ON THE EXPLICITE DEFINING RELATIONS OF
ABELIAN SCHEMES OF LEVEL THREE

HISASI MORIKAWA

Dedicated to the memory of Professor TADASI NAKAYAMA

It is known classically that abelian varieties of dimension one over the field of complex numbers may be expressed by non-singular Hesse's canonical cubic plane curves, \( X_0^3 + X_1^3 + X_2^3 - 6 \gamma X_0 X_1 X_2 = 0 \). The purpose of the present paper is to generalize this idea to higher dimensional case.

Let \( \mathbb{Z}(3) \) be the residue group of the additive group \( \mathbb{Z} \) of integers modulo \( 3\mathbb{Z} \) and \( \mathbb{Z}(3)^r \) be the \( r \)-times direct sum of \( \mathbb{Z}(3) \). We mean by \( \mathbb{Z}(3)^+_r \) the subset of \( \mathbb{Z}(3)^r \) consisting of all the elements \( (a_1^+, \ldots, a_r^+) \) such that \( a_i^+ = 0 \) or \( 1 \) \((1 \leq i \leq r)\). Then, roughly speaking, our result may be expressed as follows: a generic abelian variety with a positive divisor \( U \) such that \( l(U) = 1 \)\(^{11}\) is defined by relations of the following type

\[
\begin{align*}
(\ast) \quad & \Delta_2 Y_{a+b}Y_{-a+b} - \sum_{c \in \mathbb{Z}(3)^r} \gamma_{a,c} Y_{c+b} = 0 \\
(\ast\ast) \quad & \Delta_1 Y_{a+b}Y_{-a+b} - \sum_{c^+ \in \mathbb{Z}(3)^+_r} \beta_{a,c^+} Y_{c+b} Y_{-c+b} = 0, \quad (a, b \in \mathbb{Z}(3)^r).
\end{align*}
\]

§ 1. Formal theta functions of level \( n \) and the scheme \( A(r, n) \) associated with them

1.1. We mean by \( \mathbb{Z} \) and \( \mathbb{Q} \) the ring of integers and the field of rational numbers. We mean by \( \mathbb{Z}^r \) the \( r \)-times direct sum of the \( \mathbb{Z} \)-module \( \mathbb{Z} \) and by \( \mathbb{Q}^r \) the \( r \)-times direct sum of the \( \mathbb{Q} \)-module \( \mathbb{Q} \). Let \( \{ W(i; \alpha), W(j, l; \beta) | 1 \leq i, j, l \leq r; \alpha, \beta \in \mathbb{Q} \} \) be a system of indeterminates on which rational numbers operate such that \( W(i; \alpha)^\gamma = W(i; \alpha^\gamma), W(j, l; \beta)^\gamma = W(j, l; \beta^\gamma) \). We denote by \( I \) the ideal in the polynomial ring \( \mathbb{Z}[\{ W(i; \alpha), W(j, l; \beta) \}] \) generated by

\[
\begin{align*}
W(i; 0) - 1, \quad & W(j, l; 0) - 1, \quad W(i; n\alpha) - W(i, \alpha) \cdots W(i; \alpha), \\
W(j, l; n\beta) - W(j, l, \beta) \cdots W(j, l; \beta)
\end{align*}
\]

Received May 10, 1965.
\(^{11}\) \( l(U) \) means the rank of the module of the multiples of \(-U\).
\[ W(i; \alpha)W(i; \beta) - W(i; \alpha + \beta), \quad W(j, l; \alpha)W(j, l; \beta) - W(i, l; \alpha + \beta), \]
\[ W(i, l; \alpha) - W(l, j; \alpha), \]
\[ (1 \leq i, j, l \leq r; \quad \alpha, \beta \in \mathbb{Q}, \quad n = 1, 2, \ldots) \]

We mean by \( U^*_i \) and \( Q^*_i \) the images of \( W(i; \alpha) \) and \( W(j, l; \beta) \) in the residue ring \( B = \mathbb{Z}[\{ W(i; \alpha), \ W(j, l; \beta) \}]/I \). Then it follows the relations:

\[
\begin{align*}
(1) & \quad U^*_i^n = U^*_i \cdots U^*_i, \quad Q^*_i^n = Q^*_i \cdots Q^*_i, \quad U^*_i = 1, \quad Q^*_i = 1, \\
(2) & \quad U^*_i U^*_j = U^*_{ij}^{*}, \quad Q^*_i Q^*_j = Q^*_{ij}^{*} \\
(3) & \quad (U^*_i)^\gamma = U^*_i^\gamma, \quad (Q^*_i)^\gamma = Q^*_i^\gamma, \\
(4) & \quad Q^*_i^\gamma = Q^*_i^\gamma, \\
& \quad (1 \leq i, j, l \leq r; \quad \alpha, \beta, \gamma \in \mathbb{Q}; \quad n = 1, 2, 3, \ldots). 
\end{align*}
\]

We shall use the following brief notations:

\[
\begin{align*}
(5) & \quad U(\alpha) = \prod_{i=1}^{r} U^*_i^{\alpha_i}, \\
(6) & \quad Q(\alpha, \beta) = \prod_{i, j=1}^{r} Q^*_i Q^*_j^{\beta_j}, \\
& \quad (\alpha = (\alpha_1, \ldots, \alpha_r), \quad \beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Q}^r). 
\end{align*}
\]

Then if follows

\[
\begin{align*}
(7) & \quad U(\alpha)U(\beta) = U(\alpha + \beta), \\
(8) & \quad Q(\alpha + \beta, \gamma + \delta) = Q(\alpha, \gamma)Q(\alpha, \delta)Q(\beta, \gamma)Q(\beta, \delta), \\
& \quad (\alpha, \beta, \gamma, \delta \in \mathbb{Q}^r). 
\end{align*}
\]

1.2. We mean by \( \text{Hom} (\mathbb{Z}', \mathbb{G}_m) \) the functor of the category of commutative rings into the category of abelian groups such that \( \text{Hom} (\mathbb{Z}', \mathbb{G}_m)(A) \) means the group of all the homomorphisms of the additive group \( \mathbb{Z}' \) into the multiplicative group of the units in \( A \). By virtue of (7) \( U \) may be considered as an element in \( \text{Hom} (\mathbb{Z}', \mathbb{G}_m)(B) \). For each \( \alpha \) in \( \mathbb{Q}' \) we may construct an element \( Q(\alpha) \) in \( \text{Hom} (\mathbb{Z}', \mathbb{G}_m)(B) \) given by

\[
(9) \quad Q(\alpha)(m) = Q(\alpha, m), \quad (m \in \mathbb{Z}').
\]

We mean by \( Q(\alpha)U \) the product of \( Q(\alpha) \) and \( U \) in \( \text{Hom} (\mathbb{Z}', \mathbb{G}_m) \).

**Definition 1.** We mean by a formal rational power series in \( Q \) of restricted type.

\(^2\) We change the notation \( Q(\alpha) \) slightly. In [1] and [2] we defined \( Q(\alpha) \) by \( Q(\alpha)(m) = Q(\alpha, m)^2 \).
type a formal rational power series $\sum_{\alpha ij} \lambda_{\alpha ij} \prod_{i=1}^{tij} Q_{ij}^{\alpha ij}$ satisfying the conditions:

1° there exists a positive integer $m$ such that $m\alpha ij \in \mathbb{Z}(1 \leq i, j \leq r)$ for $(\alpha_{ij})$ satisfying $\lambda_{\alpha ij} \neq 0$,

2° for any positive integer $n$ there exist only a finite number of terms $\lambda_{\alpha ij} \prod_{i=1}^{tij} Q_{ij}^{\alpha ij}$ such that $\lambda_{\alpha ij} \neq 0$ and $\alpha_{ij} \leq n(1 \leq i \leq r)$.

All the formal rational power series in $Q$ of restricted type with coefficients in $\mathbb{Z}$ form a commutative integral domain. We denote it by $\mathbb{Z}[[Q]]$.

We shall now give the definition of formal theta functions:

**Definition 2.** We mean by a formal theta functions of level $n$ with coefficients in $\mathbb{Z}[[Q]]$ a formal power series in $U_1, U_2, \ldots, U_r$ such that

$$\varphi(U) = \sum_{m \in \mathbb{Z}^r} \lambda_m U(m)^z$$

with coefficients $\lambda_m$ in $\mathbb{Z}[[Q]]$ such that

$$\varphi(Q(m)U) = Q(m, m)^{-n}U(m)^{-2n}\varphi(U), \quad (m \in \mathbb{Z}^r).$$

1.2. We denote by $\mathbb{Z}(n)$ the residue group of $\mathbb{Z}$ modulo $n\mathbb{Z}$. We denote by $0, 1, 2, \ldots, n-1$ sometimes the elements in $\mathbb{Z}(n)$ and sometimes integers $0, 1, 2, \ldots, n-1$ in $\mathbb{Z}$ so that $a/n(a \in \mathbb{Z}(n))$ makes sense. We denote by $\mathbb{Z}(n)^r$ the $r$-times direct sum of $\mathbb{Z}(n)$. By virtue of the difference equation (10) the coefficients $\lambda_m$ of a formal theta function $\sum_{m \in \mathbb{Z}^r} \lambda_m U(m)^z$ of level $n$ are given by

$$\lambda_{nm + g} = \lambda_g Q\left(\frac{g}{n}, \frac{g}{n}\right)^{-n} Q\left(m + \frac{g}{n}, m + \frac{g}{n}\right)^n, \quad (g \in \mathbb{Z}(n)^r, m \in \mathbb{Z}^r).$$

We shall introduce the canonical system of formal theta functions of level $n$:

$$X_{n,g}(Q|U) = \sum_{m \in \mathbb{Z}^r} Q\left(m + \frac{g}{n}, m + \frac{g}{n}\right)^n U(m + \frac{g}{n})^{zn}, \quad (g \in \mathbb{Z}(n)^r).$$

Then it follows the formulae

$$X_{n,g}(Q|U^{-1}) = X_{n,-g}(Q|U) \quad (13)$$

$$X_{n,g}(Q|U) = Q\left(\frac{h}{n}, \frac{h}{n}\right)^{-n} U\left(\frac{h}{n}\right)^{-2n} X_{n,g+h}(Q|U), \quad (g, h \in \mathbb{Z}(n)^r). \quad (14)$$

From (11) it follows that a formal theta function $\sum_{m \in \mathbb{Z}^r} \lambda_m U(m)^z$ is a linear combination of $X_{n,g}(Q|U)(g \in \mathbb{Z}(n)^r)$ as follows:
Putting $U = 1$, we have a system of elements in $\mathbb{Z}[[Q]]$:

$$T_{n,g}(Q) = X_{n,g}(Q|1), \ (g \in \mathbb{Z}(n)^r).$$

From (13) it follows

$$T_{n,-g}(Q) = T_{n,g}(Q), \ (g \in \mathbb{Z}(n)^r).$$

We denote by $R(r,n)$ the graded ring $\mathbb{Z}[(T_{n,g}(Q))_{g \in \mathbb{Z}(n)^r}]$ and by $S(r,n)$ the projective scheme $\text{Proj}(R(r,n))$ of the graded ring $R(r,n)$, where $\deg T_{n,g}(Q) = n(g \in \mathbb{Z}(n)^r)$. Since $R(r,n)$ is a commutative noetherian integral domain, the scheme $S(r,n)$ is irreducible, reduced and noetherian.

By virtue of (12) and (15) it follows the fundamental property of theta functions of level $n$:

**Proposition 1.** The formal theta functions of level $n$ $X_{n,g}(Q|U) (g \in \mathbb{Z}(n)^r)$ form a base of the formal theta functions of level $n$ over any field containing $R(r,n)$.

We denote by $O_{S(r,n)}$ the structure sheaf of $S(r,n)$ and by $O_{S(r,n)}[(X_{n,g}(Q|U))_{g \in \mathbb{Z}(n)^r}]$ the graded $O_{S(r,n)}$-algebra induced by $R(r,n)$ and $S(r,n)$. Since the ring $R(r,n)$ is a noetherian integral domain, $A(r,n)$ is also a noetherian irreducible reduced scheme.

**§ 2. Formal theta functions of level three**

2.1. We shall use the following notations;

$\mathbb{Z}(3)$: the residue group of $\mathbb{Z}$ modulo $3\mathbb{Z}$, we denote by $0, 1, -1$, sometimes the elements in $\mathbb{Z}(3)$ and sometimes integers $0, 1, -1$ in $\mathbb{Z}$ so that $a/3 \ (a \in \mathbb{Z}(3))$ makes sense,

$\mathbb{Z}(3)^+$: the subset $\{0, 1\}$ in $\mathbb{Z}(3)$,

$\mathbb{Z}(3)^r$: the $r$-times direct sum,

$\mathbb{Z}(3)^{r+}$: the subset in $\mathbb{Z}(3)^r$ consisting of all the elements $(a_1^+, \ldots, a_r^+)$ such that $a_i^+ = 0$ or $1$, $(1 \leq i \leq r)$.

Since $T_{n,g}(g \in \mathbb{Z}(n)^r)$ are modular forms of degree $r$ for certain congruence subgroup, $\mathbb{Z}[T_{n,g}(Q)]_{g \in \mathbb{Z}(n)^r}$ forms a graded ring.

https://doi.org/10.1017/S0027763000011958 Published online by Cambridge University Press
a, b, c, . . . : the elements in Z(3)^r,
\[ a^+, b^+, c^+, . . . \]: the elements in Z(3)^+^r,
\[ T_a = T_{\alpha_a}(Q) = \sum_{m \in \mathbb{Z}^r} Q\left( m + \frac{a}{3}, m + \frac{a}{3}\right)^3 \quad (a \in \mathbb{Z}(3)^r), \]
\[ (T_{a^+ b^+} T_{-a^+ b^+}) : \text{the } 2^r \times 2^r \text{-matrix of which } (a^+, b^+) \text{-component is} \]
\[ T_{a^+ b^+} T_{-a^+ b^+}, \]
\[ (T_{a b} T_{-a b}) : \text{the } 3^r \times 3^r \text{-matrix of which } (a, b) \text{-component is} \]
\[ T_{a b} T_{-a b} T_{b}, \]
\[ (T_{a b}^3) : \text{the } 3^r \times 3^r \text{-matrix of which } (a, b) \text{-component is} \]
\[ T_{a b} \]
\[ \Delta_1(T) = \det (T_{a^+ b^+} T_{-a^+ b^+}), \]
\[ \Delta_2(T) = \det (T_{a b}^3), \]
\[ \alpha_{a^+ b^+}(T) = \Delta_1(T) (T_{a^+ b^+} T_{-a^+ b^+})^{-1}, \]
\[ \beta_{a^+ b^+}(T) = \Delta_2(T) (T_{a^+ b^+} T_{-a^+ b^+}) (T_{a^+ b^+} T_{-a^+ b^+})^{-1} = (T_{a^+ b^+} T_{-a^+ b^+}) (T_{a b}^3)^{-1}, \]
\[ \gamma_{a b}(T) = \Delta_2(T) (T_{a b} T_{-a b} T_{b}) (T_{a b}^3)^{-1}, \]
\[ (a^+, b^+ \in \mathbb{Z}(3)^+^r; \quad a, b \in \mathbb{Z}(3)^r). \]

We shall first show some typical relations between
\[ X_{a^+ b}(Q | U) = \sum_{m \in \mathbb{Z}^r} Q\left( m + \frac{a}{3}, m + \frac{a}{3}\right)^3 U\left( m + \frac{a}{3}\right)^6 \quad (a \in \mathbb{Z}(3)^r) \]
with coefficients in \( R(r, 3) = \mathbb{Z}[\{ T_a \}_{a \in \mathbb{Z}(3)^r}] \):

**Proposition 2.**

(18) \[ \Delta_1(T) X_{a^+ b^+}(Q | U) X_{a^+ b^+}(Q | U) \]
\[ = \sum_{c \in \mathbb{Z}(3)^r} \beta_{c^+ b^+}(Q | U) X_{a^+ b^+}(Q | U), \]
(19) \[ \Delta_2(T) X_{a^+ b^+}(Q | U) X_{a^+ b^+}(Q | U) \]
\[ = \sum_{c \in \mathbb{Z}(3)^r} \gamma_{c^+ b^+}(Q | U)^3, \quad (a, b \in \mathbb{Z}(3)^r). \]

**Proof.** As we shall see in §3 the determinants \( \Delta_1(T) \) and \( \Delta_2(T) \) are not zero. By virtue of (14) it follows
\[ \Delta_1(T) = \det(T_{a+b} - T) = \det(X_{a+b}(Q|1)X_{-a+b}(Q|1)) = \det\left(Q\left(\frac{b^*}{3}\right)X_{a+\left(Q\left(\frac{b^*}{3}\right)\right)X_{-a+\left(Q\left(\frac{b^*}{3}\right)\right)}\right) \]
\[ = \prod \det\left(X_{a+\left(Q\left(\frac{b^*}{3}\right)\right)X_{-a+\left(Q\left(\frac{b^*}{3}\right)\right)}\right) \equiv 0, \]
\[ \Delta_2(T) = \det(T_{a+b}^2) = \det\left(X_{a+b}(Q|1)^2\right) = \det\left(Q\left(\frac{b^*}{3}\right)X_{a}(Q|1)^2\right) \]
\[ = \prod \det\left(X_{a}(Q|1)^2\right) \equiv 0. \]

Since \( R(r,3) \) is an integral domain, it follows that \( \{X_{a+c}(Q|U)X_{-c}(Q|U)c^+ \in Z(3)^{+r}\} \) and \( \{X_{a+c}(Q|U)^2 \mid a \in Z(3)^{*}\} \) are sets of linearly independent formal theta functions of level 6 and 9, respectively. By virtue of \((15)\) formal theta functions of level 6 (resp. level 9) \( \sum \lambda_m U(m)^3 \) such that \( \lambda_{m+1} = \lambda_{m-1} = 0 \) \( (m \in Z^*) \) form a vector space of dimension 6 (resp. dimension 9) over the quotient field of \( R(r,3) \). Therefore we may put
\[ X_{a+c}(Q|U)X_{-c}(Q|U) = \sum_{e \in Z(3)^{+r}} \lambda_a cX_{a+c}(Q|U)X_{-c}(Q|U), \]
\[ X_{a+c}(Q|U)X_{-a}(Q|U)X_{b+}(Q|U) = \sum_{e \in Z(3)^{+r}} \mu_a cX_{a+c}(Q|U)^3. \]

Putting \( U = Q\left(\frac{c}{3}\right)(c \in Z(3)^{*}) \) we have
\[ \lambda_a, c = \Delta_1(T)^{-1}\beta_a, c(T), \mu_a, c = \Delta_2(T)^{-1}\gamma_a, c(T), \]
\( (a, c \in Z(3)^{*}, c^+ \in Z(3)^{+r}) \).

For every \( b \in Z(3)^{*} \) by virtue of \((15)\) it follows
\[ \Delta_1(T)X_{a+b}(Q|U)X_{-a+b}(Q|U) = \sum_{e \in Z(3)^{+r}} \beta_a, c(T)X_{a+c}(Q|U)X_{-c+b}(Q|U), \]
\[ \Delta_2(T)X_{a+b}(Q|U)X_{-a+b}(Q|U)X_{b+}(Q|U) = \sum_{e \in Z(3)^{+r}} \gamma_a, c(T)X_{a+c}(Q|U)^3. \]

§ 3. The explicite defining relations of abelian schemes of level three

3.1. Let \( (Y)_{a \in Z(3)^{*}} \) be a system of indeterminates and \( R(r,3)[Y] \) be the graded ring \( R(r,3)[(Y)_{a \in Z(3)^{*}}] \). Let \( I_0 \) be the homogeneous ideal in \( R(r,3)[Y] \) generated by the following homogeneous elements
\[ \{k_l, a(T)Y_{a+b}Y_{-a+b} - \sum_{e \in Z(3)^{+r}} h_{a, c+}(T)Y_{c+}bY_{-c+}b \mid h_{a, c+}(T)k_l, a(T)^{-1} \]
\[ = \beta_{a, c+}(T)\Delta_1(T)^{-1}; a, b \in Z(3)^{*}, c^+ \in Z(3)^{+r}\} \]
and
\[ \langle h_{2,a}(T)Y_{a+b}Y_{a+b} - \sum_{c \in \mathbb{Z}(3)^r} h_{4,a,c}(T)Y_{c+b}^2 h_{4,a,c}(T)h_{2,a}(T)^{-1} \rangle = \gamma_{a,c}(T)\delta_a(T)^{-1}; \quad a, b, c \in \mathbb{Z}(3)^r. \]

Since \( \delta_1(T), \delta_2(T), \beta_{a,c}(T), \gamma_{a,c}(T) \) are homogeneous elements in \( R(r, 3) \) such that \( \deg \delta_1(T) = \deg \beta_{a,c} = 2^r \) and \( \deg \delta_2(T) = \deg \gamma_{a,c}(T) \) the homogeneous ideal \( I_\nu \) of \( R(r, 3)[Y] \) induces an indeal \( \mathfrak{I}_\nu \) of the \( O_{S(r,3)} \)-algebra \( O_{S(r,3)}[Y] \). We denote by \( V(r, 3) \) the \( S(r,3) \)-projective scheme \( \text{Proj}_{S(r,3)}(O_{S(r,3)}[Y]/\mathfrak{I}_\nu) \) of the graded \( O_{S(r,3)} \)-algebra \( O_{S(r,3)}[Y]/\mathfrak{I}_\nu \).

We mean by \( (X_a)_{a \in \mathbb{Z}(3)^r} \) the image of \( (Y_a)_{a \in \mathbb{Z}(3)^r} \) in the residue ring \( R(r,3)[Y]/I_\nu \) and by \( R[X] \) the graded ring \( R(r,3)[(X_a)_{a \in \mathbb{Z}(3)^r}] \). The elements \( X_a(a \in \mathbb{Z}(3)^r) \) are characterized by the relations:

\[
(20) \quad h_{1,a}(T)X_{a+b}X_{a-b} = \sum_{c \in \mathbb{Z}(3)^r} h_{4,a,c}(T)X_{c+b}X_{c-b},
\]

\[
(21) \quad \beta_{a,c}(T)h_{1,a}(T)^{-1} = \beta_{a,c}(T)\alpha_{a,c}(T)\delta_1(T)^{-1}, \quad a, b, c \in \mathbb{Z}(3)^r,
\]

\[
(22) \quad h_{2,a}(T)X_{a+b}X_{a-b} = \sum_{c \in \mathbb{Z}(3)^r} h_{4,a,c}(T)X_{c+b},
\]

\[
(23) \quad \beta_{a,c}(T)h_{2,a}(T)^{-1} = \gamma_{a,c}(T)\delta_2(T)^{-1}, \quad a, b, c \in \mathbb{Z}(3)^r.
\]

By virtue of Proposition 2 the formal theta functions \( X_a(Q|U) (a \in \mathbb{Z}(3)^r) \) satisfy the relations (20) and (21). Hence the map: \( X_a \rightarrow X_a(Q|U) (a \in \mathbb{Z}(3)^r) \) may be extended to an \( O_{S(r,3)} \)-morphism \( \rho \) of \( O_{S(r,3)}[Y]/\mathfrak{I}_\nu \) onto \( O_{S(r,3)}[(X_a)_{a \in \mathbb{Z}(3)^r}] \). The dual \( \rho^* \) of \( \rho \) gives the injection morphism of \( A(r, 3) \) into \( V(r, 3) \).

**3.2.** When \( r = 1 \), the relation (21) is reduced to a single relation

\[
(24) \quad (T_1^4 + 2T_1^2)X_0X_{-1} = T_1T_0^2(X_0^2 + X_1 + X_0^2 + X_1^2 + X_0^2)
\]

and the relation (20) is trivial. We shall express \( \delta_1(T), \delta_2(T), \beta_{a,c}(T), \gamma_{a,c}(T) \):

\[
(25) \quad \beta_{00}(T) = \beta_{00}(T) = 0, \quad \beta_{00}(T) = \beta_{00}(T) = \delta_1(T)
\]

\[
(26) \quad \gamma_{0,\pm 1}(T) = 0, \quad \gamma_{0,\pm 1}(T) = \gamma_{0,\pm 1}(T) = \gamma_{0,\pm 1}(T)
\]

We denote by \( T_d^{(i)} \) the power series

https://doi.org/10.1017/S0027763000011958 Published online by Cambridge University Press
\[ T_d^{(i)} = X_d^{(i)}(Q_{ii} | 1) \sum_{m \in \mathbb{Z}(3)} Q_{ii}^{m+1} X_d^{m}, \ (1 \leq i \leq r; \ a \in \mathbb{Z}(3)), \]

and denote by \( X_d^{(i)} \) \((1 \leq i \leq r; \ a \in \mathbb{Z}(3))\) the quantities defined by the relation:

\[
(T_i^{(i+1)} + 2 T_i^{(i)}) X_i^{(i)} X_i^{(i)} X_i^{(i)} = T_i^{(i)} T_i^{(i)} (X_i^{(i)} + X_i^{(i+1)} + X_i^{(i)})
\]

\((1 \leq i \leq r)\).

We denote by \( R^* (1, 3) \) the subring \( \mathbb{Z}[(T_d^{(i)})_{a \in \mathbb{Z}(3)}]_{(A(T) \Delta (T))} \) of degree zero in the quotient ring of \( \mathbb{Z}[(T_d^{(i)})_{a \in \mathbb{Z}(3)}] \) with respect to \( \Delta (T) \Delta (T) \), \((1 \leq i \leq r)\) and by \( R^* (r, 3) \) the subring \( R^* (1, 3) \) of degree zero in the quotient ring of \( R \) with respect to \( \Delta (T) \Delta (T) \). We denote by \( S^* (1, 3) \) and \( S^* (r, 3) \) the affine scheme \( \text{Spec}(R^* (1, 3)) \) and \( \text{Spec}(R^* (r, 3)) \), respectively.

Since \( Q_{ii} \) \((1 \leq i \leq r)\) are indeterminates and \( T_d^{(i)} = T_d^{(i)}(Q) \) \((a \in \mathbb{Z}(3))\) are formal power series in \( Q_{ii} \), the map:

\[ T_a^{(1)} \otimes \cdots \otimes T_a^{(r)} \rightarrow T_a^{(1)} \otimes \cdots \otimes T_a^{(r)} \quad (a_1, \ldots, a_r \in \mathbb{Z}(3)) \]

induces an isomorphism of \( R^* (1, 3) \otimes \cdots \otimes R^* (r, 3) \) onto the subring \( R^{**} \) of degree zero in the quotient ring of \( \mathbb{Z}[(T_d^{(i)})_{a \in \mathbb{Z}(3)}] \) with respect to \( \Delta (T) \Delta (T) \). For the sake of simplicity we shall identify \( T_a^{(1)} \otimes \cdots \otimes T_a^{(r)} \) with \( T_a^{(1)} \otimes \cdots \otimes T_a^{(r)} \), \((a_1, \ldots, a_r \in \mathbb{Z}(3))\). Let \( F(Y) \) be an element in \( \mathbb{Z}[Y] \). Then \( F(T) = 0 \) means that \( F(T(Q)) \) is zero as a formal rational power series. Therefore an equality \( F(T) = 0 \) implies \( F(T^{(1)} \otimes \cdots \otimes T^{(r)}) = 0 \), because, replacing \( Q_{ii} \) \((i \neq j)\) by 1 in \( F(T(Q)) \), we have \( F(T^{(1)} \otimes \cdots \otimes T^{(r)}) \). This shows that the map:

\[ T(a_1, \ldots, a_r) \rightarrow T^{(1)} \otimes \cdots \otimes T^{(r)} = T^{(1)} \otimes \cdots \otimes T^{(r)} , \quad (a_1, \ldots, a_r \in \mathbb{Z}(3)) \]

induces a surjective morphism \( \lambda : R^* (r, 3) \rightarrow R^{**} = R^* (1, 3) \otimes \cdots \otimes R^* (1, 3) \).

The dual \( \lambda^* \) of \( \lambda \) is the injection morphism:

\[ S^{**} = S^* (1, 3) \times \cdots \times S^* (1, 3) \rightarrow S^* (r, 3) \]

where \( S^{**} \) is considered as the affine scheme \( \text{Spec}(R^{**}) \).

We mean by \( I^{**} \) the homogeneous ideal in \( R^{**}[Y] \) generated by

\[ \lambda(D_{a}) Y_{a+b} Y_{a+b} - \sum_{e \in \mathbb{Z}(3)^{+}} \lambda(D_{a+c}(T)) Y_{c+b} Y_{c+b}, \]

\[ \lambda(D_{a}) Y_{a+b} Y_{a+b} Y_{b} - \sum_{e \in \mathbb{Z}(3)^{+}} \lambda(D_{a+c}(T)) Y_{c+b} Y_{c+b}, \quad (a, b \in \mathbb{Z}(3)^{+}). \]

We mean by \((Z_a)_{a \in \mathbb{Z}(3)}\) the image of \((Y_a)_{a \in \mathbb{Z}(3)}\) in the residue ring \( R^{**}[Y]/I^{**} \).

Then \((Z_a)_{a \in \mathbb{Z}(3)}\) satisfies the relations:
We shall prove the isomorphism:

\((V(1, 3) \times \cdots \times V(1, 3)) \times_{\#(1, 3)} \cdots \times_{\#(1, 3)} S^* = V(r, 3) \times_{\#(r, 3)} S^*\).

**Lemma 1.**

\[\lambda(A_l(T)) = A_l(T^{(1)}) \cdots A_l(T^{(r)}),\]
\[\lambda(A_r(T)) = A_r(T^{(1)}) \cdots A_r(T^{(r)}),\]
\[\lambda(\beta(a_1, \ldots, a_r), (b_1^+, \ldots, b_r^+)(T)) = \begin{cases} (d_1(T^{(1)}) \cdots d_i(T^{(r)}), \text{for } (a_1, \ldots, a_r) \times \left(\begin{array}{c} b_1^+, \ldots, b_r^+ \end{array}\right), \text{such that } |a_i| = b_i^+ \ (1 \leq i \leq r), \\ 0 \text{ otherwise,} \end{cases}\]
\[\lambda(\gamma(a_1, \ldots, a_r), (b_1, \ldots, b_r)(T)) = \begin{cases} \gamma_{a_1}, b_1(T^{(1)}) \cdots \gamma_{a_r}, b_r(T^{(r)}), \\ \text{for } (a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{Z}(3)), \end{cases}\]

where we mean

\[|a| = \begin{cases} a & \text{for } a = 0, 1, \\ -a & \text{for } a = -1. \end{cases}\]

**Proof.** From the definition it follows:

\[\lambda(A_l(T)) = A_l(T^{(1)}) \cdots A_l(T^{(r)}) = A_l(T^{(1)}) \cdots A_l(T^{(r)}),\]
\[\lambda(\beta(a_1, \ldots, a_r), (b_1^+, \ldots, b_r^+)(T)) = \beta(a_1, \ldots, a_r), (b_1^+, \ldots, b_r^+)(T^{(1)} \cdots T^{(r)}) = \beta(a_1, b_1^+)(T^{(1)}) \cdots \beta(a_r, b_r^+)(T^{(r)}) \times \begin{cases} (d_1(T^{(1)}) \cdots d_i(T^{(r)}), \text{for } (a_1, \ldots, a_r) \times \left(\begin{array}{c} b_1^+, \ldots, b_r^+ \end{array}\right), \text{such that } |a_i| = b_i^+ \ (1 \leq i \leq r), \\ 0 \text{ otherwise,} \end{cases}\]
\[\lambda(\gamma(a_1, \ldots, a_r), (b_1, \ldots, b_r)(T)) = \begin{cases} \gamma_{a_1}, b_1(T^{(1)}) \cdots \gamma_{a_r}, b_r(T^{(r)}), \\ \text{for } (a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{Z}(3)), \end{cases}\]

**Lemma 2.** It follows the relations:

\[Z_{a_1+b_1, \ldots, a_r+b_r}Z(-a_1+b_1, \ldots, -a_r+b_r) = Z(\#(a_1+b_1, \ldots, \#(a_r+b_r)Z(- \#(a_1+b_1, \ldots, - \#(a_r+b_r)),\]
\[Z(b_1, \ldots, b_r)Z(b_1+b_1, \ldots, b_r+b_r)Z(b_1, \ldots, b_r, a+b_1, b_1+b_1, \ldots, b_r+b_r)Z(b_1, \ldots, b_r) = d(T^{(i)})^{-1} \sum_{c \in \mathbb{Z}(3)} i, c(T^{(i)})Z^2(b_1, \ldots, b_r, c+b_1, b_1+b_1, \ldots, b_r),\]

\[(1 \leq i \leq r; a_1, \ldots, a_r, b_1, \ldots, b_r) \in \mathbb{Z}(3)).\]
This is a consequence of the definition of \((Z_α)_α∈Z(3)^r\) and Lemma 1.

**Lemma 3.** Let \((c_1, \ldots, c_r)\) be a fixed element in \(Z(3)^r\) and put

\[
Z_α^{(i)} = Z(c_1, \ldots, c_i, a_{i+1}, \ldots, c_r) \quad (1 ≤ i ≤ r; \, a ∈ Z(3)).
\]

Then it follows

\[
Z_{(1 \ldots r)}Z_α^{(i)}Z_{α_3} = Z_α^{(1)} \cdots Z_α^{(r)}, \quad (α_1, \ldots, α_r ∈ Z(3)).
\]

**Proof.** From Lemma 2 it follows

\[
Z(a_1, z_1, \ldots, z_r, a_1, \ldots, a_r)Z(c_1, c_2, \ldots, c_r) = Z(a_1, c_1, \ldots, c_r, a_1, \ldots, a_r),
\]

\[
\vdots
\]

\[
Z(a_1, \ldots, a_{r-1}, a_1, a_2, c_1, \ldots, c_r) = Z^1 \cdots Z^r
\]

Hence, making the product of both sides of these equations, we have

\[
Z_{(a_1, \ldots, a_r)}Z_{(c_1, \ldots, c_r)} = Z^{(1)} \cdots Z^{(r)}, \quad (a_1, \ldots, a_r ∈ Z(3)).
\]

**Lemma 4.** \((V(1, 3) × \cdots × V(1, 3)) × S(1, 3) × \cdots × S(1, 3)S^* = V(r, 3) × S(r, 3)S^*\)

*(as \(S(r, 3)\)-scheme).

**Proof.** We denote by \(D₁\) \((Zc)\) the affine open subscheme \(z ∈ V(r, 3) × S(r, 3)S^*\)

\[
\begin{align*}
|Zc(z) ≠ 0 & \quad \text{and by } D₂(X^{(1)}_1 \cdots X^{(r)}_r) \text{ the affine open subscheme } \{u_1 × \cdots × u_r \in (V(1, 3) × \cdots × V(1, 3)) × S(1, 3) \cdots × S(1, 3)S^* | X^{(1)}_1(u_1) × \cdots × X^{(r)}_r(u_r) ≠ 0\},
\end{align*}
\]

\((X_α^{(i)})^{α = i, r, α ∈ Z(3)}\) is a system of quantities defined by

\[
\begin{align*}
\lambda(T_α^{(i)})X_{α}^{(i)}X_{α+1}^{(i)} - γ(T_α^{(i)}) &= (X_{α}^{(i)} + X_{α+1}^{(i)})^2 \quad (1 ≤ i ≤ r).
\end{align*}
\]

From Lemma 1 it follows

\[
\begin{align*}
\lambda(T^{(i)})X_{α}^{(i)}X_{α+1}^{(i)}X_{α}^{(i+1)} - γ(T^{(i)})X_{α}^{(i)}X_{α}^{(i+1)}X_{α}^{(i+2)} &= 0.
\end{align*}
\]
This shows that the map: \( Z_{(a_1, \ldots, a_r)} \to X_{(a_1, \ldots, a_r)} \) induces the injection morphism \( \psi \) of \( (V(1, 3) \times \cdots \times V(1, 3)) \times S(3)^{r*} \) into \( V(r, 3) \times S(r, 3)^{r*} \) such that \( \psi(D_r(X_{(a_1, \cdots, a_{r-1})} \times X_{(a_r)})) \subseteq D_r(Z_{(c_1, \cdots, c_r)}) \). We shall construct the inverse morphism of \( \psi \). Put \( Z_{a}^{(i)} = Z(c_1, \ldots, c_{i-1}, c_i, c_{i+1}, \ldots, c_r) \) \((1 \leq i \leq r; a \in (3))\). Then by virtue of Lemma 1 and 2 it follows

\[
Z_{a+b}^{(i)} = Z_{a}^{(i)} + bZ_{a}^{(i)} = Z(c_1, \ldots, c_{i-1}, a+b, c_{i+1}, \ldots, c_r) - Z(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_r) = 0,
\]

\[
Z_{a}^{(i)} + bZ_{a}^{(i)} - \mathcal{A}_1(T^{(i)}) \cdot \mathcal{A}_2(T^{(i)}) Z_{a}^{(i)} = Z(c_1, \ldots, c_{i-1}, a+b, c_{i+1}, \ldots, c_r) - Z(c_1, \ldots, c_{i-1}, a, c_{i+1}, \ldots, c_r) = 0.
\]

Therefore by virtue of Lemma 3 it follows that the map:

\[
X_{(a_1, \ldots, a_r)} \otimes Z_{(a_1, \ldots, a_r)} \to X_{(a_1, \ldots, a_r)}^{(i)} \cdots Z_{(a_1, \ldots, a_r)}^{(i)} \to 0
\]

induces the injective morphism \( \psi'_{(c_1, \ldots, c_r)} \) of \( D_r(Z_{(c_1, \cdots, c_r)}) \) into \( D_r(X_{(c_1, \ldots, a_r)}) \). These morphisms \( \psi \) and \( \psi'_{(c_1, \ldots, c_r)} \) are the inverse each other as \( S(r, 3) \)-morphisms between \( D_r(X_{(c_1, \ldots, a_r)}) \) and \( D_r(Z_{(c_1, \cdots, c_r)}) \). Since \( \psi \) is defined on \( (V(1, 3) \times \cdots \times V(1, 3)) \times S(3)^{r*} \), there exists an \( S(r, 3) \)-morphism \( \psi' \) such that \( \psi' | D_r(Z_{(c_1, \cdots, c_r)}) = \psi'_{(c_1, \ldots, c_r)} \). This completes the proof of Lemma 4.

3.3. We denote by \( M_{r^*}(r, 3)(Y, m) \) the \( R^*(r, 3) \)-submodule in \( R^*(r, 3)[Y] \) consisting of all the elements of degree \( m \), by \( M_{r^*}(r, 3)(X, m) \) the \( R^*(r, 3) \)-submodule in \( R^*(r, 3)[X] \) consisting of all the elements of degree \( m \) and by \( I_{r^*}(r, 3)(m) \) the \( R^*(r, 3) \)-submodule in the kernel of \( R^*(r, 3)[Y] \) onto \( R^*(r, 3)[X] \) consisting of all the elements of degree \( m \). For a point \( x \in S = S(r, 3) \) we mean by \( M_{0^*}(r, 3)(X, m) \), \( M_{0^*}(r, 3)(Y, m) \), \( I_{0^*}(r, 3)(m) \), \( M_{0^*}(r, 3)(Y, m) \), \( M_{0^*}(r, 3)(X, m) \), \( I_{0^*}(r, 3)(m) \) the tensor products

\[
M_{r^*}(r, 3)(X, m) \otimes R^*(r, 3)O_{S, x}, \quad M_{r^*}(r, 3)(X, m) \otimes R^*(r, 3)O_{S, x}, \quad I_{r^*}(r, 3) \otimes R^*(r, 3)O_{S, x}, \quad M_{0^*}(r, 3)(Y, m) \otimes O_{S, x}, \quad M_{0^*}(r, 3)(X, m) \otimes O_{S, x}, \quad I_{0^*}(r, 3)(m) \otimes O_{S, x},
\]

respectively. Then it follows the exact sequence

\[
0 \to I_{r^*}(r, 3)(m) \to M_{r^*}(r, 3)(Y, m) \to M_{r^*}(r, 3)(Y, m) \to 0
\]

(27)

\[
0 \to I_{0^*}(r, 3)(m) \to M_{0^*}(r, 3)(Y, m) \to M_{0^*}(r, 3)(X, m) \to 0
\]

(28)

\[
I_{0^*} \to M_{0^*}(Y, m) \to M_{0^*}(X, m) \to 0
\]
Lemma 4. Let $x_0$ be the generic point on $S(r, 3)$ and $y$ be any point on $S^*(r, 3)$. Then it follows:
\[ \text{rank}_{k_{s_0}} M_{k_{s_0}}(X, m) \leq \text{rank}_{k_y} M_{k_y}(X, m) \quad (m = 1, 2, 3, \ldots). \]

Proof. Since $O_{s, x_0}$ is the quotient field of $R^*(r, 3)$, it follows $k_{x_0} = O_{s, x_0}$ and the exact sequence
\[ 0 \to I_{k_{x_0}}(m) \to M_{k_{x_0}}(Y, m) \to M_{k_{x_0}}(X, m) \to 0. \]
Since $Y_a(a \in \mathbb{Z}^N(3))$ are indeterminates, it follows
\[ \text{rank}_{k_y} M_{k_y}(Y, m) = \text{rank}_{k_{x_0}} M_{k_{x_0}}(Y, m). \]
Then it is sufficient to prove the inequality
\[ \text{rank}_{k_{s_0}} I_{k_{s_0}}(m) \geq \text{rank}_{I_y}(m) \quad (m = 1, 2, 3, \ldots). \]
Let $L_1, L_2, \ldots, L_{N(m)}$ be all the monomials of degree $m$ in $Y_a(a \in \mathbb{Z}^N(3))$. Then there exists a matrix with coefficients in $R^*(r, 3)$:
\[ Q^{(m)} = (\omega_{ij}^{(m)}), \quad (1 \leq i \leq \lambda(m), 1 \leq j \leq N(m)) \]
such that
\[ \sum_{j=1}^{N(m)} \omega_{ij}^{(m)} L_j \quad (1 \leq i \leq \lambda(m)) \]
generates $I_{R^*(r, 3)}(m)$. Let $\mathfrak{p}_y$ be the prime ideal in $R^*(r, 3)$ corresponding to a point $y$ in $S^*(r, 3)$. Then it follows
\[ \text{rank}_{k_{s_0}} I_{k_{s_0}}(m) = \text{rank}_{k_{s_0}} Q^{(m)}, \quad \text{rank}_{k_y} I_{k_y}(m) = \text{rank}_{k_y}(Q^{(m)} \mod \mathfrak{p}_y), \]
\[ (m = 1, 2, 3, \ldots). \]
This implies
\[ \text{rank}_{k_{s_0}} I_{k_{s_0}}(m) \geq \text{rank}_{k_y} I_{k_y}(m), \quad (m = 1, 2, 3, \ldots). \]

Proposition 3. Let $x_0$ be the generic point on $S(r, 3)$. Then it follows
\[ V(r, 3) \times \mathbb{A}(r, 3) \times x_0 \cong A(r, 3) \times \mathbb{A}(r, 3) \times x_0. \]

Proof. Let $z_0^{(1)}$ be the generic point in $S(1, 3)$. Then $V(1, 3) \times \mathbb{A}(1, 3)$ is defined by the equation
\[ (T_0^2 + 2T_1 X_0 X_1 X_{-1} - (T_0 T_1)(X_0^2 + X_1^2 + X_{-1}^2)) = 0. \]
On the other hand the scheme $A(1, 3) \times \mathbb{A}(1, 3)$ is also defined by a cubic equation. This shows that Proposition 3 is true for $r = 1$. Let $z_0$ be the generic point in $S^*$
\[ = S^*(1, 3) \times \cdots \times S^*(1, 3). \]
Then by virtue of Lemma 3 we have
ON THE EXPLICITLY DEFINING RELATIONS OF ABELIAN SCHEMES OF LEVEL THREE

\[ V(r, 3) \times S(r, 3)z_0 = (V(1, 3) \times \cdots \times V(1, 3)) \times S(1, 3)x \times S(1, 3)z_0 \]

\[ = (V(1, 3) \times S(1, 3)z_0^{(1)}) \times \cdots \times (V(1, 3) \times S(1, 3)z_0^{(1)}) \]

\[ = (A(1, 3) \times S(1, 3)z_0^{(1)}) \times \cdots \times (A(1, 3) \times S(1, 3)z_0^{(1)}) \]

\[ = (A(1, 3) \times \cdots \times A(1, 3)) \times S(1, 3)z_0^{(1)} \times \cdots \times (A(1, 3) \times S(1, 3)z_0^{(1)}) \]

\[ = A(r, 3) \times S(r, 3)z_0. \]

From (22) it follows

\[ \text{rank}_{k_{z_0}}(X^{(1)}, m) = 3m, \quad (m = 1, 2, 3, \ldots). \]

From the above relation we have

\[ \text{rank}_{k_{z_0}}(X, m) = 3^r m', \quad (m = 1, 2, 3, \ldots). \]

Therefore from Lemma 4 it follows

\[ \text{rank}_{k_{z_0}}(X, m) \leq \text{rank}_{k_{z_0}}(X, m) \leq 3^r m', \quad (m = 1, 2, 3, \ldots). \]

On the other hand by virtue of Proposition 2 there exists an injection \( \rho^* \) of \( A(r, 3) \) into \( V(r, 3) \) as \( S(r, 3) \)-scheme. From Proposition 1 this implies

\[ \text{rank}_{k_{z_0}}(X, m) \geq 3^r m', \quad (m = 1, 2, 3, \ldots). \]

Hence we have the relations

\[ \text{rank}_{k_{z_0}}(X, m) = 3^r m', \quad \text{rank}_{k_{z_0}}(X, m) = N(m) - 3^r m' \]

\[ (m = 1, 2, 3, \ldots). \]

Let \( J_k \) be the ideal in \( k[[Y]] \) corresponding to the closed scheme \( \rho^* (A(r, 3)) \times S(r, 3)z_0 \) and \( J_k(m) \) the \( k_{z_0} \)-submodule in \( J_k \) consisting of all the elements of degree \( m \). Then by virtue of Proposition 1 and 2 it follows

\[ \text{rank} J_{k_{z_0}}(m) = N(m) - 3^r m', \quad \text{rank} J_{k_{z_0}}(m) \leq \text{rank} I_{k_{z_0}}(m), \]

\[ (m = 1, 2, 3, \ldots). \]

This implies \( J_{k_{z_0}} = I_{k_{z_0}} \). Namely we have the isomorphism between scheme \( V(r, 3) \times S(r, 3)z_0 \) and \( A(r, 3) \times S(r, 3)z_0 \).

Finally we shall state the main theorem:

**Theorem 1.** There exists an open subscheme \( U(r, 3) \) in \( S(r, 3) \) such that the \( U(r, 3) \)-scheme
§ 4. The explicite addition formula for theta functions of level three

4.1. The addition formula for theta functions of level three is simple and beautiful. First we shall introduce theta functions in \( U \otimes_R \cdots \otimes_R U \), where \( R = R(r, 3) \). We mean by a theta function of level \( n \) in \( U \otimes_R \cdots \otimes_R U \) with period \( Q \otimes_R \cdots \otimes_R Q \) a power series

\[
\varphi(U \otimes_R \cdots \otimes_R U) = \sum_{m_1, \ldots, m_l \in \mathbb{Z}} \lambda(m_1, \ldots, m_l) U(m_1)^{\varphi} \otimes_R \cdots \otimes_R U(m_l)^{\varphi}
\]

such that

\[
\varphi(Q(m_1) \otimes_R \cdots \otimes_R Q(m_l))(U \otimes_R \cdots \otimes_R U)) = Q(m_1, m_1)^{-n} \cdots Q(m_l, m_l)^{-n} (U(m_1)^{-n} \otimes_R \cdots \otimes_R U(m_l)^{-n})(U \otimes_R \cdots \otimes_R U)
\]

\((m_1, \ldots, m_l \in \mathbb{Z}^r).\)

Then, similarly as Proposition 1, the tensor products \( X_{a_1}(Q \otimes_R \cdots \otimes_R U) \) form a base of theta functions of level three in \( U \otimes_R \cdots \otimes_R U \) over the quotient field of \( R \). Similarly as Proposition 2 the tensor products \((X_{c^{+}a}(Q \otimes_R \cdots \otimes_R U) X_{a}^{-c^{+}a}(Q \otimes_R \cdots \otimes_R U)) \) form a base of theta functions of level six in \( U \otimes_R \cdots \otimes_R U \).

We mean by \( X_{a}(Q \otimes_R \cdots \otimes_R U) \) the power series

\[
\sum_{\mathbb{Z}^n} Q\left( m + \frac{a}{n}, m + \frac{a}{n} \right)^{n} \otimes_R U\left( m + \frac{a}{n} \right)^{n}
\]

and

\[
\sum_{\mathbb{Z}^n} Q\left( m + \frac{a}{n}, m + \frac{a}{n} \right)^{n} \otimes_R U\left( m + \frac{a}{n} \right)^{n}, \quad (a \in \mathbb{Z}^n).
\]

In these notation the addition formula is expressed as follows:

**Theorem 2. (The addition formula)**

\(4\) Starting with \( R_p = GF(p) [(T_{a})_{a \in \mathbb{Z}^n}] \) and \( R_p[(X_{a}(T \otimes_R \cdots \otimes_R U))]_{a \in \mathbb{Z}^n} \) we can get the result over \( GF(p) \) similar as that over \( \mathbb{Z} \) in § 2 (by the same method). This shows that \( U(r, 3) \times_p GF(p) \) is a non-empty open set.

---

https://doi.org/10.1017/S0027763000011958 Published online by Cambridge University Press
\begin{equation}
\Delta_1(T) X_{s_3,a+b}(Q | U \otimes_R U) X_{s_3,a-b}(Q | U \otimes_R U^{-1})
= \sum_{e^+,d^+ \in \mathbb{Z}(3)^+} \alpha_{e^+,d^+}(T) (X_{s_3,e^++a}(Q | U) X_{s_3,-c^+-d^+}(Q | U) \otimes_R (X_{s_3,d^++b}(Q | U) X_{s_3,-d^+-c^+}(Q | U)) \tag{29}
\end{equation}

\textit{Proof.} From the definitions it follows the relations

\begin{equation}
X_{s_3,a+b}(Q | U \otimes_R U) X_{s_3,a-b}(Q | U \otimes_R U^{-1})
= \sum_{m', m'' \in \mathbb{Z}} Q\left( m' + \frac{a+b}{3}, m' + \frac{a+b}{3} \right)^3 Q\left( m'' + \frac{a-b}{3}, m'' + \frac{a-b}{3} \right)^3
\end{equation}

\begin{equation}
\left( U\left( m' + \frac{a+b}{3} \right) \otimes_R U\left( m' + \frac{a+b}{3} \right) \right) \left( U\left( m'' + \frac{a-b}{3} \right) \otimes_R U\left( m'' + \frac{a-b}{3} \right) \right)
\end{equation}

\begin{equation}
= \sum_{m', m'' \in \mathbb{Z}} Q\left( m' + \frac{a+b}{3}, m' + \frac{a+b}{3} \right)^3 Q\left( m'' + \frac{a-b}{3}, m'' + \frac{a-b}{3} \right)^3
\end{equation}

\begin{equation}
U(3(m' + m'') + 2a)^2 \otimes_R U(3(m' - m'') + 2b)^3.
\end{equation}

This shows that $X_{s_3,a+b}(Q | U \otimes_R U) X_{s_3,a-b}(Q | U \otimes_R U^{-1})$ has an expansion

\begin{equation}
\sum_{m', m'' \in \mathbb{Z}} \lambda_{m' + m'}(Q \otimes_R U) U(m' + m')^6 \sum_{m'} U(m' + m')^6 \sum_{m''} U(m' - m'')^6.
\end{equation}

On the other hand from the difference equation (10) it follows

\begin{equation}
X_{s_3,a+b}(Q | Q \otimes_R U) X_{s_3,a-b}(Q | Q \otimes_R U^{-1})
= Q(m', m')^{-6} Q(m', m')^{-6} U(m')^{-12} \otimes_R U(m')^{-12} X_{s_3,a+b}(Q | U \otimes_R U) X_{s_3,a-b}(Q | U \otimes_R U^{-1}).
\end{equation}

Therefore we have

\begin{equation}
X_{s_3,a+b}(Q | U \otimes_R U) X_{s_3,a-b}(Q | U \otimes_R U^{-1})
= \sum_{e^+, d^+ \in \mathbb{Z}(3)^+} \lambda_{e^+, d^+} (X_{s_3,e^++a}(Q | U) X_{s_3,-c^+-d^+}(Q | U) \otimes_R (X_{s_3,d^++b}(Q | U) X_{s_3,-d^+-c^+}(Q | U)).
\end{equation}

Putting $U = Q\left( \frac{c}{3} \right)$ $(c \in \mathbb{Z}(3)^+)$, we have

\begin{equation}
\lambda_{e^+, d^+} = \Delta_1(T)^{-1} \alpha_{e^+, d^+}(T), \quad (e^+, d^+ \in \mathbb{Z}(3)^+).
\end{equation}

Reference


Mathematical Institute
Nagoya University

https://doi.org/10.1017/S0027763000011958 Published online by Cambridge University Press