BULL. AUSTRAL. MATH. SOC. 52A10, 52A40, 52A05
VOL. 56 (1997) [161–163]

Convex sets with lattice point constraints

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The first ideas of convex sets date as far back as Archimedes but it was not until the end of the last century that a systematic study was made which gave rise to the subject as an independent branch of mathematics. At the turn of the century, Minkowski [4] published his famous Convex Body Theorem which is the basis for the Geometry of Numbers. The idea is to interpret integer solutions of equations or inequalities as points with integer coordinates (lattice points).

In simple terms, Minkowski's result states that a convex set in the plane which is symmetric about the origin and contains no other lattice point in its interior must have an area not greater than 4. From the point of view of the research, Minkowski's Theorem is an example of a geometric extremal problem, that is, a problem concerning an inequality stated in terms of geometrical concepts. His work suggests a more general class of geometric extremal problems concerning sets with lattice point constraints. This thesis is a collection of new inequalities for lattice constrained convex sets in the plane.

Let $\mathcal{K}^2$ denote the set of all planar, compact, convex sets. Let K be a set in $\mathcal{K}^2$ with area A, perimeter p, diameter d, width w, inradius r and circumradius R. Let K^0 denote the interior of K. Let Λ denote the lattice generated by the vectors $\mathbf{u}$ and $\mathbf{v}$. In the case where $\mathbf{u} = (1,0)$ and $\mathbf{v} = (0,1)$, we have the integral lattice denoted by Γ. The lattice point enumerator, $G(K^0, \Lambda)$ is the number of points of Λ contained in K^0. In the case where the origin O is the only point of Λ in K^0, we say that K is Λ-admissible.

This thesis is concerned with obtaining new inequalities concerning the geometric functionals A, p, d, w, r and R for a set K having $G(K^0, \Gamma) = g$ where $g = 0, 1, 2$. The natural starting point for the research is to investigate problems for a set K having $G(K^0, \Gamma) = 0$. What geometric inequalities occur for such sets? Can these results be extended to sets having $G(K^0, \Gamma) = 1$ or $G(K^0, \Gamma) = 2$? We discover that a number of such problems may be readily solved by reducing the problem to one concerning no point of a special rectangular lattice.

Received 3rd February, 1997
Thesis submitted to The University of Adelaide, August 1996. Degree approved, October 1996. Supervisor: Associate Professor PR Scott.

https://doi.org/10.1017/S0004972700030835 Published online by Cambridge University Press

Chapter 1 of the thesis gives a brief description of the problems of the thesis. In Chapter 2, we describe methods and results which are used throughout the thesis. Thereafter, the thesis consists essentially of two parts. Part 1 (Chapters 3, 4 and 5) deals with problems concerning single geometric parameters. The problems of Chapters 3 and 4 resulted from an attempt to prove a conjecture by Scott [**7**] concerning the maximal area of a Γ-admissible set. In Chapter 3, we obtain a result concerning A for Γ-admissible sets. The result extends and generalizes Minkowski's Theorem and gives a classification of convex sets in the plane. In Chapter 4, we consider Γ-admissible sets having circumcentre O. Under certain conditions, we find the maximal circumradius of such sets and we show that the extremal set is a triangle with an edge containing two lattice points [**1**]. In Chapter 5, we obtain the maximal width of a set K having $G(K^0, \Gamma) = g$ where $g = 0, 1, 2$.

Part 2 (Chapters 6-12) deals with a larger class of problems concerning relationships between pairs of the geometric parameters A, p, d, w, r and R. In Chapter 6, we obtain inequalities relating w and d [**2**]. This problem is motivated by a width-diameter result by Scott [**5**] for a set with $G(K^0, \Gamma) = 0$. We obtain the corresponding result for the rectangular lattice and deduce results for the cases $G(K^0, \Gamma) = g$ where $g = 0, 1, 2$. In Chapter 7, we generalize inequalities by Scott [**6**] concerning the pairs (A, w), (p, w) and (R, w) to rectangular lattices. In Chapter 8, we find another inequality for the pair (A, w). In Chapter 9, we first obtain an inequality relating R, d and w for a set K with no lattice constraints. We then derive an inequality for the pair (R, d) for a set K containing no point of the rectangular lattice. We also obtain a dual inequality concerning the pair (w, r). Chapter 10 gives results concerning A and r for a set K having $G(K^0, \Gamma) = 0$ and $G(K^0, \Gamma) = 1$. We combine these inequalities with known inequalities in elementary geometry to deduce inequalities for the pairs (p, r) and (d, r) for a set K with $G(K^0, \Gamma) = 0$ [**3**].

The last two problems contained in Chapters 11 and 12 concern a set K with $G(K^0, \Gamma) = 2$ and having a special symmetry condition. In Chapter 11, we establish inequalities for the pairs (A, d) and (A, R) and in Chapter 12, we find a result for the pair (A, p). We also conjecture the corresponding results for the general class of convex sets containing two interior lattice points.

Finally in Chapter 13, we summarize the results of the thesis and make some remarks on the scope for future research in the area. It may be seen that many new and interesting problems remain in this area.

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