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GORENSTEIN DIMENSIONS MODULO A REGULAR ELEMENT

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Abstract

Let R be a commutative ring. In this paper we study the behavior of Gorenstein homological dimensions of a homologically bounded R-complex under special base changes to the rings R_x and R/xR, where x is a regular element in R. Our main results refine some known formulae for the classical homological dimensions. In particular, we provide the Gorenstein counterpart of a criterion for projectivity of finitely generated modules, due to Vasconcelos.

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1. Introduction

Throughout this paper, R is a nontrivial commutative ring with a unit element and x is an element of R that is neither a zero divisor nor invertible. In [7], we investigated the relation between homological behavior of the ring R and those of rings R_x and R/xR. See, for instance, [7, Theorems 3.4 and 3.7]. It was also proved that for a complex of R-modules M, the following equalities hold:

$$\mathrm{fd}_R\, M = \max\{\mathrm{fd}_{R/xR}(R/xR \otimes^{\mathbf{L}}_R M), \mathrm{fd}_{R_x} M_x\}, \tag{1.1}$$

$$\mathrm{id}_R M = \max\{\mathrm{id}_{R/xR}(R/xR \otimes^{\mathbf{L}}_R M), \mathrm{id}_{R_x} M_x\}. \tag{1.2}$$

See [7, Theorem 3.2 and Theorem 4.2] for detailed statements.
In this paper we prove the Gorenstein counterparts of Equations (1.1) and (1.2). More precisely, we prove that for every homologically bounded complex M over a coherent ring R, the following equality holds:

$$\mathrm{Gfd}_R M = \max\{\mathrm{Gfd}_{\overline{R}}\overline{M}, \mathrm{Gfd}_{R_x} M_x\}, \tag{1.3}$$

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where $\overline{R}$ denotes the factor ring $R/(x)$ and $\overline{M}$ is a complex of $\overline{R}$-modules; see Section 2 for details. A similar formula holds for Gorenstein injective dimensions when R is noetherian with a dualizing complex; see Theorem 3.5.

In [8], Vasconcelos proved a criterion for projectivity of finitely generated modules, that is, it was proved that (see [8, Theorem 1.6]) when M is a finitely generated module over R and x is a nonzero divisor on both R and M, M is projective over R if and only if M/xM is projective over R/x and M_x is projective over R_x. It is therefore natural to ask whether the same criterion for Gorenstein projectivity is true. In fact, when M is finitely generated and R is noetherian, a very special case of Equation (1.3) gives an affirmative answer to this question; see Remark 3.3.

2. Prerequisites

In this short section, we fix our notation and prove some easy lemmas that will be used later. Throughout, R is a nontrivial commutative ring with a unit element and x is an element of R that is neither a zero divisor nor invertible. We sometimes write 'R-complex' in place of 'a complex of R-modules'. Complexes are graded homologically. Thus, an R-complex M has the form

$$\cdots \to M_{\ell+1} \xrightarrow{\partial^M_{\ell+1}} M_\ell \xrightarrow{\partial^M_\ell} M_{\ell-1} \to \cdots .$$

Modules are considered to be complexes concentrated in degree zero. We write ΣM for a complex with

$$(\Sigma M)_n = M_{n-1} \quad \text{and} \quad \partial^{\Sigma M} = -\partial^M.$$

The *supremum* and *infimum* of M are defined as follows:

$$\sup(M) = \sup\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(M) \neq 0\},$$
$$\inf(M) = \inf\{\ell \in \mathbb{Z} \mid \mathrm{H}_\ell(M) \neq 0\},$$

with the usual conventions that one sets $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

The derived category is written $\mathcal{D}(R)$ and the subscript '$\square$' signifies the homological boundedness condition. Thus, $\mathcal{D}_\square(R)$ denotes the full subcategory of $\mathcal{D}(R)$ of homologically bounded complexes.

For each R-complex M, we set

$$\overline{M} = R/xR \otimes^{\mathbf{L}}_R M.$$

Note that $\overline{M}$ is an R/xR-complex and, as an R-complex, it is quasi-isomorphic to the mapping cone of the homothety morphism $x_M : M \to M$. We also denote by $\overline{R}$ the ring R/xR.

Remark 2.1. Let M be a complex of R-modules. It follows from [7, Lemma 2.1] that:

(i) $\mathrm{H}(\overline{M})$ and $\mathrm{H}(M_x)$ are bounded if and only if $\mathrm{H}(M)$ is bounded;
(ii) $\mathrm{H}(\overline{M})$ and $\mathrm{H}(M_x)$ are trivial if and only if $\mathrm{H}(W)$ is trivial.

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2.1. Auslander and Bass classes. Recall that a complex $C \in \mathcal{D}^f_\square(R)$ is said to be *semidualizing* provided that the canonical map $R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism.

The *Auslander category* (see [1]) with respect to a semidualizing complex C is the full subcategory $\mathcal{A}_R(C)$ of $\mathcal{D}_\square(R)$ consisting of all R-complexes M such that $C \otimes^{\mathbf{L}}_R M \in \mathcal{D}_\square(R)$ and the canonical morphism $M \to \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R M)$ is an isomorphism.

Dually, the *Bass category* with respect to a semidualizing complex C is the full subcategory $\mathcal{B}_R(C)$ of $\mathcal{D}_\square(R)$ consisting of all R-complexes N such that $\mathbf{R}\mathrm{Hom}_R(C, N) \in \mathcal{D}_\square(R)$ and the canonical morphism $C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, N) \to N$ is an isomorphism.

Let C be a semidualizing complex. For R-complexes M and N, set

$$\triangle_C(M) = \mathrm{Cone}(M \longrightarrow \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R M))$$

and

$$\Lambda_C(N) = \mathrm{Cone}(C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, N) \longrightarrow N).$$

It is clear that the following bi-implications hold when M and N are homologically bounded:

$$M \in \mathcal{A}_R(C) \iff C \otimes^{\mathbf{L}}_R M \in \mathcal{D}_\square(R) \wedge \mathrm{H}(\triangle_C(M)) = 0$$

and

$$N \in \mathcal{B}_R(C) \iff \mathbf{R}\mathrm{Hom}_R(C, N) \in \mathcal{D}_\square(R) \wedge \mathrm{H}(\Lambda_C(N)) = 0.$$

Note also that for any semidualizing complex C,

$$\mathrm{fd}_R\, M \leq \infty \Rightarrow M \in \mathcal{A}_R(C)$$

and

$$\mathrm{id}_R N \leq \infty \Rightarrow N \in \mathcal{B}_R(C)$$

(see [1, 4.4]).

Lemma 2.2. *Let C be a semidualizing complex for R and $\varphi : R \to S$ a ring homomorphism such that S belongs to $\mathcal{A}_R(C)$. Then $C \otimes^{\mathbf{L}}_R S$ is a semidualizing complex for S.*

Proof. See [2, Theorem 5.1]. □

Corollary 2.3. *If C is a semidualizing complex for R, then $\overline{C}$ and C_x are semidualizing for $\overline{R}$ and R_x, respectively.*

Proposition 2.4. *Let C be a semidualizing complex for R and suppose that M is a homologically bounded complex of R-modules. Then:*

(i) $M \in \mathcal{A}_R(C) \iff \overline{M} \in \mathcal{A}_{\overline{R}}(\overline{C}) \wedge M_x \in \mathcal{A}_{R_x}(C_x)$;

(ii) $M \in \mathcal{B}_R(C) \iff \overline{M} \in \mathcal{B}_{\overline{R}}(\overline{C}) \wedge M_x \in \mathcal{B}_{R_x}(C_x)$.

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PROOF. We only prove (i). A dual argument proves (ii).

'⇐': Since $\mathrm{H}(\overline{M})$ and $\mathrm{H}(M_x)$ are bounded, (2.1) shows that $\mathrm{H}(M)$ is bounded. Similarly, $H(C \otimes_R^{\mathbf{L}} M)$ is also bounded. On the other hand, $\mathrm{H}(\Delta_{\overline{C}}(\overline{M})) = H(\overline{\Delta_C(M)})$ and $\mathrm{H}(\Delta_{C_x} M_x) = H((\Delta_C M)_x)$ are both trivial and so is $H(\Delta_C M)$ by (2.1). Therefore, $M \in \mathcal{A}_R(C)$.

'⇒': This follows from [2, Proposition 5.8]. □

REMARK 2.5. Recall that a *dualizing complex* is a semidualizing complex with finite injective dimension. It follows from Corollary 2.3 and [7, Theorem 4.2] that if D is a dualizing complex for R, then D_x and $\overline{D}$ are dualizing for R_x and $\overline{R}$, respectively.

3. Main results

It is well established that Gorenstein homological dimensions refine many results in classical homological theory of modules. It is also believed that any result in the classical theory has a Gorenstein counterpart. Our main results are the Gorenstein counterparts of some formulae given in [7].

The literature on Gorenstein homological algebra is rich and extensive. Thus, recollecting the basic definitions and facts (in this short paper) seemed to us out of place. We quote here just what we need in establishing our formulae. The reader is referred to the literature for more information. See for example [1, 4, 6].

In [4], Christensen *et al.* showed that the Gorenstein injective dimension $\mathrm{Gid}_R M$ of a homologically bounded complex M can be computed by the following formula:

$$\mathrm{Gid}_R M = \sup\{-\sup \mathbf{R}\mathrm{Hom}_R(U, M) - \sup(U) \mid U \in \mathcal{I}(R) \wedge \mathrm{H}(U) \neq 0\}$$

and, when R is coherent,

$$\mathrm{Gfd}_R M = \sup\{\sup(U \otimes_R^{\mathbf{L}} M) - \sup U \mid U \in \mathcal{I}(R) \wedge \mathrm{H}(U) \neq 0\},$$

where $\mathcal{I}(R)$ denotes the class of all R-complexes with finite injective dimension.

LEMMA 3.1. *Let $\varphi : R \to S$ be a ring homomorphism of coherent rings with* $\mathrm{fd}_R S < \infty$. *Then, for any homologically bounded R-complex M,*

$$\mathrm{Gfd}_S(S \otimes_R^{\mathbf{L}} M) \leq \mathrm{Gfd}_R M.$$

PROOF. By assumption, the forgetful functor $\mathcal{D}(S) \to \mathcal{D}(R)$ gives an embedding $\mathcal{I}(S) \to \mathcal{I}(R)$. For each S-complex U with finite injective dimension,

$$\sup(U \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} M)) - \sup(U) = \sup(U \otimes_R^{\mathbf{L}} M) - \sup(U).$$

Thus,

$$\begin{aligned}\mathrm{Gfd}_S(S \otimes_R^{\mathbf{L}} M) &= \sup\{\sup(U \otimes_S^{\mathbf{L}} (S \otimes_R^{\mathbf{L}} M)) - \sup(U) \mid U \in \mathcal{I}(S) \wedge \mathrm{H}(U) \neq 0\} \\ &= \sup\{\sup(U \otimes_R^{\mathbf{L}} M) - \sup(U) \mid U \in \mathcal{I}(S) \wedge \mathrm{H}(U) \neq 0\} \\ &\leq \sup\{\sup(U \otimes_R^{\mathbf{L}} M) - \sup(U) \mid U \in \mathcal{I}(R) \wedge \mathrm{H}(U) \neq 0\} \\ &= \mathrm{Gfd}_R M,\end{aligned}$$

where the inequality holds because U ranges over two different classes, one of which is larger than the other. □

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Theorem 3.2. *Let R be a coherent ring and M a homologically bounded complex. Then*

$$\mathrm{Gfd}_R M = \max\{\mathrm{Gfd}_{\overline{R}}\overline{M}, \mathrm{Gfd}_{R_x} M_x\}.$$

If every flat R-module has finite projective dimension and M has finitely presented homologies, then

$$\mathrm{Gpd}_R M = \max\{\mathrm{Gpd}_{\overline{R}}\overline{M}, \mathrm{Gpd}_{R_x} M_x\}.$$

Proof. '$\leq$': For each R-complex U,

$$\mathrm{H}(\overline{U \otimes^{\mathbf{L}}_R M}) \cong \mathrm{H}(\overline{U} \otimes^{\mathbf{L}}_{\overline{R}} \overline{M})$$

and

$$\mathrm{H}(U \otimes^{\mathbf{L}}_R M)_x \cong \mathrm{H}(U_x \otimes^{\mathbf{L}}_{R_x} M_x).$$

Thus, it follows from [7, Lemma 2.2] that

$$\begin{aligned}\sup(U \otimes^{\mathbf{L}}_R M) - \sup(U) &= \max\{\sup(\overline{U \otimes^{\mathbf{L}}_R M}) - 1, \sup(U \otimes^{\mathbf{L}}_R M)_x\} - \sup U\\ &= \max\{\sup(\overline{U} \otimes^{\mathbf{L}}_{\overline{R}} \overline{M}) - 1 - \sup U, \sup(U_x) \otimes^{\mathbf{L}}_{R_x} M_x - \sup U\}\\ &\leq \max\{\sup(\overline{U} \otimes^{\mathbf{L}}_{\overline{R}} \overline{M}) - \sup \overline{U}, \sup(U_x \otimes^{\mathbf{L}}_R M_x) - \sup U_x\}\\ &\leq \max\{\mathrm{Gfd}_{\overline{R}}\overline{M}, \mathrm{Gfd}_{R_x} M_x\}.\end{aligned}$$

The other inequality '$\geq$' follows from Lemma 3.1.

For the last assertion, see [4, Theorem 3.8(b)]. □

Remark 3.3. Let R be a coherent ring and M be an R-module. When x is a nonzero divisor on M, $\overline{M}$ and M/xM are indistinguishable in the derived category $\mathcal{D}(R)$. Thus, the previous theorem shows that M is a Gorenstein flat R-module if and only M/xM and M_x are Gorenstein flat modules over the rings $\overline{R}$ and R_x, respectively. Similarly, one has a criterion for Gorenstein projectivity of finitely presented modules. In particular, if M is finitely generated and R is noetherian, then M is Gorenstein projective if and only if M/xM and M_x are Gorenstein projective over $\overline{R}$ and R_x, respectively. This is the Gorenstein counterpart of a result of Vasconcelos [8, Theorem 1.6], where he proved the criterion for projectivity without assuming that R is noetherian. It is therefore natural to ask about the validity of the criterion (respectively the formulae for Gfd and Gpd) without any condition on the commutative ring R.

The following properties of Gorenstein injective dimension are used in proof of the next theorem.

Remark 3.4. For a homologically bounded complex M over a noetherian ring R with dualizing complex D, $\mathrm{Gid}_R M$ is finite if and only if M belongs to $\mathcal{B}_R(D)$. Also, if $\mathrm{Gid}_R M$ happens to be finite, then

$$\mathrm{Gid}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec}(R)\};$$

see [4, Theorem 4.4] and [5, Theorem 2.2].

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Theorem 3.5. *Let R be a noetherian ring with dualizing complex D. The following equality holds for any homologically bounded complex M:*

$$\mathrm{Gid}_R M = \max\{\mathrm{Gid}_{\overline{R}}\overline{M} + 1, \mathrm{Gid}_{R_x} M_x\}.$$

Proof. By Remarks 3.4 and 2.5 and Proposition 2.4, we may assume that both sides are finite. To prove '≤', choose a prime ideal $\mathfrak{p}$ such that

$$\mathrm{Gid}_R M = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}.$$

We divide the proof into two cases.

Case I. If x does not belong to $\mathfrak{p}$, set $\mathfrak{q} = \mathfrak{p}_x \in \operatorname{Spec}(R_x)$. Then

$$\begin{aligned}\mathrm{Gid}_R M &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}\\ &= \operatorname{depth} (R_x)_{\mathfrak{q}} - \operatorname{width}_{(R_x)_{\mathfrak{q}}} (M_x)_{\mathfrak{q}}\\ &\le \mathrm{Gid}_{R_x} M_x.\end{aligned}$$

Therefore, in this case, the desired inequality holds.

Case II. If $x \in \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}/(x) \in \operatorname{Spec}(\overline{R})$ and

$$\begin{aligned}\mathrm{Gid}_R M &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}\\ &= \operatorname{depth} \overline{R_{\mathfrak{q}}} + 1 - \operatorname{width}_{\overline{R_{\mathfrak{q}}}} (\overline{M})_{\mathfrak{q}}\\ &\le \mathrm{Gid}_{\overline{R}}\overline{M} + 1.\end{aligned}$$

For the other inequality '≥', choose $\mathfrak{q} \in \operatorname{Spec}(\overline{R})$ such that $\mathrm{Gid}_{\overline{R}}\overline{M} = \operatorname{depth}_R(\overline{R})_{\mathfrak{q}} - \operatorname{width}_{(\overline{R})_{\mathfrak{q}}}(\overline{M})_{\mathfrak{q}}$. There exists $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p}/(x) = \mathfrak{q}$ and

$$\begin{aligned}\mathrm{Gid}_{\overline{R}}\overline{M} &= \operatorname{depth}_R(\overline{R})_{\mathfrak{q}} - \operatorname{width}_{(\overline{R})_{\mathfrak{q}}}(\overline{M})_{\mathfrak{q}}\\ &= \operatorname{depth} R_{\mathfrak{p}} - 1 - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}\\ &\le \mathrm{Gid}_R M - 1.\end{aligned}$$

It remains to prove $\mathrm{Gid}_R M \ge \mathrm{Gid}_{R_x} M_x$. This can be proved in the exact same manner as in [1, Proposition 6.2.13]. □

Remark 3.6. Restricted homological dimensions for complexes are defined in [3]. Exactly the same argument as given in the proof of Theorem 3.2 shows that the equality

$$\mathrm{Rfd}_R M = \max\{\mathrm{Rfd}_{\overline{R}}\overline{M}, \mathrm{Rfd}_{R_x} M_x\}$$

hold for each homologically bounded complex M. Turning to the restricted injective dimensions, it is natural to ask if they satisfy the same equality as the one given in Theorem 3.5. The method used in our proof of Theorem 3.5 needs a Chouinard-type formula. But, to the best of the authors' knowledge, this formula only holds under some restricting hypotheses (see [3, Remark 5.12]).

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