \tilde{\sigma}_R^{-j}[\{\tilde{R}\}]$, where $\tau : \tilde{\Sigma} \rightarrow \tilde{\Sigma}_L$ is the projection to the non-positive coordinates. Let $\{\widehat{p}_R\}_{\tilde{\Sigma}_L}$ be the measures constructed in Theorem 5.1. By steps 2 and 3 in Theorem 5.1,

$$\widehat{p}_R(g) = \lim_{j \rightarrow \infty} \sum_{\tilde{\sigma}_R^j \underline{S} = \tilde{R}} e^{\tilde{\phi}_j(\underline{S})} \delta_{\sigma^{-j} \underline{S}^{\pm, (j)}}(g). \tag{7}$$

Moreover, by step 3 in Theorem 5.1, this limit is independent of the choice of $\{\underline{S}^{\pm, (j)}\}_{j \geq 0}$. Thus, similarly,

$$\widehat{p}_R(g \circ \widehat{\Gamma}_{\tilde{R}}) = \lim_{j \rightarrow \infty} \sum_{\tilde{\sigma}_R^j \tilde{S} = \tilde{R}} e^{\tilde{\phi}_j(\tilde{S})} \delta_{\widehat{\Gamma}_{\tilde{R}}(\sigma^{-j} \underline{S}^{\pm, (j)})}(g), \tag{8}$$

where for all $j \geq 0$, for all $\tilde{S} \in \tilde{\sigma}_R^{-j}[\{\tilde{R}\}]$, $\tau(\sigma^j \widehat{\Gamma}_{\tilde{R}}(\sigma^{-j} \underline{S}^{\pm, (j)})) = \tilde{S}$.

Let $C > 0$, $\gamma \in (0, 1)$ be the Hölder constant and Hölder exponent of $\tilde{\phi}$, respectively. For all $j \geq 0$, for all $\underline{S} \in \tilde{\sigma}_R^{-j}[\{\tilde{R}\}]$, write $\tilde{S} := \tau(\sigma^j \widehat{\Gamma}_{\tilde{R}}(\sigma^{-j} \underline{S}^{\pm, (j)})) \in \tilde{\sigma}_R^{-j}[\{\tilde{R}\}]$. Write $d(\underline{R}, \tilde{R}) = e^{-n}$, $n \geq 1$. Then,

$$e^{\tilde{\phi}_j(\underline{S})} = e^{\pm C \sum_{k \geq 0} \gamma^{k+n}} \cdot e^{\tilde{\phi}_j(\tilde{S})}.$$

Write $\tilde{C}_\phi := \exp(C \cdot \sum_{k \geq 0} \gamma^k) \in (0, \infty)$. Then,

$$\text{for all } h \in C(\{v \in \tilde{\Sigma} : v_i = \tilde{R}_i, \text{ for all } i \leq 0\}), \widehat{p}_R(h) = (\tilde{C}_\phi)^{\pm \gamma^n} \cdot (\widehat{p}_R \circ \widehat{\Gamma}_{\tilde{R}}^{-1})(h). \tag{9}$$

□

Remark. The proof of Corollary 5.1 shows something a little stronger than absolute continuity, it shows that $\{\widehat{p}_R\}_{R \in \tilde{\Sigma}_L}$ is a continuous family with a uniform modulus of continuity for the holonomies.

Definition 5.3. Let $\tilde{\Sigma}$ be a maximal irreducible component of $\widehat{\Sigma}$, and let $\{\widehat{p}_R\}_{R \in \tilde{\Sigma}_L}$ be the family of measures given by Theorem 5.1, when $P_G(\phi|_{\tilde{\Sigma}_L}) < \infty$. Define for all $\underline{R} \in \tilde{\Sigma}_L$, $\widehat{\mu}_R := \psi(\underline{R}) \cdot \widehat{p}_R$, where ψ is the ϕ -harmonic function on $\tilde{\Sigma}_L$ which is fixed in Theorem 5.1.

COROLLARY 5.4. *The family of measures $\{\widehat{\mu}_R\}_{R \in \tilde{\Sigma}_L}$ from Definition 5.3 satisfies:*

$\widehat{\mu}_R \circ \sigma^{-1} = e^{-P_G(\phi)} \sum_{\tilde{\sigma}_R \underline{S} = \underline{R}} e^{\phi(\underline{S})} \widehat{\mu}_S$. In addition, $\{\widehat{\mu}_R\}_{R \in \tilde{\Sigma}_L}$ is an absolutely continuous family of measures.

The invariance follows from Definition 5.3 and from equation (6). The absolute continuity follows from Corollary 5.2.

COROLLARY 5.5. *When ϕ is recurrent, the family $\{\widehat{\mu}_R\}_{R \in \tilde{\Sigma}_L}$ is the unique (up to scaling) continuous family of measures which satisfies $\widehat{\mu}_R \circ \sigma^{-1} = e^{-P_G(\phi)} \sum_{\tilde{\sigma}_R \underline{S} = \underline{R}} e^{\phi(\underline{S})} \widehat{\mu}_S$.*

Proof. Let p and ψ be versions of the (unique up to scaling) ϕ -conformal measure and ϕ -harmonic function on $\tilde{\Sigma}_L$, respectively. Then, $\widehat{\mu} = \int \widehat{\mu}_R dp(\underline{R})$ is a σ -invariant measure on $\tilde{\Sigma}$. By [Sar01, Theorem 1], the recurrence of ϕ implies that p is conservative

and finite on cylinders. Thus, $\widehat{\mu}$ must be conservative as well. Fix $[R] \subseteq \widetilde{\Sigma}_L$, and let $[R, R_{-n+2}, \dots, R_{-1}, R]$ be an admissible cylinder. Then,

$$\begin{aligned} &\widehat{\mu}([R, R_{-n+2}, \dots, R_{-1}, R]) \\ &= \int \widehat{\mu}_R([R, R_{-n+2}, \dots, R_{-1}, R]) dp \propto (\psi \cdot p)([R, R_{-n+2}, \dots, R_{-1}, R]). \end{aligned} \tag{10}$$

This determines $\widehat{\mu}_R$ for p -a.e. R (up to a scaling constant), and since p gives a positive volume to every cylinder [Sar99, Sar01], the continuity of the family $\{\widehat{\mu}_R\}_{R \in \widetilde{\Sigma}_L}$ determines the family uniquely. \square

Definition 5.6. (Leaf measures) Let φ be a Grassmann–Hölder continuous potential, let $\widetilde{\Sigma}_L$ be a maximal irreducible component of $\widehat{\Sigma}_L$, and let $R \in \widetilde{\Sigma}_L$. The corresponding leaf measure is $\mu_R := \widehat{\mu}_R \circ \widehat{\pi}^{-1}$, where $\widehat{\mu}_R$ is as in Definition 5.3.

THEOREM 5.7. *Let $\varphi : \text{RWT}_\chi \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Let $H_\chi(p)$ be an ergodic homoclinic class of a periodic and χ -hyperbolic point p . Then there exists a corresponding family of leaf measures which is φ -invariant, and is absolutely continuous.*

- (1) *Leaf measures: for all $R \in \widetilde{\Sigma}_L$, μ_R is carried by $\widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}] \subseteq V^u(\underline{R})$.*
- (2) *φ -invariance up to a bounded coboundary: for all $R \in \widetilde{\Sigma}_L$,*

$$\mu_R \circ f^{-1} = e^{-P_{H_\chi(p)}(\varphi)} \cdot \sum_{\widetilde{\sigma}_{R\underline{S}=R}} e^{A-A \circ f^{-1}} \cdot e^\varphi \cdot \mu_{\underline{S}},$$

where $A : \text{RWT}_\chi \rightarrow \mathbb{R}$ is bounded and given by Theorem 4.4.

- (3) *Absolute continuity: for any cylinder $[R] \subseteq \widetilde{\Sigma}_L$, and for any two chains $\underline{R}, \underline{S} \in \widetilde{\Sigma}_L \cap [R]$, let*

$$\Gamma_{\underline{R}\underline{S}} : \widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}] \rightarrow \widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = S_i, \text{ for all } i \leq 0\}]$$

be the holonomy map along the stable leaves of $\widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}]$ (it is well defined on the rectangle $\widehat{\pi}[[R]]$). Then, $(\mu_R \circ \Gamma_{\underline{R}\underline{S}}^{-1})(g) = C_\phi^{\pm 1} \cdot \mu_{\underline{S}}(g)$ for every $g \in C(V^u(\underline{S}))$, where $C_\phi > 0$ is a global constant depending only on ϕ (and a fixed corresponding harmonic function).

Here, $\mathcal{F}_{H_\chi(p)}(\varphi) := \{\mu_R\}_{R \in \widetilde{\Sigma}_L}$ is a φ -invariant family of leaf measures (see Definition 2.16).

Proof. Let $A, \phi : \text{RWT}_\chi \rightarrow \mathbb{R}$ be given by Theorem 4.4 such that $\varphi = \varphi^* + A - A \circ f^{-1}$, where A is bounded and $\phi = \varphi^* \circ \widehat{\pi} : \widehat{\Sigma} \rightarrow \mathbb{R}$ is well defined and weakly Hölder continuous by the remark after Theorem 4.4; and $\phi(\underline{R})$ depends only on the negative coordinates of \underline{R} .

Let $\widetilde{\Sigma}$ be a maximal irreducible component of $\widehat{\Sigma}$ such that $\widehat{\pi}[\widetilde{\Sigma}^\#] = H_\chi(p)$ modulo all conservative measures (see Proposition 3.16). Let $[R] \subseteq \widetilde{\Sigma}_L$, and let $\underline{R}, \underline{S} \in [R] \cap \widetilde{\Sigma}_L$.

Since φ is bounded, so is ϕ . By the spectral decomposition, we may assume without loss of generality that $(\widetilde{\Sigma}, \sigma)$ is topologically mixing (see [Sar15, Theorem 2.5]). Thus, by the variational principle, see [Sar09, Theorem 4.4] (and the fact that $\widehat{\pi}|_{\widetilde{\Sigma}^\#}$ is

finite-to-one which allows to lift probabilities while preserving the entropy), $P_G(\phi|\underline{\Sigma}) = P_{H_x(p)}(\varphi) < \infty$. (Every invariant probability measure can be decomposed into its ergodic components, and each component lifts to an invariant probability measure. Then this lift is N -to-one for some $N \in \mathbb{N}$ since the cardinality of fibers is an invariant function. Finite extensions preserve entropy, and entropy is affine.) Then we may assume without loss of generality that $P_G(\phi|\underline{\Sigma}_L) = 0$, otherwise replace ϕ by $\phi - P_G(\phi|\underline{\Sigma}_L)$.

(1) We have

$$\begin{aligned} &\mu_R(\widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}]) \\ &= \widehat{\mu}_R \circ \widehat{\pi}^{-1}(\widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}]) \\ &\geq \widehat{\mu}_R(\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}) = \widehat{\mu}_R(1) = \mu_R(1). \end{aligned}$$

Thus, $\widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}] \subseteq V^u(\underline{R})$ has full measure.

(2) By the relation $\widehat{\pi} \circ \sigma = f \circ \widehat{\pi}$ and Corollary 5.4,

$$\begin{aligned} \mu_R \circ f^{-1} &= \widehat{\mu}_R \circ \widehat{\pi}^{-1} \circ f^{-1} = (\widehat{\mu}_R \circ \sigma^{-1}) \circ \widehat{\pi}^{-1} \\ &= \left(\sum_{\widetilde{\sigma}_R \underline{Q} = \underline{R}} e^{\phi(\underline{Q})} \widehat{\mu}_{\underline{Q}} \right) \circ \widehat{\pi}^{-1} = \sum_{\widetilde{\sigma}_R \underline{Q} = \underline{R}} e^{\phi(\underline{Q})} \mu_{\underline{Q}}. \end{aligned}$$

For a continuous test function $g \in C(V^u(\underline{Q}))$,

$$\begin{aligned} e^\phi \cdot \mu_{\underline{Q}}(g) &= \int_{\substack{\underline{Q}' \in \widetilde{\Sigma}: \\ Q_i = Q'_i, \text{ for all } i \leq 0}} e^{\varphi^* \circ \widehat{\pi}} g \circ \widehat{\pi} \, d\widehat{\mu}_{\underline{Q}} \\ &= \int_{\substack{\underline{Q}' \in \widetilde{\Sigma}: \\ Q_i = Q'_i, \text{ for all } i \leq 0}} (e^{\varphi^*} g) \circ \widehat{\pi} \, d\widehat{\mu}_{\underline{Q}} = (e^{\varphi+A-A \circ f^{-1}} \cdot \mu_{\underline{Q}})(g). \end{aligned}$$

This completes the φ -invariance up to a bounded coboundary.

(3) Let $\Gamma_{\underline{R}\underline{S}} : \widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}] \rightarrow \widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = S_i, \text{ for all } i \leq 0\}]$ be the holonomy map along the stable leaves of points in $\widehat{\pi}[\{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\}]$. Let $\widehat{\Gamma}_{\underline{R}\underline{S}} : \{\underline{v} \in \widetilde{\Sigma} : v_i = R_i, \text{ for all } i \leq 0\} \rightarrow \{\underline{v} \in \widetilde{\Sigma} : v_i = S_i, \text{ for all } i \leq 0\}$ be a continuous and invertible map such that $\underline{v} \mapsto (S_i)_{i \leq 0} \cdot (v_i)_{i \geq 0}$, where \cdot denotes an admissible concatenation. Let $g \in C(V^u(\underline{S}))$, and write $\widehat{g} := g \circ \widehat{\pi} \in C(\{\underline{v} \in \widetilde{\Sigma} : v_i = S_i, \text{ for all } i \leq 0\})$. It follows that $\Gamma_{\underline{R}\underline{S}} \circ \widehat{\pi} = \widehat{\pi} \circ \widehat{\Gamma}_{\underline{R}\underline{S}}$, and so $(\mu_{\underline{R}} \circ \Gamma_{\underline{R}\underline{S}}^{-1})(g) = (\widehat{\mu}_{\underline{R}} \circ \widehat{\Gamma}_{\underline{R}\underline{S}}^{-1})(\widehat{g}) = \psi(\underline{R}) \cdot (\widehat{p}_{\underline{R}} \circ \widehat{\Gamma}_{\underline{R}\underline{S}}^{-1})(\widehat{g})$ and $\mu_{\underline{S}}(g) = \psi(\underline{S}) \cdot \widehat{p}_{\underline{S}}(\widehat{g})$. By Lemma 4.7, $\text{Var}_1(\log \psi) < \infty$. Thus, by equation (9), we are done. \square

Remark. Absolute continuity implies a local product structure. Other related important properties of hyperbolic equilibrium states of Hölder continuous potentials are true, such as being Bernoulli up to a period. This has been proven when $\dim M = 2$ in [Sar11] and for the setup of countable Markov shifts in [Dao13]. Sarig’s proof extends, as it only uses the fact that the equilibrium state can be coded as an equilibrium state on a countable Markov

shift of a weakly Hölder continuous potential. In [Dao13], Daon relaxes the assumption on the regularity of the potential to the Walters property.

6. Proofs of the main results

6.1. Local equilibrium states and the leaf condition

THEOREM 6.1. *Let M be a compact Riemannian manifold without boundary and of dimension $d \geq 2$. Let $f \in \text{Diff}^{1+\beta}(M)$, $\beta > 0$. Fix $\chi > 0$ and $\epsilon = \epsilon_\chi$ as in Lemma 2.6. Let $\varphi : M \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential. Let p be a χ -hyperbolic periodic point. Then, there exists a χ -hyperbolic local equilibrium state of φ on $H_\chi(p)$ if and only if the following two conditions hold:*

(1) we have

$$\sum_{n \geq 1} \sum_{f^n(q)=q, q \in H_\chi(p) \cap \Lambda_l} e^{\varphi_n(q) - n \cdot P_{H_\chi(p)}(\varphi)} = \infty,$$

where Λ_l is a level set such that $p \in \Lambda_{l, e^{-2 \cdot 3\epsilon}}$; and

(2) there exists a leaf measure $\mu \in \mathcal{F}_{H_\chi(p)}(\varphi)$ such that

$$\mu(\text{RWT}_\chi^{\text{PR}}) > 0,$$

where $\mathcal{F}_{H_\chi(p)}(\varphi)$ is any φ -invariant family of measures given by Theorem 5.7.

If a χ -hyperbolic local equilibrium state of φ exists on $H_\chi(p)$, then its conditional measures on unstable leaves (in the explicit sense of equation (11)) are proportional to the members of the φ -invariant family of measures; and it is unique.

Proof. The ‘only if’ direction is straightforward: let μ be a χ -hyperbolic local equilibrium state of φ on $H_\chi(p)$. Let $\tilde{\Sigma}$ be a maximal irreducible component which lifts all conservative measures on $H_\chi(p)$ (see Proposition 3.16). Lift μ to an ergodic, σ -invariant, probability measure on $\tilde{\Sigma}$, $\hat{\mu}$, such that $\hat{\mu} \circ \hat{\pi}^{-1} = \mu$, and $h_{\hat{\mu}}(\sigma) + \int \varphi \circ \hat{\pi} d\hat{\mu} = h_\mu(f) + \int \varphi d\mu$. This is possible by choosing a suitable ergodic component of $\int 1/|N_x| \sum_{v \in N_x} \delta_v d\mu(x)$, where $N_x := \hat{\pi}^{-1}[\{x\}] \cap \tilde{\Sigma}^\#$. Let $\phi : \tilde{\Sigma}_L \rightarrow \mathbb{R}$ be given by Theorem 4.4 and the remark which follows it, and let $\{\hat{\mu}_R\}_{R \in \tilde{\Sigma}_L}$ be given by Definition 5.3.

By the spectral decomposition (see [Sar15, Theorem 2.5]), we may assume without loss of generality that $(\tilde{\Sigma}_L, \tilde{\sigma}_R)$ is topologically mixing. It follows that

$$\begin{aligned} h_{\hat{\mu}}(\sigma) + \int \phi d\hat{\mu} &= \sup\{h_{\hat{\nu}}(\sigma) + \int \phi d\hat{\nu} : \hat{\nu} \text{ is an invariant probability measure on } \tilde{\Sigma}\} \\ &= \sup\{h_{\hat{\nu} \circ \tau^{-1}}(\tilde{\sigma}_R) + \int \phi d\hat{\nu} \circ \tau^{-1} : \hat{\nu} \text{ is an invariant probability measure on } \tilde{\Sigma}\}, \end{aligned}$$

where $\tau : \tilde{\Sigma} \rightarrow \tilde{\Sigma}_L$ is the projection onto the non-positive coordinates. Every invariant probability ν on $\tilde{\Sigma}_L$ can be lifted to an invariant probability measure on $\tilde{\Sigma}$, $\hat{\nu}$, such that $\hat{\nu} \circ \tau^{-1} = \nu$ and $h_{\hat{\nu}}(\tilde{\sigma}_R) = h_{\hat{\nu}}(\sigma)$. Thus,

$$\begin{aligned} h_{\hat{\mu} \circ \tau^{-1}}(\tilde{\sigma}_R) + \int \phi d\hat{\mu} \circ \tau^{-1} \\ = \sup\{h_\nu(\sigma) + \int \phi d\nu : \nu \text{ is an invariant probability measure on } \tilde{\Sigma}_L\}. \end{aligned}$$

Therefore, by [BS03], ϕ is recurrent on $\tilde{\Sigma}_L$. Thus, by Lemma 4.8, $\sum_{n \geq 1} \sum_{\substack{f^n(q)=q \\ q \in H_\chi(p) \cap \Lambda_\ell}} e^{\varphi_n(q) - n \cdot P_{H_\chi(p)}(\varphi)} = \infty$.

By [Sar09, Theorem 4.5], $\hat{\mu} \circ \tau^{-1} = \psi \cdot p$, where ψ is the ϕ -harmonic function and p is the ϕ -conformal measure, scaled such that $\psi \cdot p(1) = 1$, and ψ, p are unique up to a scaling. Then, by equation (10), $\hat{\mu} = \int_{\tilde{\Sigma}_L} \hat{\mu}_R dp(R)$. Thus, $\mu = \int \mu_R dp$, where $\{\mu_R\}_{R \in \tilde{\Sigma}_L}$ is the ϕ -invariant family of leaf measures as in Theorem 5.7. In addition, μ is carried by $\text{RWT}_\chi^{\text{PR}}$ since it is an invariant probability measure. This concludes the ‘only if’ direction.

We now turn to the ‘if’ direction: ϕ is recurrent on $\tilde{\Sigma}_L$ by Lemma 4.8. Let p be a ϕ -conformal measure carried by $\tilde{\Sigma}_L^\#$, and let $\{\mu_R\}_{R \in \tilde{\Sigma}_L}$ be the ϕ -invariant family of leaf measures given by Theorem 5.7. By [Sar01, Theorem 1], p is ergodic and conservative. Let

$$\mu := \int_{\tilde{\Sigma}_L} \mu_R dp(R). \tag{11}$$

Recall that $\{\mu_R\}_{R \in \tilde{\Sigma}_L}$ are carried by local unstable leaves, and are the projection of a family of measures given by a measurable partition of $\tilde{\Sigma}$ parameterized by $\tilde{\Sigma}_L$. It follows that μ is an ergodic, f -invariant, and conservative measure on $H_\chi(p)$, and that μ is finite on Pesin level sets. This measure is finite if and only if ϕ is positive recurrent. For full details of the proof of these properties, see [BO21, Theorem 6.2]. (In that proof, ϕ denotes the geometric potential, but the proof is identical for general weakly Hölder continuous functions.)

Note that $\mu(1) = p(\psi)$, where ψ is the unique (up to scaling) ϕ -harmonic function on $\tilde{\Sigma}_L$. By [Sar99, Theorem 8], if $h_{\text{top}}(\sigma) < \infty$, then $\mu(1) < \infty$ if and only if $\psi \cdot p$ is an equilibrium state of ϕ on $\tilde{\Sigma}_L$ (and thus μ is a local χ -hyperbolic equilibrium state of ϕ on $H_\chi(p)$).

By assumption, there exists $R \in \tilde{\Sigma}_L$ such that $\mu_R(\text{RWT}_\chi^{\text{PR}}) > 0$. By [BO21, Theorem 6.8], μ is finite. This concludes the ‘if’ direction.

The proof so far shows that the local equilibrium state can always be disintegrated into conditional measures on unstable leaves, where the conditional measures are $\{\mu_R\}_{R \in \tilde{\Sigma}_L}$.

We proceed to show the uniqueness of the χ -hyperbolic local equilibrium state, when it exists. We saw that $\mu = \hat{\mu} \circ \hat{\pi}^{-1}$ where $\hat{\mu}$ is an equilibrium state for ϕ on $\tilde{\Sigma}$. By [BS03], $\hat{\mu}$ is unique and satisfies a variational principle on $\tilde{\Sigma}$. It follows that a χ -hyperbolic local equilibrium state is unique on $H_\chi(p)$. □

Remark. If there exists $p \in \Lambda_\ell$ such that $\sum_{n \geq 1} \sum_{\substack{f^n(q)=q, q \in H_\chi(p) \cap \Lambda_\ell}} e^{\varphi_n(q) - n P_{H_\chi(p)}(\varphi)} = \infty$, then $\sum_{n \geq 1} \sum_{\substack{f^n(q)=q, q \in H_\chi(p) \cap \Lambda_\ell}} e^{\varphi_n(q) - n P_{H_\chi(p)}(\varphi)} = \infty$. Similarly, if $p \in \Lambda_{\ell e^{-2\beta\epsilon}}$, then $p \in \Lambda_\ell$. Then the condition in Theorem 6.1(2) can be relaxed so $p \in \Lambda_\ell$.

COROLLARY 6.2. *Let μ be a local equilibrium state of a Grassmann–Hölder continuous potential, then μ has a local product structure.*

Proof. We show that every Pesin level set can be covered by finitely many disjoint small sets with a local product structure, on which the measure has a product structure. The proof is given in two steps. In the first step, we write μ as the projection of an equilibrium state

of a specifically chosen maximal irreducible component of $\widehat{\Sigma}$. The second step is to use the first step to show the local product structure.

Step 1. Recall Definition 3.14, and let $\langle R \rangle$ be the equivalence class of $R \in \mathcal{R}$ with respect to \sim , whenever there exists n such that $f^{-n}[R] \cap R \neq \emptyset$. Write $\Lambda_{\langle R \rangle} := \bigcup_{S \in \langle R \rangle} \{x \in S : \text{for all } i \in \mathbb{Z}, R(f^i(x)) \in \langle R \rangle\}$, and note that $\Lambda_{\langle R \rangle}$ is f -invariant. In addition, there exist at most countably many different such sets, which exhaust all invariant probability measures which are carried by RWT_χ . Therefore, there exists $R \in \mathcal{R}$ such that $\mu(\Lambda_{\langle R \rangle}) = 1$.

Recall Definition 3.21. Let $\widetilde{\Sigma} := \langle R \rangle^{\mathbb{Z}} \cap \widehat{\Sigma}$, and note that $\mu \circ \underline{R}^{-1}$ is a shift-invariant probability measure on $\widetilde{\Sigma}$, which is carried by $\widetilde{\Sigma} \cap \widehat{\Sigma}^\circ$ (recall Definition 3.18). Moreover, one can check that $\mu \circ \underline{R}^{-1}$ is an equilibrium state for ϕ , where μ is a local equilibrium state for φ , and ϕ is given by Theorem 4.4. Therefore, ϕ is positive recurrent on $\widetilde{\Sigma}_L$, and as in Theorem 6.1, $\mu \circ \underline{R}^{-1} = \int_{\widetilde{\Sigma}_L} \widehat{\nu}_{\underline{S}} dq(\underline{S})$, where q is a ϕ -conformal measure on $\widetilde{\Sigma}_L$. Thus, in particular, q is carried by $\widetilde{\Sigma}_L^\circ := \widetilde{\Sigma}_L \cap \widehat{\Sigma}_L^\circ$ and $\widehat{\nu}_{\underline{S}}$ is carried by $\widetilde{\Sigma}^\circ := \widetilde{\Sigma} \cap \widehat{\Sigma}^\circ$ for q -a.e. $\underline{S} \in \widetilde{\Sigma}_L^\circ$.

Step 2. Let K be a Pesin level set such that $\mu(K) > 0$. We cover $K \bmod \mu$ by finitely many partition elements $\{R_i\}_{i=1, \dots, N}$ where $R_i \in \langle R \rangle$. We show that for (without loss of generality) $R := R_1$, $\mu|_R$ has a product structure.

First, note that $\mu = \int_{\widetilde{\Sigma}_L^\circ} \widehat{\nu}_{\underline{S}} \circ \widehat{\pi}^{-1} dq(\underline{S})$, and so for q -a.e. $\underline{R} \in \widetilde{\Sigma}_L^\circ$, if $\widehat{\nu}_{\underline{R}} \circ \widehat{\pi}^{-1}(\widehat{\pi}[\tau^{-1}[\{S\}] \cap \widetilde{\Sigma}^\circ]) > 0$, then there exists $\underline{S}^\pm \in \tau^{-1}[\{S\}] \cap \widetilde{\Sigma}^\circ$ and $\underline{R}^\pm \in \tau^{-1}[\{\underline{R}\}] \cap \widetilde{\Sigma}^\circ$ such that $\widehat{\pi}(\underline{R}^\pm) = \widehat{\pi}(\underline{S}^\pm)$; whence $\underline{S}^\pm = \underline{R}^\pm$, and so $\underline{S} = \underline{R}$.

Therefore, $\mu = \sum_{S \in \langle R \rangle} \int_{\{S\} \cap \widetilde{\Sigma}_L^\circ} \widehat{\nu}_{\underline{S}} \circ \widehat{\pi}^{-1} dq(\underline{S})$, where the sum is over mutually singular measures. Then $\mu|_R = \int_{\{R\} \cap \widetilde{\Sigma}_L^\circ} \widehat{\nu}_{\underline{S}} \circ \widehat{\pi}^{-1} dq(\underline{S})$ is given by its disintegration into mutually singular measures on local unstable leaves. By Theorem 5.7, $\mu|_R$ has a product structure. □

6.2. Ledrappier–Young property for hyperbolic equilibrium states. In the celebrated results of [LY85a, LY85b], Ledrappier and Young prove a general formula for the entropy of invariant probability measures of diffeomorphisms in terms of the local dimensions of their conditional measures. One very important application of their theory is the characterization of SRB measures as those which satisfy Pesin’s entropy formula, extending the previous same result of Ledrappier for hyperbolic SRB measures [Led84]. That is, an ergodic and hyperbolic invariant probability measure, μ , which satisfies the following two properties must have conditional measures on unstable local manifolds which are absolutely continuous with respect to the induced Riemannian volume measure. Let $\varphi(x) := -\log \text{Jac}(d_x f|_{H^u(x)}) : \text{RWT}_\chi \rightarrow \mathbb{R}$ (the geometric potential), then:

(1) we have

$$h_\mu(f) + \int \varphi d\mu = \sup \left\{ h_\nu(f) + \int \varphi d\nu : \nu \text{ is an } \begin{matrix} \text{erg.} \\ \text{hyp.} \\ \text{f-inv.} \end{matrix} \text{ prob.} \right\},$$

where the geometric potential in the integrand is well defined almost everywhere for every hyperbolic invariant probability measure;

(2) $\sup\{h_\nu(f) + \int \varphi d\nu : \nu \text{ is an erg. hyp. f-inv. prob.}\} = 0$.

Note that $P_G(\phi|_{\tilde{\Sigma}_L}) \leq 0$ is always true, because of the Ruelle–Margulis inequality [Rue78]. The fact that SRB measures satisfy items (1), (2) is due to Ledrappier and Strelcyn [LS82]. A different way to write this characterization, is the following. Let μ be an ergodic hyperbolic invariant probability measure. Then μ is an SRB measure if and only if:

- (1) ϕ is positive recurrent when lifted to an irreducible component $\tilde{\Sigma}_L \subseteq \widehat{\Sigma}_L$ (see §4.2), where ϕ is given by Theorem 4.4 applied to $\varphi(x)$;
- (2) $P_G(\phi|_{\tilde{\Sigma}_L}) = 0$.

It follows that conditions (1) and (2) above are satisfied $\Leftrightarrow \mu_{V^u} \ll m_{V^u}$, where μ_{V^u} is a conditional measure of μ with respect to a measurable partition of local unstable leaves $\{V^u\}$, and m_{V^u} is the induced Riemannian volume of V^u (the conformal measure of the geometric potential φ on V^u).

In this section, we extend this result by replacing the geometric potential φ by a general potential which is Grassmann–Hölder continuous, and by replacing the induced Riemannian volume m_{V^u} by any φ -conformal family. We obtain a new proof different to that of Ledrappier in the hyperbolic case [Led84], and to that of Ledrappier and Young in the general case [LY85a].

THEOREM 6.3. *Let M be a compact Riemannian manifold without boundary, and of dimension $d \geq 2$. Let $f \in \text{Diff}^{1+\beta}(M)$, $\beta > 0$, let $\chi > 0$ and $\epsilon = \epsilon_\chi > 0$ as in Lemma 2.6. Let $\varphi : WT_\chi^\epsilon \rightarrow \mathbb{R}$ be a Grassmann–Hölder continuous potential (recall Definition 2.10). Let q be a χ -hyperbolic periodic point. Assume that $H_\chi(q)$ admits a (unique) χ -hyperbolic equilibrium state of φ , ν . For any family of conditional measures as in Theorem 6.1, $\mathcal{F}_{H_\chi(q)}(\varphi)$, there is a sub-family \mathcal{F}' which disintegrates ν as in Theorem 6.1, and for which the following holds. Fix $\mu \in \mathcal{F}'$ which is carried by a local unstable leaf V^u . Assume that $H_\chi(q)$ admits a φ -conformal system of measures $\mathcal{C} := \{m_{W^u}^\varphi : W^u \text{ is a local unstable leaf of } H_\chi(q)\}$. Then, $\mu \ll m_{V^u}^\varphi$. In particular, $m_{V^u}^\varphi$ gives a positive measure to $\text{RWT}_\chi^{\text{PR}}$.*

Proof. Let $\phi : \text{RWT}_\chi \rightarrow \mathbb{R}$ be given by Theorem 4.4. As in the proof of Theorem 5.7, let $\tilde{\Sigma}$ be a maximal irreducible component of $\widehat{\Sigma}$ such that $\widehat{\pi}[\tilde{\Sigma}^\#] = H_\chi(q)$ modulo conservative measures; and write $\mathcal{F}_{H_\chi(q)}(\varphi) = \{\mu_R\}_{R \in \tilde{\Sigma}_L}$.

Denote the lift of ν to $\tilde{\Sigma}$ by $\widehat{\nu}$. As in the proof of Theorem 6.1, ϕ is positive recurrent on $\tilde{\Sigma}$ and $\widehat{\nu} = \int_{\tilde{\Sigma}_L} \widehat{\mu}_R dp(R)$, where p is the unique (up to scaling) ϕ -conformal measure on $\tilde{\Sigma}_L$. In [Sar99], Sarig showed that when ϕ is positive recurrent, p is conservative, and is carried by $\tilde{\Sigma}_L^\#$. Set $\mathcal{F}' := \{\mu_R\}_{R \in \tilde{\Sigma}_L^\#}$. Write $\mu = \mu_R$, where $R \in \tilde{\Sigma}_L^\#$. (Note that the absolute continuity of holonomies and the existence of a measure $\mu' \in \mathcal{F}'$ such that $\mu'(\text{RWT}_\chi^{\text{PR}}) > 0$ imply that also $\mu(\text{RWT}_\chi^{\text{PR}}) > 0$.) Let $S \in \mathcal{R}$ be a symbol which repeats infinitely often in R .

Let $K \in \mathbb{N}$ be $N(R_0) \cdot N(S)$ (see Definition 3.6). This is a bound on the number of sequences in $\widehat{\Sigma}^\#$ which code the same point and such that this point can be coded by a sequence which has R_0 repeat infinitely often in the future and S repeat infinitely often in the past (see [BO18, Theorem 1.3]). Choose $n \geq 1$ and $\underline{S}, \underline{Q} \in \tilde{\Sigma}_L$ such that $\sigma_R^n \underline{S} = \sigma_R^n \underline{Q} = \underline{R}$ and $S_0 = Q_0 = R_0$. Let $\underline{S}^\pm, \underline{Q}^\pm \in \tilde{\Sigma}$ which return to $[R_0]$ infinitely

often in the future and such that $S_i^\pm = S_i$, for all $i \leq 0$ and $Q_i^\pm = Q_i$, for all $i \leq 0$. Define $x := \widehat{\pi}(\underline{S}^\pm)$, $y := \widehat{\pi}(\underline{Q}^\pm)$, $z := \widehat{\pi}([\underline{S}^\pm, \underline{Q}^\pm]_{R_0})$, where $[\underline{S}^\pm, \underline{Q}^\pm]_R$ is the Smale bracket of \underline{S}^\pm and \underline{Q}^\pm in $[R_0]$. If $V^u(\underline{S}) \cap V^u(\underline{Q}) \neq \emptyset$, then by [BO18, Proposition 3.12], $x = z = y$.

Then for each $\underline{S} \in \widetilde{\sigma}_R^{-n}[\{\underline{R}\}] \cap [R_0]$, the number of $\underline{Q} \in \widetilde{\sigma}_R^{-n}[\{\underline{R}\}] \cap [R_0]$ such that $V^u(\underline{S}) \cap V^u(\underline{Q}) \neq \emptyset$ is bounded by K . That is,

$$\#\{\underline{S}' \in \widetilde{\Sigma}_L : S'_0 = R_0, \sigma_R^n \underline{S}' = \underline{R}, V^u(\underline{S}') \cap V^u(\underline{S}) \neq \emptyset\} \leq K.$$

Define on $V^u(\underline{R})$ the density function $\rho_n := \sum_{\widetilde{\sigma}_R^n \underline{S}' = \underline{R}, S'_0 = R} \mathbb{1}_{V^u(\underline{S}')} \circ f^n$, then for any $n \geq 1$,

$$0 \leq \rho_n \leq K. \tag{12}$$

Let $c_{R_0} := \min\{\sup\{m_{V^u(\underline{R}')}^\varphi(1) : \underline{R}' \in [R_0]\}^{-1}, \inf\{m_{V^u(\underline{R}')}^\varphi(1) : \underline{R}' \in [R_0]\}\}$. This is positive by the continuity of \mathcal{C} , since $[R_0]$ is compact and $\underline{R}' \mapsto V^u(\underline{R}')$ is continuous in C^1 -norm (see Definition 2.15).

Let $g \in C(V^u(\underline{R}))$ such that g is L -Lipschitz, and $\|g\|_\infty \leq 1$. Write $\widehat{g} := g \circ \widehat{\pi}$, and so $\mu_{\underline{R}}(g) = \widehat{\mu}_{\underline{R}}(\widehat{g})$. Let

$$\widehat{\omega}_n := \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_{[R_0]} \circ \sigma^k. \tag{13}$$

Note that $|\widehat{\omega}_n| \leq 1$. By Birkhoff’s ergodic theorem, and the absolute continuity of $\{\widehat{\mu}_{\underline{R}}\}_{\underline{R} \in \widetilde{\Sigma}}$ with respect to holonomies (see Corollary 5.2), for $\widehat{\mu}_{\underline{R}}$ -a.e. \underline{R}^\pm , $\widehat{\omega}_n(\underline{R}^\pm) \xrightarrow[n \rightarrow \infty]{} \widehat{\nu}([R_0])$, and so $\widehat{g}(\underline{R}^\pm) \cdot \widehat{\omega}_n(\underline{R}^\pm) \xrightarrow[n \rightarrow \infty]{} \widehat{\nu}([R_0]) \cdot \widehat{g}(\underline{R}^\pm)$. Then, by Lebesgue’s dominated convergence theorem,

$$\widehat{\mu}_{\underline{R}}(\widehat{g} \cdot \widehat{\omega}_n) \xrightarrow[n \rightarrow \infty]{} \widehat{\nu}([R_0]) \cdot \widehat{\mu}_{\underline{R}}(\widehat{g}) = \widehat{\nu}([R_0]) \cdot \mu_{\underline{R}}(g). \tag{14}$$

Since $\widehat{\pi}$ is Hölder continuous, and g is L -Lipschitz, \widehat{g} is Hölder continuous as well. Let $H_L > 0$, $\theta \in (0, 1)$ such that $d(\underline{R}^\pm, \underline{S}^\pm) \leq e^{-n} \Rightarrow |\widehat{g}(\underline{R}^\pm) - \widehat{g}(\underline{S}^\pm)| \leq H_L \cdot \theta^n$.

For every $n \geq 1$ and $\underline{S} \in \widetilde{\Sigma}_L$ such that $\sigma_R^n \underline{S} = \underline{R}$, fix $\underline{S}^\pm \in \widetilde{\Sigma}$ such that $S_i^\pm = S_i$ for all $i \leq 0$. By [BO18, Proposition 4.4],

$$\text{diam}_{V^u(\underline{R})}(f^{-n}[V^u(\underline{S})]) \leq 4e^{-\chi \cdot n/2}, \tag{15}$$

where $\text{diam}_{V^u(\underline{R})}$ denotes the diameter with respect to the induced Riemannian metric on $V^u(\underline{R})$.

Let $\underline{W} = (R_0, W_1, \dots, W_{n-2}, R_0)$ be an admissible word of length $n \geq 1$. We estimate $\widehat{\mu}_{\underline{R}}([\underline{W}])$. First, we write $[\underline{W}] = \sigma^{-n} \sigma^n [\underline{W}]$. Next, by Corollary 5.5,

$$\begin{aligned} \widehat{\mu}_{\underline{R}}([\underline{W}]) &= \widehat{\mu}_{\underline{R}}(\sigma^{-n} \sigma^n [\underline{W}]) = e^{-nP_G(\phi)} \sum_{\widetilde{\sigma}_R^n \underline{S} = \underline{R}} e^{\phi_n(\underline{S})} \widehat{\mu}_{\underline{S}}(\sigma^n [\underline{W}]) \\ &= e^{-nP_G(\phi) + \phi_n(\underline{R}, \underline{W})} \widehat{\mu}_{\underline{R}, \underline{W}}(1), \end{aligned}$$

where \cdot denotes an admissible concatenation. Recall that $\widehat{\mu}_{R'}(1) = \psi(R')$, where ψ is the unique (up to scaling) ϕ -harmonic function on $\widetilde{\Sigma}_L$. Then for $C_{R_0} := \max_{[R_0]} \{\psi, \psi^{-1}\}$,

$$\widehat{\mu}_{\underline{R}}([W]) = C_{R_0}^{\pm 1} \cdot e^{-nP_G(\phi) + \phi_n(\underline{R} \cdot W)}. \tag{16}$$

Since \mathcal{C} is a φ -conformal family, we get that for all $n \geq 1$ and for every $\underline{S} \in \widetilde{\sigma}_R^{-n}[\{R\}]$,

$$m_{V^u(\underline{R})}^\varphi(f^{-n}[V^u(\underline{S})]) = \int e^{\varphi_n(x) - nP_{H_X(q)}(\varphi)} dm_{V^u(\underline{S})}^\varphi.$$

We wish to estimate $\varphi_n(\cdot)$ on $V^u(\underline{S})$. For that, we need the non-trivial fact that $d_{\text{TM}}(f^{-i}(x), f^{-i}(y))$ decreases exponentially fast in $i \geq 0$. This is true since $V^u(\underline{S})$ is contained in the graph of a function with a Hölder continuous derivative (and $d(f^{-i}(x), f^{-i}(y)) \leq 4e^{-\chi/2i}$), and the Hölder constant and exponent do not depend on the choice of $\underline{S} \in [R_0]$. For more details, see [BO18, Definition 3.1]. Thus, since φ is Grassmann–Hölder continuous, we get that there exists $C_\varphi > 0$ such that for all $x, y \in V^u(\underline{S})$, $\sup_{n' \geq 0} |\varphi_{n'}(x) - \varphi_{n'}(y)| \leq C_\varphi$. Using the fact that $V^u(\underline{S})$ contains a point with a coding in $\widetilde{\Sigma}^\#$, together with Theorem 4.4, where $\varphi = \phi + A - A \circ f^{-1}$ with $\|A\|_\infty < \infty$, we get for all $n \geq 1$,

$$\begin{aligned} m_{V^u(\underline{R})}^\varphi(f^{-n}[V^u(\underline{S})]) &= e^{\pm(C_\varphi + 2\|A\|)} e^{\phi_n(\underline{S}) - nP_{H_X(q)}(\varphi)} \cdot m_{V^u(\underline{S})}(\underline{1}) \\ &= (c_{R_0}^{-1} e^{C_\varphi + 2\|A\|})^{\pm 1} e^{\phi_n(\underline{S}) - nP_G(\phi)}. \end{aligned} \tag{17}$$

In the last equality, we used the fact that $P_G(\phi) = P_{H_X(q)}(\varphi)$ (recall equation (2)).

The last identity we need before the main computation of the proof is the following: for any $n \geq 1$,

$$(\mathbb{1}_{[R_0]} \circ \sigma^n) \cdot \widehat{\mu}_{\underline{R}} = \sum_{\substack{|W|=n \\ W_0=W_{n-1}=R_0}} \widehat{\mu}_{\underline{R}}|_{[W]}. \tag{18}$$

On the left-hand side, $\mathbb{1}_{[R_0]} \circ \sigma^n$ acts as a density for $\widehat{\mu}_{\underline{R}}$, and on the right-hand side, $\widehat{\mu}_{\underline{R}}|_{[W]}$ is the restriction of the measure to the respective cylinder.

We are now able to use the estimates and identities from equations (12), (15), (16), (17), and (18) to get for all $n \geq 1$,

$$\begin{aligned} &K \cdot m_{V^u(\underline{R})}^\varphi(g) \geq (\rho_n \cdot m_{V^u(\underline{R})}^\varphi)(g) \quad (\because \text{equation (12)}) \\ (\because g \text{ is Lip, equation (15)}) &= \pm 4LK \cdot e^{-\chi \cdot n/2} m_{V^u(\underline{R})}^\varphi(1) \\ &\quad + \sum_{\substack{\widetilde{\sigma}_R^n \underline{S} = \underline{R}, S_0 = R_0}} m_{V^u(\underline{R})}^\varphi(f^{-n}[V^u(\underline{S})]) \cdot g(f^{-n} \circ \widehat{\pi}(\underline{S}^\pm)) \\ (\because \text{equation (17)}) &= \pm 4LK \cdot e^{-\chi \cdot n/2} c_{R_0}^{-1} \\ &\quad + (c_{R_0}^{-1} e^{C_\varphi + 2\|A\|})^{\pm 1} \sum_{\substack{\widetilde{\sigma}_R^n \underline{S} = \underline{R}, S_0 = R_0}} e^{\phi_n(\underline{S}) - nP_G(\phi)} \cdot \widehat{g}(\sigma^{-n}(\underline{S}^\pm)) \\ (\because \text{equation (16)}) &= \pm 4KL \cdot c_{R_0}^{-1} e^{-\chi \cdot n/2} \\ &\quad + (c_{R_0}^{-1} e^{C_\varphi + 2\|A\|} C_{R_0})^{\pm 1} \sum_{|W|=n, W_{n-1}=R_0} \widehat{\mu}_{\underline{R}}([W]) \cdot \widehat{g}(\sigma^{-n}(\underline{S}^\pm)) \end{aligned}$$

$$(\cdot : \widehat{g} \text{ is Lip, equation (18)}) = \pm 4LKc_{R_0}^{-1} \cdot e^{-\chi \cdot n/2} \pm e^{C_\varphi + 2\|A\|} c_{R_0}^{-1} C_{R_0} \cdot H_L \theta^n + (c_{R_0}^{-1} e^{C_\varphi + 2\|A\|} C_{R_0})^{\pm 1} \cdot \widehat{\mu}_R(\mathbb{1}_{[R_0]} \circ \sigma^n \cdot \widehat{g}).$$

Write $\theta_1 := \max\{\theta, e^{-\chi/2}\} \in (0, 1)$, $\widetilde{C}_{R_0} := 4e^{C_\varphi + 2\|A\|} c_{R_0}^{-1} C_{R_0} \cdot \sum_{k \geq 0} \theta_1^k < \infty$. Thus, for all $n \geq 1$,

$$K \cdot m_{V^u(\underline{R})}^\varphi(g) \geq 2LK \cdot H_L \cdot \widetilde{C}_{R_0} + \widetilde{C}_{R_0}^{\pm 1} \cdot \widehat{\mu}_R(\mathbb{1}_{[R_0]} \circ \sigma^n \cdot \widehat{g}). \tag{19}$$

By summing and averaging equation (19) N times, and by equation (14) (recall the definition of $\widehat{\omega}_N$ in equation (13)),

$$K \cdot m_{V^u(\underline{R})}^\varphi(g) \geq \frac{1}{N} \sum_{n=1}^N (\rho_n \cdot m_{V^u(\underline{R})}^\varphi)(g) = \pm \frac{1}{N} 2LK \cdot H_L \cdot \widetilde{C}_{R_0} + \widetilde{C}_{R_0}^{\pm 1} \cdot \widehat{\mu}_R(\widehat{\omega}_N \widehat{g}) \xrightarrow{N \rightarrow \infty} \widehat{\nu}([R_0]) \widetilde{C}_{R_0}^{\pm 1} \mu_R(g).$$

Since $\text{Lip}(V^u(\underline{R}))$ is dense in $\|\cdot\|_\infty$ -norm in $C(V^u(\underline{R}))$, by the Riesz–Kakutani–Markov representation theorem, $\mu_R \leq \widehat{\nu}([R_0])^{-1} \cdot K \cdot \widetilde{C}_{R_0} \cdot m_{V^u(\underline{R})}^\varphi$. \square

Remarks. We make the following remarks.

- (1) In fact, the proof shows that not only $\mu_R \ll m_{V^u(\underline{R})}^\varphi$, but that $\mu_R = (\widehat{\nu}([R_0])^{-1} \cdot K \cdot \widetilde{C}_{R_0})^{\pm 1} \cdot m_{V^u(\underline{R})}^\varphi|_{E_R^{R_0}}$, where $E_R^{R_0} := \widehat{\pi}[\{\underline{R}^\pm \in \widetilde{\Sigma} : \text{for all } i \leq 0, R_i^\pm = R_i \text{ and } \#\{j \geq 0 : R_j^\pm = R_0\} = \infty\}]$ carries μ_R .
- (2) In addition, the proof shows that the φ -conformal family of measures is unique up to equivalence of the leaf measures, when restricted to sets like $E_R^{R_0}$; and it works for a φ -conformal family where the transformation law is not exactly $e^{\varphi_n - n \cdot P_{H_\chi(\rho)}(\varphi)}$, but merely up to a multiplicative constant uniform in n .

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A. Appendix. Special notation

WT	weakly temperable points (Definition 2.4)
RWT	recurrently weakly temperable points (Definition 2.4, Definition 2.7)
Σ	infinite-to-one Markov extension (Theorem 3.2)
$\widehat{\Sigma}$	finite-to-one Markov extension (Definition 3.3)
\mathcal{R}	Markov partition (Definition 3.3)
$\widetilde{\Sigma}$	maximal irreducible component of $\widehat{\Sigma}$ (Definition 3.14)
$\widehat{\Sigma}^\circ$	itineraries of orbits in the Markov partition (Definition 3.18)
$X_L, X \in \{\Sigma, \widehat{\Sigma}, \widetilde{\Sigma}, \Sigma^\circ\}$	$\{(x_i)_{i \leq 0} : (x_i)_{i \in \mathbb{Z}} \in X\}$

$X^\#, X \in \{\Sigma, \widehat{\Sigma}, \widetilde{\Sigma}\}$	$\{x \in X : \text{there exists } a, b : x_i = a, x_{-j} = b \text{ for infinitely many positive } i \text{ and positive } j\}$
$X_L^\#, X \in \{\Sigma, \widehat{\Sigma}, \widetilde{\Sigma}\}$	$\{x \in X_L : \text{there exists } a : x_{-i} = a \text{ for infinitely many positive } i\}$
σ_R	the right-shift (Definition 3.17)
$\widetilde{\sigma}_R$	the restriction of the right-shift to $\widetilde{\Sigma}_L$ (Definition 4.1)

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