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FIXED POINT THEOREMS FOR POINT-TO-SET MAPPINGS

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ABSTRACT. Let (X, d) be a metric space. Let X^c be the family of all non-empty closed subsets of X endowed with the Hausdorff metric D induced by d. Let f be a non-expansive mapping of X into X^c ($D(f(x), f(y)) \leq d(x, y)$ for all x, y in X). Some fixed point theorems are obtained by imposing certain conditions on f and X.

Let (X, d) be a bounded complete metric space. Let f be a mapping of X into X^c. F. E. Browder [6] proves that f has a fixed point if (a) $f(x)$ is a singleton for each x in X, and (b) there exists a monotonically non-decreasing function α of $[0, \infty)$ into $[0, \infty)$ such that $\alpha(t) < t$ for all $t > 0$, α is continuous from the right and f is α-contractive, i.e. $D(f(x), f(y)) \leq \alpha(d(x, y))$ for all x, y in X. We shall generalize this result to the following theorem. Comparing the above result of Browder with Theorem 1 in [4], we note that for a monotonically non-decreasing function α of $[0, \infty)$ into $[0, \infty)$, α is continuous from the right if and only if α is upper semicontinuous from the right.

THEOREM 1. *Let (X, d) be a bounded complete metric space. Let f be a mapping of X into X^c. Suppose that there exists a monotonically non-decreasing function α of $[0, \infty)$ into $[0, \infty)$ such that*

(a) α is upper semicontinuous from the right and $\alpha(t) < t$ for all $t > 0$;

(b) f is α-contractive on X.

Then there exists a unique non-empty closed subset Y of X such that

$$\text{cl} \cup \{f(x) : x \in Y\} = Y.$$

Proof. Consider the mapping F on X^c defined by

$$F(A) = \text{cl} \cup \{f(a) : a \in A\}, \qquad A \in X^c.$$

Then F is a mapping of X^c into X^c. We shall prove that

$$D(F(A), F(B)) \leq \alpha(D(A, B)), \qquad A, B \in X^c.$$

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Let $A, B \in X^c$, $a \in A$, $b \in B$, $x \in f(a)$. Then

$$\begin{aligned} d(x, F(B)) &\leq d(x, f(b)) \\ &\leq D(f(a), f(b)) \\ &\leq \alpha(d(a, b)). \end{aligned}$$

So

$$d(x, F(B)) \leq \inf\{\alpha(d(a, b)) : b \in B\}.$$

Let $s = d(a, B)$. Then there exists a sequence $\{b_n\}$ in B such that the sequence $\{d(a, b_n)\}$ is monotonically non-increasing and converges to s. Since α is continuous from the right,

$$\begin{aligned} \alpha(d(a, B)) &= \alpha(s) \\ &= \lim_{n\to\infty} \alpha(d(a, b_n)) \\ &\geq \inf\{\alpha(d(a, b)) : b \in B\}. \end{aligned}$$

So

$$d(x, F(B)) \leq \alpha(d(a, B)).$$

Since α is monotonically non-decreasing and $d(a, B) \leq D(A, B)$,

$$d(x, F(B)) \leq \alpha(D(A, B)).$$

Thus

$$\sup\{d(x, F(B)) : x \in f(a), a \in A\} \leq \alpha(D(A, B)).$$

Since the function $z \to d(z, F(B))$ is continuous on X,

$$\sup\{d(x, F(B)) : x \in F(A)\} \leq \alpha(D(A, B)).$$

Similarly,

$$\sup\{d(F(A), y) : y \in F(B)\} \leq \alpha(D(A, B)).$$

Therefore

$$D(F(A), F(B)) \leq \alpha(D(A, B)).$$

By Proposition 4.1.3. in [8], X^c is complete. Hence by Theorem 1 in [4], there exists a unique Y in X^c such that $F(Y) = Y$, i.e.

$$\text{cl} \cup \{f(x) : x \in Y\} = Y.$$

For any A in X^c, we shall use $\delta(A)$ to denote the diameter of A, i.e.

$$\delta(A) = \sup\{d(x, y) : x, y \in A\}.$$

To see the connection of Theorem 1 with the fixed point theory, we make the following remarks. (1) Suppose that f is single-valued. Then $\delta(f(A)) < \delta(A)$ for every A in X^c with $\delta(A) > 0$. But for the Y in Theorem 1,

$$\begin{aligned} \delta(Y) &= \delta(\text{cl} \cup \{f(x) : x \in Y\}) \\ &= \delta(\cup \{f(x) : x \in Y\}) \\ &= \delta(f(Y)). \end{aligned}$$

So $\delta(Y)=0$ and Y is a singleton, say $\{x_0\}$. Hence

$$\{x_0\} = Y = \text{cl} \cup \{f(x): x \in Y\} = \{f(x_0)\}$$

and x_0 is a fixed point of f. (2) Suppose that Y (but not necessarily X) is compact. Then from Theorem 1, the restriction of f to Y is a mapping of Y into Y^c. For any distinct x, y in Y, $D(f(x), f(y)) \leq \alpha(d(x, y)) < d(x, y)$. Hence f, restricted to Y, is a multi-valued contractive mapping. Since Y is compact, by Theorem 4.2 of [8], Y^c is compact. By Theorem 4 of [7], f has a fixed point in Y. We thank the referee for this short proof. In fact, for this case, the conclusion of Theorem 1 can be improved: Since f is continuous, $\{f(x): x \in Y\}$ is compact. So by Theorem 2.5.2′ in [8], $\cup \{f(x): x \in Y\}$ is compact. Hence from Theorem 1,

$$\cup \{f(x): x \in Y\} = Y.$$

(3) As it is seen from (1) and (2), the unique Y is used to catch a fixed point of f. In practice, Y in Theorem 1 can be constructed so that the properties of Y can actually be checked. Indeed, by Theorem 1 of [4] and the proof of Theorem 1, for any A in X^c, Y is the limit of the sequence $\{F^n(A)\}$ of iterates of A. Since X is bounded, X^c is also bounded and therefore by Theorem 1 in [6], on X^c, $\{F^n(A)\}$ converges to Y uniformly.

In Theorem 1, the condition that α is upper semicontinuous from the right is guaranteed if X is metrically convex (for any distinct x, y in X, there exists z in X, different from x and y, such that $d(x, y)=d(x, z)+d(z, y)$). This can be seen from the following lemma.

Let f be a mapping of a metric space (X, d) into a metric space (Y, d'). $\|f\|$ will denote the number $\sup\{d'(f(x), f(y))/d(x, y): x, y \in X, x \neq y\}$.

LEMMA 1. *Let (X, d) be a complete metrically convex metric space. Let (Y, d') be a metric space. Let f be a mapping of X into Y such that $\|f\| < \infty$. Then the function α on $d(X \times X)$ defined by*

$$\alpha(t) = \sup\{d'(f(x), f(y)): x, y \in X, d(x, y) \leq t\}$$

for each t in $d(X \times X)$ is subadditive, monotonically non-decreasing, continuous from the right and satisfies $\alpha(t) \leq \|f\| t$ for all $t \geq 0$.

Proof. We shall merely prove that α is subadditive. Let $s, t \in d(X \times X)$, $x, y \in X$. Suppose that $d(x, y) \leq s+t$. Then there exist u, v in $[0, \infty)$ such that

$$d(x, y) = u+v \quad \text{and} \quad u \leq s, \quad v \leq t.$$

Since X is metrically convex, by a result of Menger [3, p. 41], there exists z in X such that

$$d(x, z) = u, \qquad d(z, y) = v.$$

So

$$\begin{aligned} d'(f(x), f(y)) &\leq d'(f(x), f(z)) + d'(f(z), f(y)) \\ &\leq \alpha(s) + \alpha(t). \end{aligned}$$

Varying x, y, we obtain

$$\alpha(s+t) \le \alpha(s)+\alpha(t).$$

We remark here that the function α in Lemma 1 is modified from Lemma 2 in [4]. From Lemma 1 and Theorem 1, we can prove that

THEOREM 2. *Let (X, d) be a bounded metrically convex complete metric space. Let f be a mapping of X into X^c. Suppose that there exists a monotonically non-decreasing function α of $[0, \infty)$ into $[0, \infty)$ such that*

(a) $\alpha(t)<t$ for $t>0$:

(b) f is α-contractive on X.

Then there exists a unique non-empty closed subset Y of X such that

$$\text{cl} \cup \{f(x):x \in Y\} = Y.$$

Let K be a non-empty weakly compact convex subset of a Banach space B. Let f be a non-expansive mapping of K into itself. It is not known if f must have a fixed point [1], [2], [5]. One of the difficulties in obtaining a fixed point of f is that $I-f$ does not have a convexity structure to match the convexity structure of K, where I is the identity function on K. For any subset A of B, $\|A\|$ will denote the number $\inf\{\|x\|:x \in A\}$.

DEFINITION. Let A be a convex subset of a Banach space B. Let g be a function of A into B^c. The function g is almost convex if for any sequence $\{x_n\}$ in A, there exists a number r in $(0, \infty)$ such that for any $n\ge 1$ and any x in $\text{Co}(\{x_k:k\ge n\})$,

$$\|g(x)\| \le r \sup\{\|g(x_k)\|:k \ge n\}.$$

We remark here that, with the notations as in the above definition, g is almost convex if for any x, y in A and any t in $(0, 1)$,

$$\|g((1-t)x+ty)\| \le (1-t)\ \|g(x)\| + t\ \|g(y)\|.$$

THEOREM 3. *Let K be a non-empty weakly compact convex subset of a Banach space B. Let f be a nonexpansive mapping of K into the family of all non-empty weakly closed subsets of K. Suppose that $I-f$ is almost convex. Then f has a fixed point.*

Proof. By Lemma 4 in [9], there exists a sequence $\{x_n\}$ in K such that $\{\|x_n-f(x_n)\|\}$ converges to 0. For each k, let $A_k=\text{cl Co}\{x_n:n\ge k\}$. Since A_k is closed and convex, it is weakly closed. So by the weak compactness of K, $\cap \{A_k:k\ge 1\}$ is non-empty. Let $x \in \cap \{A_k:k\ge 1\}$. Then there exists a sequence $\{y_n\}$ such that for each k,

$$y_k \in \text{Co}\{x_n:n \ge k\} \quad \text{and} \quad \|y_k-x\| < 1/k.$$

Since $I-f$ is almost convex, there exists a number r in $(0, \infty)$ such that for each n,

$$\|y_n-f(y_n)\| \le r \sup\{\|x_k-f(x_k)\|:k \ge n\}.$$

Let $n \geq 1$. Since $f(y_n)$ is weakly compact, there exists z_n in $f(y_n)$ such that $\|y_n - z_n\| = \|y_n - f(y_n)\|$. Since $f(x)$ is weakly compact, there exists w_n in $f(x)$ such that $\|w_n - z_n\| = \|z_n - f(x)\|$. Since

$$\begin{aligned}\|x-w_n\| &\leq \|x-y_n\| + \|y_n-z_n\| + \|z_n-w_n\| \\ &< 1/n + \|y_n-f(y_n)\| + \|z_n-f(x)\| \\ &\leq 1/n + r\sup\{\|x_k-f(x_k)\| : k \geq n\} + D(f(y_n), f(x)) \\ &< 2/n + r\sup\{\|x_k-f(x_k)\| : k \geq n\},\end{aligned}$$

$\|x-w_n\| \to 0$ as $n \to \infty$. Since $w_n \in f(x)$ for each n, it follows that $x \in f(x)$.

THEOREM 4. *Let K be a non-empty weakly compact convex subset of a Banach space B. Let $\mathscr{F}$ be a family of nonexpansive mappings of K into the family of all non-empty weakly closed subsets of K. Suppose that*

(*a*) *for each f in $\mathscr{F}$, $I-f$ is almost convex;*

(*b*) *for any distinct f, g in $\mathscr{F}$ and for any x, y in K, $y \notin g(x)$ if $x \in f(x)$ and $y \notin f(y)$.*

Then $\mathscr{F}$ has a common fixed point.

Proof. Let $f\, \mathscr{F}$. Let F_f be the set of all fixed points of f. Let x, y be distinct points in F_f. Consider the sequence $\{x_n\}$ in K defined by

$$x_{2n} = x \quad \text{and} \quad x_{2n-1} = y \quad \text{for each } n.$$

Since $I-f$ is almost convex, there exists a number r in $(0, \infty)$ such that for any t in $(0, 1)$, the vector $x_t = (1-t)x + ty$ satisfies

$$\|x_t - f(x_t)\| \leq r\sup\{\|x-f(x)\|, \|y-f(y)\|\}.$$

So $x_t \in F_f$ for each $t \in (0, 1)$. Therefore F_f is convex. Now let $\{x_n\}$ be a sequence in F_f which converges to some x in K. By continuity of f, $\{f(x_n)\}$ converges to $f(x)$. So $\{d(x_n, f(x))\}$ converges to 0. Thus there exists a sequence $\{y_n\}$ in $f(x)$ such that $\{d(x_n, y_n)\}$ converges to 0. So $\{d(x, y_n)\}$ converges to 0. By the weak compactness of $f(x)$, $x \in f(x)$. So F_f is closed. Hence F_f is weakly closed. From (b), for any g in $\mathscr{F}$ with $f \neq g$, $g(x) \subset F_f$ for each x in F_f. So we can apply Theorem 3 to f, g/F_f. This proves that $F_f \cap F_g \neq \varnothing$. Repeating the same argument, we conclude that the family $\{F_f : f \in \mathscr{F}\}$ has the finite intersection property. By the weak compactness of K, $\{F_f : f \in \mathscr{F}\}$ has nonempty intersection. Hence $\mathscr{F}$ has a common fixed point.

THEOREM 5. *Let K be a nonempty weakly compact convex subset of a Banach space B. Let $\mathscr{F}$ be a commuting family of nonexpansive mappings of K into K. Suppose that for each f in $\mathscr{F}$, $I-f$ is almost convex. Then the monoid generated by $\mathscr{F}$ has a common fixed point.*

Proof. Apply Theorem 4 to the monoid.

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