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WEIGHTED INTEGRAL INEQUALITIES OF OSTROWSKI, ČEBYŠEV AND LUPAŞ TYPE WITH APPLICATIONS

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Abstract

We establish some weighted integral inequalities of Ostrowski, Čebyšev and Lupaş type and give applications for continuous probability density functions supported on infinite intervals.

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1. Introduction

For two *Lebesgue integrable* functions $f,\ g : [a,b] \to \mathbb{R}$, we define the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{(b-a)^2}\int_a^b f(t)\,dt \int_a^b g(t)\,dt. \tag{1.1}$$

In 1935, Grüss [9] showed that

$$|C(f,g)| \le \tfrac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exist real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for almost all } t \in [a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.2) in the sense that it cannot be replaced by a smaller quantity. A less known result, even though it was obtained by Čebyšev [1] in 1882, states that

$$|C(f,g)| \le \tfrac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.4}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case. The Čebyšev inequality (1.4) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be *absolutely continuous*, $f',\ g' \in L_\infty[a,b]$ and $\|f'\|_\infty = \text{ess sup}_{t\in[a,b]}|f'(t)|$.

A mixture between the results of Grüss (1.2) and Čebyšev (1.4) is the following inequality obtained by Ostrowski [13] in 1970:

$$|C(f,g)| \le \tfrac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

provided that f is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5). The case of *Euclidean norms* of the derivative was considered by Lupaş [11], where he proved that

$$|C(f,g)| \le \frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a), \tag{1.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $1/\pi^2$ is best possible. For other inequalities of Grüss' type, see, for example, [2–8, 10, 12].

In order to extend the above results to infinite intervals, we establish some weighted integral inequalities of Ostrowski, Čebyšev and Lupaş type. We give applications for continuous probability density functions supported on infinite intervals along with two examples.

2. Weighted inequalities

We define, as above,

$$C_{h'}(f,g) := \frac{1}{h(b)-h(a)}\int_a^b f(t)g(t)h'(t)\,dt - \frac{1}{[h(b)-h(a)]^2}\int_a^b f(t)h'(t)\,dt\int_a^b g(t)h'(t)\,dt,$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a,b]$ and such that the integrals exist. The following weighted version of Ostrowski's inequality (1.5) holds.

THEOREM 2.1. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f is Lebesgue integrable and satisfies the condition $m \le f(t) \le M$ for $t \in [a,b]$, $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and g'/h' is essentially bounded, that is, $g'/h' \in L_\infty[a,b]$, then*

$$|C_{h'}(f,g)| \le \frac{1}{8}[h(b)-h(a)](M-m)\left\|\frac{g'}{h'}\right\|_{[a,b],\infty}. \tag{2.1}$$

The constant $\frac{1}{8}$ is best possible.

PROOF. Assume that $[c,d] \subset [a,b]$. If $g : [c,d] \to \mathbb{C}$ is absolutely continuous on $[c,d]$, then $g \circ h^{-1} : [h(c), h(d)] \to \mathbb{C}$ is absolutely continuous on $[h(c), h(d)]$ and using the chain rule and the derivative of inverse functions yields

$$(g \circ h^{-1})'(z) = (g' \circ h^{-1})(z)(h^{-1})'(z) = \frac{(g' \circ h^{-1})(z)}{(h' \circ h^{-1})(z)} \tag{2.2}$$

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for almost every $z \in [h(c), h(d)]$. If $x \in [c, d]$, then taking $z = h(x)$ gives

$$(g \circ h^{-1})'(z) = \frac{(g' \circ h^{-1})(h(x))}{(h' \circ h^{-1})(h(x))} = \frac{g'(x)}{h'(x)}.$$

Since $g'/h' \in L_\infty[c, d]$, it follows that $(g \circ h^{-1})' \in L_\infty[h(c), h(d)]$ and

$$\|(g \circ h^{-1})'\|_{[h(c),h(d)],\infty} = \left\|\frac{g'}{h'}\right\|_{[c,d],\infty}.$$

Now, if we use Ostrowski's inequality (1.5) for the functions $f \circ h^{-1}$ and $g \circ h^{-1}$ on the interval $[h(a), h(b)]$,

$$\begin{aligned}\Big|\frac{1}{h(b) - h(a)} &\int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u)\, du \\ &- \frac{1}{[h(b) - h(a)]^2} \int_{h(a)}^{h(b)} f \circ h^{-1}(u)\, du \int_{h(a)}^{h(b)} g \circ h^{-1}(u)\, du\Big| \\ &\le \frac{1}{8}[h(b) - h(a)](M - m)\|(g \circ h^{-1})'\|_{[h(a),h(b)],\infty}, \end{aligned} \tag{2.3}$$

since $m \le f \circ h^{-1}(u) \le M$ for all $u \in [h(a), h(b)]$. The change of variable $t = h^{-1}(u)$, $u \in [g(a), g(b)]$, gives $u = h(t)$ and $du = h'(t)\, dt$ and so

$$\int_{h(a)}^{h(b)} (f \circ h^{-1})(u)\, du = \int_a^b f(t)h'(t)\, dt,$$

$$\int_{h(a)}^{h(b)} g \circ h^{-1}(u)\, du = \int_a^b g(t)h'(t)\, dt,$$

$$\int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u)\, du = \int_a^b f(t)g(t)h'(t)\, dt$$

and

$$\|(g \circ h^{-1})'\|_{[h(a),h(b)],\infty} = \left\|\frac{g'}{h'}\right\|_{[a,b],\infty}.$$

By making use of (2.3) we then get the desired result (2.1).

The best constant follows by Ostrowski's inequality (1.5). □

If $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \to [0, \infty)$, defined by $W(x) := \int_a^x w(s)\, ds$, is strictly increasing and differentiable on (a, b) and $W'(x) = w(x)$ for any $x \in (a, b)$.

Corollary 2.2. *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$, f is Lebesgue integrable and satisfies the condition $m \le f(t) \le M$ for $t \in [a, b]$, $g : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ and g'/w is essentially bounded, that is, $g'/w \in L_\infty[a, b]$. Then*

$$|C_w(f, g)| \le \frac{1}{8}(M - m)\left\|\frac{g'}{w}\right\|_{[a,b],\infty} \int_a^b w(s)\, ds. \tag{2.4}$$

The constant $\frac{1}{8}$ is best possible.

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REMARK 2.3. Under the assumptions of Corollary 2.2 and if there exists a constant $K > 0$ such that $|g'(t)| \le Kw(t)$ for almost all $t \in [a, b]$, then, by (2.4),

$$|C_w(f, g)| \le \frac{1}{8}(M - m)K \int_a^b w(s)\, ds.$$

We also have the following weighted version of Čebyšev's inequality.

THEOREM 2.4. *Let $h : [a, b] \to [h(a), h(b)]$ be a continuous strictly increasing function differentiable on (a, b). If $f,\ g : [a, b] \to \mathbb{R}$ are absolutely continuous on $[a, b]$ and $f'/h', g'/h' \in L_\infty[a, b]$, then*

$$|C_{h'}(f, g)| \le \frac{1}{12}[h(b) - h(a)]^2 \left\| \frac{f'}{h'} \right\|_{[a,b],\infty} \left\| \frac{g'}{h'} \right\|_{[a,b],\infty}. \tag{2.5}$$

The constant $\frac{1}{12}$ is best possible.

The proof follows by the use of Čebyšev's inequality (1.4) for the functions $f \circ h^{-1}$ and $g \circ h^{-1}$ on the interval $[h(a), h(b)]$.

COROLLARY 2.5. *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$. Suppose that $f,\ g : [a, b] \to \mathbb{R}$ are absolutely continuous on $[a, b]$ and $f'/w, g'/w \in L_\infty[a, b]$. Then*

$$|C_w(f, g)| \le \frac{1}{12} \left\| \frac{f'}{w} \right\|_{[a,b],\infty} \left\| \frac{g'}{w} \right\|_{[a,b],\infty} \left(\int_a^b w(s)\, ds \right)^2. \tag{2.6}$$

The constant $\frac{1}{12}$ is best possible.

REMARK 2.6. Under the assumptions of Corollary 2.5 and if there are constants K, $L > 0$ such that $|f'(t)| \le Lw(t)$, $|g'(t)| \le Kw(t)$ for almost all $t \in [a, b]$, then, by (2.6),

$$|C_w(f, g)| \le \frac{1}{12} LK \left(\int_a^b w(s)\, ds \right)^2.$$

Finally, we have the following weighted version of Lupaş' inequality.

THEOREM 2.7. *Let $h : [a, b] \to [h(a), h(b)]$ be a continuous strictly increasing function differentiable on (a, b). If $f,\ g : [a, b] \to \mathbb{R}$ are absolutely continuous on $[a, b]$ and $f'/(h')^{1/2}, g'/(h')^{1/2} \in L_2[a, b]$, then*

$$|C_{h'}(f, g)| \le \frac{1}{\pi^2} \left\| \frac{f'}{(h')^{1/2}} \right\|_{[a,b],2} \left\| \frac{g'}{(h')^{1/2}} \right\|_{[a,b],2} [h(b) - h(a)]. \tag{2.7}$$

The constant $1/\pi^2$ is best possible.

PROOF. From the identity (2.2),

$$\int_{h(a)}^{h(b)} |(g \circ h^{-1})'(u)|^2\, du = \int_{h(a)}^{h(b)} \left| \frac{(g' \circ h^{-1})(u)}{(h' \circ h^{-1})(u)} \right|^2 du.$$

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Make the change of variable $t = h^{-1}(u)$, $u \in [h(a), h(b)]$. Then $u = h(t)$ and $du = h'(t)dt$. Therefore,

$$\int_{h(a)}^{h(b)} \left| \frac{(g' \circ h^{-1})(u)}{(h' \circ h^{-1})(u)} \right|^2 du = \int_b^b \left| \frac{g'(t)}{h'(t)} \right|^2 h'(t)\, dt = \left\| \frac{g'}{(h')^{1/2}} \right\|_{[a,b],2}^2.$$

In a similar way,

$$\int_{h(a)}^{h(b)} \left| \frac{(f' \circ h^{-1})(u)}{(h' \circ h^{-1})(u)} \right|^2 du = \left\| \frac{f'}{(h')^{1/2}} \right\|_{[a,b],2}^2.$$

By making use of Lupaş' inequality (1.6) for the functions $f \circ h^{-1}$ and $g \circ h^{-1}$ on the interval $[h(a), h(b)]$,

$$\begin{aligned} &\left| \frac{1}{h(b) - h(a)} \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u)\, du \right. \\ &\qquad \left. - \frac{1}{[h(b) - h(a)]^2} \int_{h(a)}^{h(b)} f \circ h^{-1}(u)\, du \int_{h(a)}^{h(b)} g \circ h^{-1}(u)\, du \right| \\ &\le \frac{1}{\pi^2} \|(f \circ h^{-1})'\|_{[h(a),h(b)],2} \|(g \circ h^{-1})'\|_{[h(a),h(b)],2} [h(b) - h(a)], \end{aligned}$$

which together with the above calculations produces the desired result (2.7). □

Corollary 2.8. *Assume that* $w : [a, b] \to (0, \infty)$ *is continuous on* $[a, b]$. *Suppose that* $f, g : [a, b] \to \mathbb{R}$ *are absolutely continuous on* $[a, b]$ *and* $f'/w^{1/2}, g'/w^{1/2} \in L_2[a, b]$. *Then*

$$|C_w(f, g)| \le \frac{1}{\pi^2} \left\| \frac{f'}{w^{1/2}} \right\|_{[a,b],2} \left\| \frac{g'}{w^{1/2}} \right\|_{[a,b],2} \int_a^b w(s)\, ds.$$

The constant $1/\pi^2$ *is best possible.*

3. Applications for probability density functions

The above results can be extended for infinite intervals I by assuming that the function $f : I \to \mathbb{C}$ is locally absolutely continuous on I.

For instance, suppose that $I = [a, \infty)$, $w(s) > 0$ for $s \in [a, \infty)$ with $\int_a^\infty w(s)\, ds = 1$, that is, w is a probability density function on $[a, \infty)$, f is *Lebesgue measurable* and satisfies the condition $m \le f(t) \le M$ for $t \in [a, \infty)$ and $g : [a, \infty) \to \mathbb{R}$ is locally absolutely continuous on $[a, \infty)$ with $g'/w \in L_\infty[a, \infty)$. Define the functional

$$C_w(f, g) := \int_a^\infty w(t) f(t) g(t)\, dt - \int_a^\infty w(t) f(t)\, dt \int_a^\infty w(t) g(t)\, dt.$$

Then, from (2.1),

$$|C_w(f, g)| \le \frac{1}{8}(M - m) \left\| \frac{g'}{w} \right\|_{[a,\infty),\infty}. \tag{3.1}$$

Moreover, if $f'/w \in L_\infty[a, \infty)$, then, by (2.5),

$$|C_w(f, g)| \le \frac{1}{12} \left\| \frac{f'}{w} \right\|_{[a,\infty),\infty} \left\| \frac{g'}{w} \right\|_{[a,\infty),\infty}. \tag{3.2}$$

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If $f'/w^{1/2}, g'/w^{1/2} \in L_2[a,\infty)$, then, by (2.7),

$$|C_w(f,g)| \le \frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,\infty),2}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,\infty),2}. \tag{3.3}$$

In probability theory and statistics, the *beta prime distribution* (also known as the *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x > 0$ with two parameters α and β, having the probability density function

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)},$$

where B is the *beta function*

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad \text{for } \alpha,\beta > 0.$$

The cumulative distribution function is

$$W_{\alpha,\beta}(x) = I_{x/(1+x)}(\alpha,\beta),$$

where I is the *regularised incomplete beta function* defined by

$$I_z(\alpha,\beta) := \frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.$$

Here $B(\cdot;\alpha,\beta)$ is the *incomplete beta function* defined by

$$B(z;\alpha,\beta) := \int_0^z t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad \text{for } \alpha,\beta,z > 0.$$

Consider the functional

$$\begin{aligned}C_{B,\alpha,\beta}(f,g) &:= B(\alpha,\beta)\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta}f(t)g(t)\,dt \\ &\quad - \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta}f(t)\,dt\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta}g(t)\,dt,\end{aligned}$$

where $\alpha,\beta > 0$. By (3.1)–(3.3), for $\ell(t) = t$,

$$|C_{B,\alpha,\beta}(f,g)| \le \tfrac{1}{8}(M-m)B^3(\alpha,\beta)\|g'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta}\|_{[0,\infty),\infty},$$

provided that $m \le f(t) \le M$ for $t \in [0,\infty)$ and $g'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta} \in L_\infty[0,\infty)$. Therefore,

$$|C_{B,\alpha,\beta}(f,g)| \le \tfrac{1}{12}B^4(\alpha,\beta)\|f'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta}\|_{[0,\infty),\infty}\|g'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta}\|_{[0,\infty),\infty},$$

provided that $f'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta}, g'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta} \in L_\infty[0,\infty)$ and

$$\begin{aligned}|C_{B,\alpha,\beta}(f,g)| \le \frac{1}{\pi^2}B^3(\alpha,\beta)&\|f'\ell^{(1-\alpha)/2}(1+\ell)^{(\alpha+\beta)/2}\|_{[0,\infty),2} \\ &\times \|g'\ell^{(1-\alpha)/2}(1+\ell)^{(\alpha+\beta)/2}\|_{[0,\infty),2},\end{aligned}$$

provided that $f'\ell^{(1-\alpha)/2}(1+\ell)^{(\alpha+\beta)/2}, g'\ell^{(1-\alpha)2}(1+\ell)^{(\alpha+\beta)/2} \in L_2[0,\infty)$.

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Similar results can be stated for probability distributions that are supported on the whole axis $\mathbb{R} = (-\infty, \infty)$. Namely, if $I = (-\infty, \infty)$, $f : \mathbb{R} \to \mathbb{C}$ is locally absolutely continuous on $\mathbb{R}$ and $w(s) > 0$ for $s \in \mathbb{R}$ with $\int_{-\infty}^{\infty} w(s)\,ds = 1$, that is, w is a probability density function on $(-\infty, \infty)$, f is *Lebesgue measurable* and satisfies the condition $m \le f(t) \le M$ for $t \in (-\infty, \infty)$ and $g : (-\infty, \infty) \to \mathbb{R}$ is locally absolutely continuous on $(-\infty, \infty)$ with $g'/w \in L_\infty(-\infty, \infty)$, then, by considering the functional

$$C_w(f, g) := \int_{-\infty}^{\infty} w(t)f(t)g(t)\,dt - \int_{-\infty}^{\infty} w(t)f(t)\,dt \int_{-\infty}^{\infty} w(t)g(t)\,dt,$$

we find that

$$|C_w(f, g)| \le \frac{1}{8}(M - m)\left\|\frac{g'}{w}\right\|_{(-\infty,\infty),\infty}. \tag{3.4}$$

Moreover, if $f'/w \in L_\infty(-\infty, \infty)$, then

$$|C_w(f, g)| \le \frac{1}{12}\left\|\frac{f'}{w}\right\|_{(-\infty,\infty),\infty}\left\|\frac{g'}{w}\right\|_{(-\infty,\infty),\infty} \tag{3.5}$$

and, if $f'/w^{1/2}, g'/w^{1/2} \in L_2(-\infty, \infty)$, then

$$|C_w(f, g)| \le \frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{(-\infty,\infty),2}\left\|\frac{g'}{w^{1/2}}\right\|_{(-\infty,\infty),2}. \tag{3.6}$$

We illustrate these results with an example. The probability density of the *normal distribution* on $(-\infty, \infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation* and σ^2 is the *variance*. The cumulative distribution function is

$$W_{\mu,\sigma^2}(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt.$$

Consider the functional

$$\begin{aligned} C_{N,\sigma,\mu}(f, g) := \sqrt{2\pi}\sigma \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)f(t)g(t)\,dt \\ - \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)f(t)\,dt \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)g(t)\,dt \end{aligned}$$

with the parameters μ and σ as above. By (3.4)–(3.6),

$$|C_{N,\sigma,\mu}(f, g)| \le \frac{1}{8}(M - m)(\sqrt{2\pi}\sigma)^3\left\|g' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),\infty},$$

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provided that $m \le f(t) \le M$ for $t \in (-\infty, \infty)$ and $g' \exp((\ell-\mu)^2/(2\sigma^2)) \in L_\infty(-\infty,\infty)$. Moreover, if $f' \exp((\ell-\mu)^2/2\sigma^2) \in L_\infty(-\infty,\infty)$, then

$$|C_{N,\sigma,\mu}(f,g)| \le \frac{1}{12}(\sqrt{2\pi}\sigma)^4 \left\| f' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),\infty} \left\| g' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),\infty}.$$

Finally, if $f' \exp((\ell-\mu)^2/2\sigma^2)$, $g' \exp((\ell-\mu)^2/2\sigma^2) \in L_2(-\infty,\infty)$, then

$$|C_{N,\sigma,\mu}(f,g)| \le \frac{1}{\pi^2}(\sqrt{2\pi}\sigma)^3 \left\| f' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),2} \left\| g' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),2}.$$

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