12
Strong coupling and string duality

One of the most striking results of the mid-1990s was the realisation that all of the superstring theories are in fact dual to one another at strong coupling[149]. This also brought eleven dimensional supergravity into the picture and started the search for M-theory, the dynamical theory within which all of those theories would fit as various effective descriptions of perturbative limits. All of this is referred to as the 'Second Superstring Revolution'. Every revolution is supposed to have a hero or heroes. We shall consider branes to be cast in that particular role, since they (and D-branes especially) supplied the truly damning evidence of the strong coupling fate of the various string theories.

We shall discuss aspects of this in the present section. We simply study the properties of D-branes in the various string theories, and then trust to that fact that as they are BPS states, many of these properties will survive at strong coupling.

12.1 Type IIB/type IIB duality

12.1.1 D1-brane collective coordinates

Let us first study the D1-brane. This will be appropriate to the study of type IIB and the type I string by Ω-projection. Its collective dynamics as a BPS soliton moving in flat ten dimensions is captured by the 1+1 dimensional world-volume theory, with 16 or 8 supercharges, depending upon the theory we are in. (See figure 12.1(a).)

It is worth first setting up a notation and examining the global symmetries. Let us put the D1-brane to lie along the x^1 direction, as we will do many times in what is to come. This arrangement of branes breaks the

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Lorentz group up as follows:

$$SO(1,9) \supset SO(1,1)_{01} \times SO(8)_{2-9}. \tag{12.1}$$

Accordingly, the supercharges decompose under (12.1) as

$$\mathbf{16} = \mathbf{8}_+ \oplus \mathbf{8}_- \tag{12.2}$$

where $\pm$ subscripts denote a chirality with respect to $SO(1,1)$.

For the 1–1 strings, there are eight Dirichlet–Dirichlet (DD) directions, the Neveu–Schwarz (NS) sector has zero point energy $-1/2$. The massless excitations form vectors and scalars in the 1+1 dimensional model. For the vectors, the Neumann–Neumann (NN) directions give a gauge field A^μ. Now, the gauge field has no local dynamics, so the only contentful bosonic excitations are the transverse fluctuations. These come from the eight Dirichlet–Dirichlet (DD) directions x^m, $m = 2,\ldots,9$, and are

$$\phi^m(x^0, x^1): \quad \lambda_\phi \psi^m_{-\frac{1}{2}}|0\rangle. \tag{12.3}$$

The fermionic states ξ from the Ramond (R) sector (with zero point energy 0, as always) are built on the vacua formed by the zero modes $\psi^i_0, i{=}0,\ldots,9$. This gives the initial $\mathbf{16}$. The GSO projection acts on the vacuum in this sector as:

$$(-1)^F = e^{i\pi(S_0+S_1+S_2+S_3+S_4)}. \tag{12.4}$$

A left- or right-moving state obeys $\Gamma^0\Gamma^1\xi_\pm = \pm\xi_\pm$, and so the projection onto $(-1)^F\xi{=}\xi$ says that left- and right-moving states are odd and (respectively) even under $\Gamma^2\ldots\Gamma^9$, which is to say that they are either in the $\mathbf{8_s}$ or the $\mathbf{8_c}$. So we see that the GSO projection simply correlates world-sheet chirality with spacetime chirality: ξ_- is in the $\mathbf{8}_c$ of $SO(8)$ and ξ_+ is in the $\mathbf{8}_s$.

So we have seen that for a D1-brane in type IIB string theory, the right-moving spinors are in the $\mathbf{8_s}$ of $SO(8)$, and the left-moving spinors in the $\mathbf{8_c}$. These are the same as the fluctuations of a fundamental IIB string, in static gauge[26], and here spacetime supersymmetry is manifest. (It is in 'Green–Schwarz' form[108].) There, the supersymmetries Q_α and $\tilde{Q}_\alpha$ have the same chirality. Half of each spinor annihilates the F-string and the other half generates fluctuations. Since the supersymmetries have the same $SO(9,1)$ chirality, the $SO(8)$ chirality is correlated with the direction of motion.

So far we have been using the string metric. We can switch to the Einstein metric, $g^{(\mathrm{E})}_{\mu\nu} = e^{-\Phi/2}g^{(\mathrm{S})}_{\mu\nu}$, since in this case gravitational action

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has no dependence on the dilaton, and so it is invariant under duality. The tensions in this frame are:

$$\begin{aligned} &\text{F-string:} \quad g_s^{1/2}/2\pi\alpha' \\ &\text{D-string:} \quad g_s^{-1/2}/2\pi\alpha'. \end{aligned} \tag{12.5}$$

Since these are BPS states, we are able to trust these formulae at arbitrary values of g_s.

Let us see what interpretation we can make of these formulae. At weak coupling the D-string is heavy and the F-string tension is the lightest scale in the theory. At strong coupling, however, the D-string is the lightest object in the theory (a dimensional argument shows that the lowest dimensional branes have the lowest scale[150]), and it is natural to believe that the theory can be reinterpreted as a theory of weakly coupled D-strings, with $g_s' = g_s^{-1}$. One cannot prove this without a non-perturbative definition of the theory, but quantising the light D-string implies a large number of the states that would be found in the dual theory, and self-duality of the IIB theory seems by far the simplest interpretation – given that physics below the Planck energy is described by some specific string theory, it seems likely that there is a unique extension to higher energies. This agrees with the duality deduced from the low energy action and other considerations[149, 164]. In particular, the NS–NS and R–R two-form potentials, to which the D- and F-strings respectively couple, are interchanged by this duality.

This duality also explains our remark about the strong and weak coupling limits of the three string junction depicted in figure 11.2. The roles of the D- and F-strings are swapped in the $g_s \to 0, \infty$ limits, which fits with the two limiting values $\alpha \to \pi/2, 0$.

12.1.2 S-duality and $SL(2,\mathbb{Z})$

The full duality group of the $D = 10$ type IIB theory is expected to be $SL(2,\mathbb{Z})$[151, 153]. This relates the fundamental string not only to the R–R string but to a whole set of strings with the quantum numbers of p F-strings and q D-strings for p and q relatively prime[133]. The bound states found in section 11.3 are just what is required for $SL(2,\mathbb{Z})$ duality[26]. As the coupling and the R–R scalar are varied, each of these strings becomes light at the appropriate point in moduli space. We shall study this further in section 16.1, on the way to uncovering ‘F-theory’, a tool for generating very complicated type IIB backgrounds by geometrising the $SL(2,\mathbb{Z})$ symmetry.

12.2 $SO(32)$ Type I/heterotic duality

12.2.1 D1-brane collective coordinates

Let us now consider the D1-brane in the type I theory. We must modify our previous analysis in two ways. First, we must project onto Ω-even states.

As in section 2.6, the $U(1)$ gauge field A is in fact projected out, since ∂_{t} is odd under Ω. The normal derivative ∂_{n}, is even under Ω, and hence the Φ^m survive. Turning to the fermions, we see that Ω acts as $e^{i\pi(S_1+S_2+S_3+S_4)}$ and so the left-moving $\mathbf{8_c}$ is projected out and the right-moving $\mathbf{8_s}$ survives.

Recall that D9-branes must be introduced after doing the Ω projection of the type IIB string theory. These are the $SO(32)$ Chan–Paton factors. This means that we must also include the massless fluctuations due to strings with one end on the D1-brane and the other on a D9-brane (see figure 12.1(b)). The zero point energy in the NS sector for these states is 1/2, and so there is way to make a massless state. The R sector has zero point energy zero, as usual, and the ground states come from excitations in the x^0, x^1 direction, since it is in the NN sector that the modes are integer. The GSO projection $(-)^F = \Gamma^0\Gamma^1$ will project out one of these, λ_-, while the right-moving one will remain. The Ω projection simply relates 1–9 strings to 9–1 strings, and so places no constraint on them. Finally, we should note that the 1–9 strings, as they have one end on a D9-brane, transform as vectors of $SO(32)$.

Now, by the argument that we saw in the case of the type IIB string, we should deduce that this string becomes a light fundamental string in some dual string theory at strong coupling. We have seen such a string before in section 7.2. It is the 'heterotic' string, which has $(0,1)$ spacetime supersymmetry, and a left-moving family of 32 fermions transforming as the $\mathbf{32}$ of $SO(32)$. They carry a current algebra which realises the

(a) (b)

Fig. 12.1. D1-branes (a) in type IIB theory its fluctuations are described by 1–1 strings; (b) in type I string theory, there are additional contributions from 1–9 strings.

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$SO(32)$ as a spacetime gauge symmetry. The other ten dimensional heterotic string, with gauge group $E_8 \times E_8$, has a strong coupling limit which we will examine shortly, using the fact that upon compactifying on a circle, the two heterotic string theories are perturbatively related by T-duality (see section 8.1.3)[173, 174].

We have obtained the $SO(32)$ string here with spacetime supersymmetry and with a left-moving current algebra $SO(32)$ in fermionic form[162]. As we learned in section 7.2, we can bosonise these into the 16 chiral bosons which we then used to construct the heterotic string in the first instance. This also fits rather well with the fact that we had already noticed that we could have deduced that such a string theory might exist just by looking at the supergravity sector in section 7.3. This is just how type I/heterotic duality was deduced first[153, 164] and then D-brane constructions were used to test it more sharply[162]. We shall see that considerations of the strong coupling limit of various other string theories will again point to the existence of the heterotic string. We have already seen hints of that in chapter 7, as discussed in insert 7.5. Of course, the heterotic strings were discovered by direct perturbative construction, but it is amusing to thing that, in another world, they may be discovered by string duality.

We end with a brief remark about some further details that we shall not pursue. Recall that it was mentioned at the end of section 7.2, the fermionic $SO(32)$ current algebra requires a GSO projection. By considering a closed D1-brane we see that the Ω projection removes the $U(1)$ gauge field, but in fact allows a discrete gauge symmetry: a holonomy ± 1 around the D1-brane. This discrete gauge symmetry is the GSO projection, and we should sum over all consistent possibilities. The heterotic strings have spinor representations of $SO(32)$, and we need to be able to make them in the Type I theory, in order for duality to be correct. In the R sector of the discrete D1-brane gauge theory, the 1–9 strings are periodic. The zero modes of the fields Ψ^i, representing the massless 1–9 strings, satisfy the Clifford algebra $\{\Psi_0^i, \Psi_0^j\} = \delta^{ij}$, for $i, j = 1, \ldots, 32$, and so just as for the fundamental heterotic string we get spinors $\mathbf{2^{31}} \oplus \overline{\mathbf{2^{31}}}$. One of them is removed by the discrete gauge symmetry to match the spectrum with a single massive spinor which we uncovered directly using lattices in section 7.2.1.

12.3 Dual branes from 10D string–string duality

There is an instructive way to see how the D-string tension turns into that of an F-string. In terms of supergravity fields, part of the duality

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transformation (7.46) involves

$$G_{\mu\nu} \to e^{-\tilde{\Phi}} \tilde{G}_{\mu\nu}, \qquad \Phi \to -\tilde{\Phi}, \tag{12.6}$$

where the quantities on the right, with tildes, are in the dual theory. This means that in addition to $g_{\rm s} = \tilde{g}_{\rm s}^{-1}$, for the relation of the string coupling to the dual string coupling, there is also a redefinition of the string length, via

$$\alpha' = \tilde{g}_{\rm s}^{-1} \tilde{\alpha}',$$

which is the same as

$$\alpha' g_{\rm s}^{-1} = \tilde{\alpha}'.$$

Starting with the D-string tension, these relations give:

$$\tau_1 = \frac{1}{2\pi\alpha' g_{\rm s}} \to \frac{1}{2\pi\tilde{\alpha}'} = \tau_1^{\rm F},$$

precisely the tension of the fundamental string in the dual string theory, measured in the correct units of length.

One might understandably ask the question about the fate of other branes under S-dualities[165]. For the type IIB's D3-brane:

$$\tau_3 = \frac{1}{(2\pi)^3 \alpha'^2 g_{\rm s}} \to \frac{1}{(2\pi)^3 \tilde{\alpha}'^2 \tilde{g}_{\rm s}} = \tau_3,$$

showing that the dual object is again a D3-brane. For the D5-brane, in either type IIB or type I theory:

$$\tau_5 = \frac{1}{(2\pi)^5 \alpha'^3 g_{\rm s}} \to \frac{1}{(2\pi)^5 \tilde{\alpha}'^3 \tilde{g}_{\rm s}^2} = \tau_5^{\rm F}.$$

This is the tension of a fivebrane which is *not* a D5-brane. This is interesting, since for both dualities, the R–R two-form $C^{(2)}$ is exchanged for the NS–NS two-form $B^{(2)}$, and so this fivebrane is magnetically charged under the latter. It is in fact the magnetic dual of the fundamental string. Its $g_{\rm s}^{-2}$ behaviour identifies it as a soliton of the NS–NS sector.

So we conclude that there exists in both the type IIB and $SO(32)$ heterotic theories such a brane, and in fact such a brane can be constructed directly as a soliton solution. They should perhaps be called 'F5-branes', since they are magnetic duals to fundamental strings or 'F1-branes', but this name never stuck. They go by various names like 'NS5-brane', since they are made of NS–NS sector fields, or 'solitonic fivebrane', and so on. As they are constructed completely out of closed string fields, T-duality along a direction parallel to the brane does not change its dimensionality, as would happen for a D-brane. We conclude therefore that they also exist in the T-dual type IIA and $E_8 \times E_8$ string theories. Let us study them a bit further.

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12.3.1 The heterotic NS-fivebrane

For the heterotic cases, the soliton solution also involves a background gauge field, which is in fact an instanton. This follows from the fact that in type I string theory, the D5-brane is an instanton of the D9-brane gauge fields as we saw with dramatic success in section 9.2. We shall have even more to say about this later, when we uncover more properties of how to probe branes with branes. As we saw there and in chapter 7, through equation (7.38), $\mathrm{Tr}(F \wedge F)$ and $\mathrm{tr}(R \wedge R)$ both magnetically source the two-form potential $C_{(2)}$, since by taking one derivative:

$$d\tilde{G}^{(3)} = -\frac{\alpha'}{4}\Big[\mathrm{Tr}F^2 - \mathrm{tr}R^2\Big].$$

By strong/weak coupling duality, this must be the case for the NS–NS two form $B_{(2)}$. To leading order in α', we can make a solution of the heterotic low energy equation of motion with these clues quite easily as follows. Take for example an $SU(2)$ instanton (the very object described in insert 9.4 when we reminded ourselves about non-trivial second Chern class) embedded in an $SU(2)$ subgroup of the $SO(4)$ in the natural decomposition: $SO(32) \supset SO(28) \times SO(4)$. As we said, this will source some dH, which in turn will source the metric and the dilaton. In fact, to leading order in α', the corrections to the metric away from flat space will not give any contribution to $\mathrm{tr}(R \wedge R)$, which has more derivatives than $\mathrm{Tr}(F \wedge F)$, and is therefore subleading in this discussion. The result should be an object which is localised in $\mathbb{R}^4$ with a finite core size (the 'dressed' instanton), and translationally invariant in the remaining $5+1$ directions. This deserves to be called a fivebrane. A solution realising this logic can be found, and it can be written as[72, 73]:

$$\begin{aligned}
ds^2 &= -dt^2 + (dx^1)^2 + \cdots + (dx^5)^2 + e^{2\Phi}\left(dr^r + r^2 d\Omega_3^2\right)\\
e^{2\Phi} &= g_{\rm s}^2\left(1 + \alpha'\frac{(r^2+2\rho^2)}{(r^2+\rho^2)^2} + O(\alpha'^2)\right), \quad H_{\mu\nu\lambda} = -\epsilon_{\mu\nu\lambda}{}^{\sigma}\partial_\sigma\Phi\\
A_\mu &= \left(\frac{r^2}{r^2+\rho^2}\right) g^{-1}\partial_\mu g, \quad g = \frac{1}{r}\begin{pmatrix} x^6 + ix^7 & x^8 + ix^9\\ x^8 - ix^9 & x^6 - ix^7\end{pmatrix},
\end{aligned} \tag{12.7}$$

showing its structure as an $SU(2)$ instanton localised in x^6, x^7, x^8, x^9, with core size ρ. As before, r^2 is the radial coordinate, and $d\Omega_3^2$ is a metric on a round S^3.

Once we have deduced the existence of this object in the $SO(32)$ heterotic string, it is straightforward to see that it must exist in the $E_8 \times E_8$ heterotic string too. We simply compactify on a circle in a world-volume direction where there is no structure at all. Shrinking it away takes us to the other heterotic theory, with an NS5-brane of precisely the same sort of structure as above. Alternatively, we could have just constructed the fivebrane directly using the ideas above without appealing to T-duality at all.

12.3.2 The type IIA and type IIB NS5-brane

As already stated, similar reasoning leads one to deduce that there must be an NS5-brane in type II string theory*. We can deduce its supergravity fields by using the ten dimensional S-duality transformations to convert the case $p = 5$ of equations (10.38), (10.39), to give[72, 73]:

$$\begin{aligned} ds^2 &= -dt^2 + (dx^1)^2 + \cdots + (dx^5)^2 + \tilde{Z}_5 \left(dr^r + r^2 d\Omega_3^2 \right) \\ e^{2\Phi} &= g_s^2 \tilde{Z}_5 = g_s^2 \left(1 + \frac{\alpha' N}{r^2} \right), \\ B_{(6)} &= (\tilde{Z}_5^{-1} - 1) g_s dx^0 \wedge \cdots \wedge dx^5. \end{aligned} \tag{12.8}$$

This solution has N units of the basic magnetic charge of $B_{(2)}$, and is a point in x^6, x^7, x^8, x^9. Note that the same sort of transformation will give a solution for the fields around a fundamental IIB string, by starting with the $p = 1$ case of (10.38)[163, 164]. We shall do this in chapter 16.

For the same reasons as for the heterotic string, once we have made an NS5-brane for the type IIB string, it is easy to see that we can use T-duality along a world-volume direction (where the solution is trivial) in order to make one in the type IIA string theory as well.

A feature worth considering is the world-volume theory describing the low energy collective motions of these type II branes. This can be worked out directly, and string duality is consistent with the answer: from the duality, we can immediately deduce that the type IIB's NS5-brane must have a vector multiplet, just like the D5-brane. Also as with D5-branes, there is enhanced $SU(N)$ gauge symmetry when N coincide[160], the extra massless states being supplied by light D1-branes stretched between them. (See figure 12.2.) The vector multiplet can be read off from table 7.1 as $(\mathbf{2},\mathbf{2})+4(\mathbf{1},\mathbf{1})+2(\mathbf{1},\mathbf{2})+2(\mathbf{2},\mathbf{1})$. There are four scalars, which are the four transverse positions of the brane in ten dimensions. The fermionic content can be seen to be manifestly non-chiral giving a $(1, 1)$ supersymmetry on the world-volume.

* In the older literature, it is sometimes called a 'symmetric fivebrane', after its left-right symmetric σ-model description, in constrast to that of the heterotic NS5-brane.

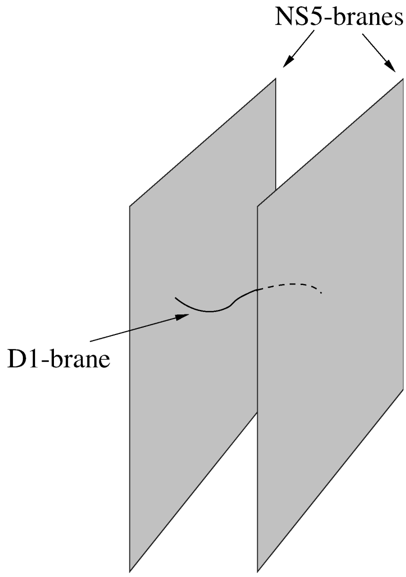

Fig. 12.2. D1-branes stretched between NS5-branes in type IIB string theory will give extra massless vectors when the NS5-branes coincide.

For the type IIA NS5-brane, things are different. Following the T-duality route mentioned above, it can be seen that the brane actually must have a chiral (0,2) supersymmetry. So it cannot have a vector multiplet any more, and instead there is a six dimensional tensor multiplet on the brane. So there is a two-form potential instead of a one-form potential, which is interesting. The tensor multiplet can be read off from table 7.1 as $(\mathbf{1}, \mathbf{3}) + 5(\mathbf{1}, \mathbf{1}) + 4(\mathbf{2}, \mathbf{1})$, with a manifestly chiral fermionic content. There are now *five* scalars, which is suggestive, since in their interpretation as collective coordinates for transverse motions of the brane, there is an implication of an *eleventh* direction. This extra direction will become even more manifest in section 12.4.

There is an obvious $U(1)$ gauge symmetry under the transformation $B^+_{(2)} \to B^+_{(2)} + d\Lambda_{(1)}$, and the question arises as to whether there is a non-Abelian generalisation of this when many branes coincide. On the D-brane side of things, it is clear how to construct the extra massless states as open strings stretched between the branes whose lengths can shrink to zero size in the limit. Here, there is a similar, but less well-understood phenomenon. The tensor potential on the world-volume is naturally sourced by six dimensional strings, which are in fact the ends of open D2-branes ending on the NS5-branes. The mass or tension of these strings is set by the amount that the D2-branes are stretched between

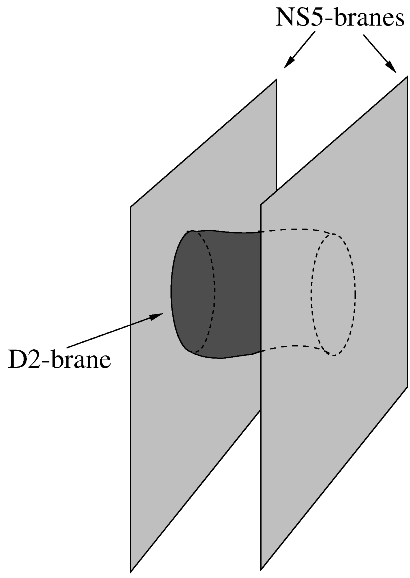

Fig. 12.3. D2-branes stretched between NS5-branes in type IIA string theory will give extra massless self-dual tensors when the NS5-branes coincide.

two NS-branes, by precise analogy with the D-brane case. So we are led to the interesting case that there are tensionless strings when many NS5-branes coincide, forming a generalised enhanced gauge tensor multiplet. (See figure 12.3.) These strings are not very well understood, it must be said. They are not sources of a gravity multiplet, and they appear not to be weakly coupled in any sense that is understood well enough to develop an intrinsic perturbation theory for them[†].

However, the theory that they imply for the branes is apparently well-defined. The information about how it works fits well with the dualities to better understood things, as we have seen here, and as we will see later when we shall say a little more about it in chapter 18, since it can be indirectly defined using the AdS/CFT correspondence. It should be noted that we do not have to use D-branes or duality to deduce a number of the features mentioned above for the world-volume theories on the NS5-branes. That there is either a (1,1) vector multiplet or a (0,2) tensor multiplet was first uncovered by direct analysis of the collective dynamics of the NS5-branes as supergravity solitons in the type II theories[159].

[†] The cogniscenti will refer to theories of non-Abelian 'gerbes' at this point. The reader should know that these are not small furry pets, but well-defined mathematical objects. They are (reportedly) a generalisation of the connection on a vector bundle, appropriate to two-form gauge fields[90].

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12.4 Type IIA/M-theory duality

Let us turn our attention to the type IIA theory and see if at strong coupling we can see signs of a duality to a useful weakly coupled theory. In doing this we will find that there are even stranger dualities than just a string–string duality (which is strange and beautiful enough as it is!), but in fact a duality which points us firmly in the direction of the unexplored and the unknown.

12.4.1 A closer look at D0-branes

Notice that, in the IIA theory, the D0-brane has a mass $\tau_0 = \alpha'^{-1/2} g_{\rm s}$, as measured in the string metric. As $g_{\rm s} \to \infty$, this mass becomes light, and eventually becomes the lightest scale in the theory, lighter even than that of the fundamental string itself.

We can trust the extrapolation of the mass formula done in this way because the D0-brane is a BPS object, and so the formula is protected from, for example, levelling off to some still not-too-light scale by loop corrections, etc. So we are being shown new features of the theory here, and it would be nice to make sense of them. Notice that in addition, we have seen in section 11.5 that n D0-branes have a single supersymmetric bound state with mass $n\tau_0$. So in fact, these are genuine physical particles, charged under the $U(1)$ of the R–R one-form $C_{(1)}$, and forming an evenly spaced tower of mass states which is become light as we go further to strong coupling. How are we to make sense of this in ten dimensional string theory?

In fact, the spectrum we just described is characteristic of the appearance of an additional dimension[152, 153], where the momentum (Kaluza–Klein) states have masses n/R and form a continuum is $R \to \infty$. Here, $R = \alpha'^{1/2} g_{\rm s}$, so weak coupling is small R and the theory is effectively ten dimensional, while strong coupling is large R, and the theory is eleven dimensional. We saw such Kaluza–Klein behaviour in section 4.2. The charge of the nth Kaluza–Klein particle corresponds to n units of momentum $1/R$ in the hidden dimension. In this case, this $U(1)$ is the R–R one form of type IIA, and so we interpret D0-brane charge as eleven dimensional momentum.

12.4.2 Eleven dimensional supergravity

In this way, we are led to consider eleven dimensional supergravity as the strong coupling limit of the type IIA string. This is only for *low energy*, of course, and the issue of the complete description of the short distance physics at strong coupling to complete the 'M-theory', is yet to be settled.

It cannot be simply eleven dimensional supergravity, since that theory (like all purely field theories of gravity) is ill-defined at short distances. A most widely examined proposal for the structure of the short distance physics is 'Matrix Theory'[157], and we shall briefly discuss it in chapter 16.

In the absence of a short distance theory, we have to make do with the low-energy effective theory, which is a graviton, and antisymmetric 3-form tensor gauge field $A_{(3)}$, and their superpartners. Notice that this theory has the same number of bosonic and fermionic components as the type II theory. Take type IIA and note that the NS–NS sector has 64 bosonic components as does the R–R sector, giving a total of 128. Now count the number of physical components of a graviton, together with a three-form in eleven dimensions. The answer is $9\times10/2-1=44$ for the graviton and $9\times8\times7/(3\times2)=84$ for the three-form. The superpartners constitute the same number of fermionic degrees of freedom, of course, giving an $\mathcal{N}=1$ supersymmetry in eleven dimensions, equivalent to 32 supercharges. In fact, a common trick to be found in many discussions for remembering how to write the type IIA Lagrangian[5] is simply to dimensionally reduce the eleven dimensional supergravity Lagrangian. Now we see that a physical reason lies behind it. The bosonic part of the action is:

$$S_{\rm IID}=\frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-G}\left(R-\frac{1}{48}(F_{(4)})^2\right)-\frac{1}{12\kappa_{11}^2}\int A_{(3)}\wedge F_{(4)}\wedge F_{(4)}, \tag{12.9}$$

and we shall work out $2\kappa_{11}^2=16\pi G_{\rm N}^{11}$ shortly.

To relate the type IIA string coupling to the size of the eleventh dimension we need to compare the respective Einstein–Hilbert actions[153], ignoring the rest of the actions for now:

$$\frac{1}{2\kappa_0^2g_{\rm s}^2}\int d^{10}x\,\sqrt{-G_{\rm s}}R_{\rm s}=\frac{2\pi R}{2\kappa_{11}^2}\int d^{10}x\,\sqrt{-G_{11}}R_{11}. \tag{12.10}$$

The string and eleven dimensional supergravity metrics are equal up to an overall rescaling,

$$G_{{\rm s}\mu\nu}=\zeta^2G_{11\mu\nu} \tag{12.11}$$

and so $\zeta^8=2\pi R\kappa_0^2g_{\rm s}^2/\kappa_{11}^2$. The respective masses are related $n/R=m_{11}=\zeta m_{\rm s}=n\zeta\tau_0$ or $R=\alpha'^{1/2}g_{\rm s}/\zeta$. Combining these with the result (7.44) for κ_0, we obtain

$$\zeta=g_{\rm s}^{1/3}\left[2^{7/9}\pi^{8/9}\alpha'\kappa_{11}^{-2/9}\right] \tag{12.12}$$

and the radius in eleven dimensional units is:

$$R=g_{\rm s}^{2/3}\left[2^{-7/9}\pi^{-8/9}\kappa_{11}^{2/9}\right]. \tag{12.13}$$

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In order to emphasise the basic structure we hide in braces numerical factors and factors of κ_{11} and α'. The latter factors are determined by dimensional analysis, with κ_{11} having units of (11D supergravity length$^{9/2}$) and α' (string theory length^2). We are free to set $\zeta = 1$, using the same metric and units in M-theory as in string theory. In this case

$$\kappa_{11}^2 = g_\mathrm{s}^3 \left[2^7 \pi^8 \alpha'^{9/2}\right], \qquad \text{and then} \quad R = g_\mathrm{s}\ell_\mathrm{s}. \tag{12.14}$$

The reason for not always doing so is that when we have a series of dualities, as below, there will be different string metrics. For completeness, let us note that if we define Newton's constant via $2\kappa_{11}^2 = 16\pi G_\mathrm{N}^{11}$, then we have:

$$\kappa_{11}^2 = 2^7\pi^8\ell_\mathrm{p}^9; \qquad \ell_\mathrm{p} = g_\mathrm{s}^{1/3}\sqrt{\alpha'} = g_\mathrm{s}^{1/3}\ell_\mathrm{s}. \tag{12.15}$$

See insert 12.1 for more about the Kaluza–Klein reduction.

12.5 $E_8 \times E_8$ heterotic string/M-theory duality

We have deduced the duals of four of the five ten dimensional string theories. Let us study the final one, the $E_8 \times E_8$ heterotic string, which is T-dual to the $SO(32)$ string[173, 174].

Compactify on a large radius R_HA and turn on a Wilson line which breaks $E_8 \times E_8$ to $SO(16) \times SO(16)$. As we learned in section 8.1.3, this is T-dual to the $SO(32)$ heterotic string, again with a Wilson line breaking the group to $SO(16) \times SO(16)$. The couplings and radii are related

$$\begin{aligned} R_\mathrm{HB} &= \frac{\ell_\mathrm{s}^2}{R_\mathrm{HA}}, \\ g_\mathrm{s,HB} &= g_\mathrm{s,HA}\frac{\ell_\mathrm{s}}{R_\mathrm{HA}}. \end{aligned} \tag{12.16}$$

Now use type I/heterotic duality to write this as a type I theory with[153]

$$\begin{aligned} R_\mathrm{IB} &= g_\mathrm{s,HB}^{-1/2} R_\mathrm{HB} = g_\mathrm{s,HA}^{-1/2}\frac{\ell_\mathrm{s}^{3/2}}{R_\mathrm{HA}^{1/2}}, \\ g_\mathrm{s,IB} &= g_\mathrm{s,HB}^{-1} \ = \ g_\mathrm{s,HA}^{-1}\frac{R_\mathrm{HA}}{\ell_\mathrm{s}}. \end{aligned} \tag{12.17}$$

The radius is very small, so it is useful to make another T-duality, to the 'type I′' or 'type IA' theory. The compact dimension is then a segment of

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Insert 12.1. Kaluza–Klein relations

It is amusing to work out the relationship between the metric in Einstein frame, and the metric and scalar in one dimension fewer, in either Einstein or another frame. The general case might be useful, so we will work it out, bearing in mind that for the eleven to ten case, the scalar is the type IIA dilaton Φ, but in other cases it is simply an additional modulus (there may already be a dilaton). We want to get to D dimensions, reducing on x^D, and the higher dimensional metric shall be written in Kaluza–Klein form:

$$G^{(D+1)}_{MN}dx^M dx^N = e^{2\alpha\Phi}\left(G^{(D)}_{\mu\nu}dx^\mu dx^\nu + e^{2\Phi}(dx^D + A_\mu dx^\mu)^2\right),$$

where we have included the possibility that we will have to do a rescaling to change frames in the lower dimensions, by multiplying by $e^{2\alpha\Phi}$. This results in the Ricci scalar of the new metric being multiplied by $e^{-2\alpha\Phi}$. The determinant of the original metric becomes $e^{2\alpha(D+1)\Phi}e^{2\Phi}\det[-G^{(D)}]$, and so the reduced action is

$$\int d^Dx\, e^{\alpha(D+1)\Phi}e^{\Phi}\det{}^{\frac{1}{2}}[-G^{(D)}]e^{-2\alpha\Phi}R^{(D)}.$$

The total power of e^Φ which appears is $\alpha(D+1)+1-2\alpha = 1+\alpha(D-1)$. So now we can dial up whatever frame we desire. String frame would have an $e^{-2\Phi}$, and so we get $1+\alpha(D-1) = -2$, i.e. $\alpha = -3/(D-1)$. For the case the $D=10$, $\alpha=-1/3$ and this means that

$$G^{(11)}_{MN}dx^M dx^N = e^{-\frac{2}{3}\Phi}G^{(10)}_{\mu\nu}dx^\mu dx^\nu + e^{\frac{4}{3}\Phi}(dx^{10}+A_\mu dx^\mu)^2.$$

length $\pi R_{\rm IA}$ with eight D8-branes and O8-planes at each end, and

$$\begin{aligned} R_{\rm IA} &= \frac{\ell_{\rm s}^2}{R_{\rm IB}} = g_{\rm s,HA}^{1/2}R_{\rm HA}^{1/2}\ell_{\rm s}^{1/2}, \\ g_{\rm s,IA} &= g_{\rm s,IB}\frac{\ell_{\rm s}}{R_{\rm IB}\sqrt{2}} = g_{\rm s,HA}^{-1/2}\frac{R_{\rm HA}^{3/2}}{\ell_{\rm s}^{3/2}\sqrt{2}}. \end{aligned} \tag{12.18}$$

It is worth drawing a picture of this arrangement, and it is displayed in figure 12.4. Notice that since the charge of an O8-plane is precisely that of eight D8-branes, the charge of the R–R sector is locally cancelled at

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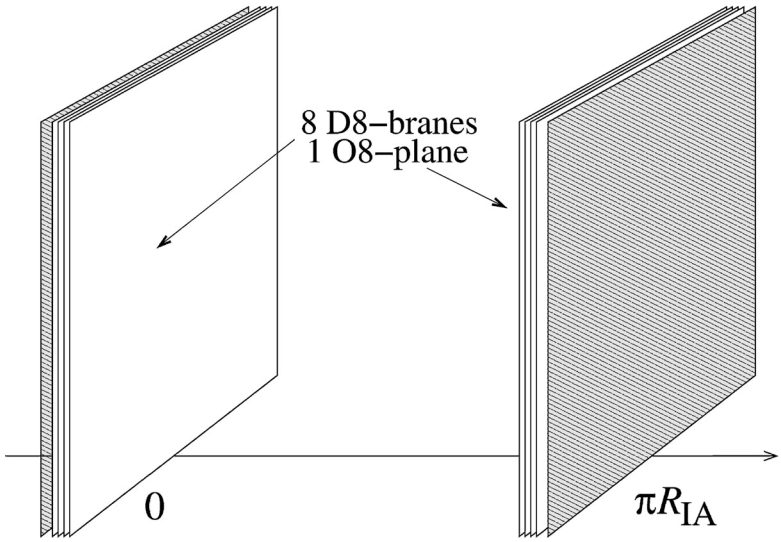

Fig. 12.4. The type IA configuration of two groups of eight D8-branes and O8-planes resulting from a $SO(16) \times SO(16)$ Wilson line.

each end. There is therefore no R–R flux in the interior of the interval and so crucially, we see that the physics between the ends of the segment is given locally by the IIA string. Now we can take $R_{\rm HA} \to \infty$ to recover the original ten dimensional theory (in particular the Wilson line is irrelevant and the original $E_8 \times E_8$ restored). Both the radius and the coupling of the type IA theory become large. Since the bulk physics is locally that of the IIA string‡, the strongly coupled limit is eleven dimensional. Taking into account the transformations (12.11) and (12.13), the radii of the two compact dimensions in M-theory units are

$$\begin{aligned} R_9 &= \zeta_{\rm IA}^{-1} R_{\rm IA} = g_{\rm s}^{2/3} \left[2^{-11/18} \pi^{-8/9} \kappa_{11}^{2/9}\right] \\ R_{10} &= g_{\rm s,IA}^{2/3} \left[2^{-7/9} \pi^{-8/9} \kappa_{11}^{2/9}\right] = g_{\rm s,HA}^{-1/3} R_{\rm HA} \left[2^{-10/9} \pi^{-8/9} \alpha'^{-1/2} \kappa_{11}^{2/9}\right]. \end{aligned} \tag{12.19}$$

Again, had we chosen $\zeta_{\rm IA} = 1$, we would have

$$R_{10} = R_{\rm HA} 2^{-1/3}, \quad R_9 = g_{\rm s} \ell_{\rm s} 2^{1/6}. \tag{12.20}$$

‡ Notice that this is not the case if the D8-branes are placed in a more general arrangement where the charges are not cancelled locally. For such arrangements, the dilaton and R–R nine-form is allowed to vary piecewise linearly between neighbouring D8-branes. The supergravity between the branes is the 'massive' supergravity considered by Romans[97]. This is a very interesting topic in its own right, which we shall not have room to touch upon here. A review of some aspects, with references, is given in the bibliography[181].

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As $R \to \infty$, $R_{10} \to \infty$ also, while R_9 remains fixed and (for g_s large) large compared to the Planck scale. This suggests that in the strongly coupled limit of the ten dimensional $E_8 \times E_8$ heterotic string an eleventh dimension again appears: it is a line segment of length R_9, with one E_8 ninebrane factor on each endpoint[169].

We have not fully completed the argument, since we only have argued for $SO(16)$ at each end. One way to see how the E_8 arises is to start from the other end and place eleven dimensional supergravity on a line segment. This theory is anomalous, but the anomaly can be cancelled by having 248 vector fields on each ten dimensional boundary[169]. So the **120** of $SO(16)$ is evidently joined by 128 new massless states at strong coupling. As we saw in section 7.2 in the decomposition of E_8 to $SO(16)$, the adjoint breaks up as $\mathbf{248} = \mathbf{120} \oplus \mathbf{128}$, where the **128** is the spinor representation of $SO(16)$. Now we see why we could not construct this in perturbative type IA string theory. Spinor representations of orthogonal groups cannot be made with Chan–Paton factors. However, we can see these states as massive D0–D8 bound states, T-dual to the D1–D9 spinors we were able to make in the $SO(32)$ case in section 12.2. Now, with $SO(16)$ at each end, we can make precisely the pair of **128**s we need.

12.6 M2-branes and M5-branes

12.6.1 Supergravity solutions

Just as in the other supergravities, we can make extended objects in the theory. The most natural one to consider first, given what we have displayed as the content of the theory is one which carries the charge of the higher rank gauge field, $A_{(3)}$. This is a two dimensional brane (a membrane) which we shall call the '*M2-brane*', and the solution is[166]:

$$ds^2 = f_3^{-2/3}\left(-dt^2 + (dx^1)^2 + (dx^2)^2\right) + f_3^{1/3}(dr^2 + r^2 d\Omega_7^2)$$
$$f_3 = \left(1 + \frac{\pi N \ell_p^3}{r^3}\right), \qquad A_{(3)} = f_3^{-1} dt \wedge dx^1 \wedge dx^2, \quad (12.21)$$

where the eleven dimensional Planck length $\ell_{\rm p}$ is given by equation (12.15).

By eleven dimensional Hodge duality, it is easy to see that there is another natural object, a fivebrane which is magnetically dual to the M2-brane, called the '*M5-brane*'[167]:

$$ds^2 = f_5^{-1/3}\left(-dt^2 + (dx^1)^2 + \cdots + (dx^5)^2\right) + f_5^{2/3}(dr^2 + r^2 d\Omega_4^2)$$
$$f_5 = \left(1 + \frac{32\pi^2 N \ell_{\rm p}^6}{r^6}\right), \quad A_{(6)} = f_5^{-1} dt \wedge dx^1 \wedge \cdots \wedge dx^5. \quad (12.22)$$

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The tensions of the single (i.e. $N = 1$) M2- and M5-branes of eleven dimensional supergravity are:

$$\tau_2^{\mathrm{M}} = (2\pi)^{-2}\ell_{\mathrm{p}}^{-3}; \qquad \tau_5^{\mathrm{M}} = (2\pi)^{-5}\ell_{\mathrm{p}}^{-6}. \tag{12.23}$$

The product of the M-branes' tensions gives

$$\tau_2^{\mathrm{M}}\tau_5^{\mathrm{M}} = 2\pi(2\pi)^{-8}\ell_{\mathrm{p}}^{-9} = \frac{2\pi}{2\kappa_{11}^2} \tag{12.24}$$

and so is the minimum allowed by the quantum theory, in close analogy with what we know for D-branes from equation (8.20).

12.6.2 From D-branes and NS5-branes to M-branes and back

It is interesting to track the eleven dimensional origin of the various branes of the IIA theory[154]. The D0-branes are, as we saw above, are Kaluza–Klein states[153]. The F1-branes, the IIA strings themselves, are wrapped M2-branes of M-theory. The D2-branes are M2-branes transverse to the eleventh dimension X^{10}. The D4-branes are M5-theory wrapped on X^{10}, while the NS5-branes are M5-branes transverse[§] to X^{10}. The D6-branes, being the magnetic duals of the D0-branes, are Kaluza–Klein monopoles[168, 152] (we shall see this directly later in section 15.2). As mentioned before, the D8-branes have a more complicated fate. To recapitulate, the point is that the D8-branes cause the dilaton to diverge within a finite distance[162], and must therefore be a finite distance from an orientifold plane, which is essentially a boundary of spacetime as we saw in section 4.11. As the coupling grows, the distance to the divergence and the boundary necessarily shrinks, so that they disappear into it in the strong coupling limit: they become part of the gauge dynamics of the nine dimensional boundary of M-theory[169], used to make the $E_8 \times E_8$ heterotic string, as discussed in more detail above. This raises the issue of the strong coupling limit of orientifolds in general. There are various results in the literature, but since the issues are complicated, and because the techniques used are largely strongly coupled field theory deductions, which take us well beyond the scope of this book, except for an O6-plane in section 15.3 and the O7-plane in sections 16.1.11 and 16.1.12, we will have to refer the reader to the literature[235].

One can see further indication of the eleventh dimension in the world-volume dynamics of the various branes. We have already seen this in

[§] The reader might like to check, using the Kaluza–Klein relations given at the end of insert 12.1, that the D2-brane and NS5-brane metrics can be obtained from the M2- and M5-brane metrics and vice versa by reduction or the reverse, 'oxidation'.

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section 12.3.2 where we saw that the type IIA NS5-brane has a chiral tensor multiplet on its world-volume, the five scalars of which are indicative of an eleven dimensional origin. We saw in the above that this is really a precursor of the fact that it lifts to the M5-brane with the same world-volume tensor multiplet, when type IIA goes to strong coupling. The world-volume theory is believed to be a 5+1 dimensional fixed point theory (see insert 3.1). Consider as another example the D2-brane. In $2+1$ dimensions, the vector field on the brane is dual to a scalar, through Hodge duality of the field strength, $*F_2 = d\phi$. This scalar is the eleventh embedding dimension[155]. It joins the other seven scalars already defining the collective modes for transverse motion to show that there are *eight* transverse dimensions. Carrying out the duality in detail, the D2-brane action is therefore found to have a hidden eleven dimensional Lorentz invariance. We shall see this feature in certain probe computations later on in section 15.2. So we learn that the M2-brane, which it becomes, has a 2+1 dimensional theory with eight scalars on its world-volume. The existence of this theory may be inferred in purely field theory terms as being an infra-red fixed point (see insert 3.1) of the 2+1 dimensional gauge theory[180].

12.7 U-duality

A very interesting feature of string duality is the enlargement of the non-perturbative duality group under further toroidal compactification. There is a lot to cover, and it is somewhat orthogonal to most of what we want to do for the rest of the book, so we will err on the side of brevity (for a change). The example of the type II string on a five-torus T^5 is useful, since it is the setting for the simplest black hole state counting that we'll study in chapter 17, and we have already started discussing it in section 7.5.

12.7.1 Type II strings on T^5 and $E_{6(6)}$

As we saw in section 7.5, the T-duality group is $O(5,5,\mathbb{Z})$. The 27 gauge fields split into 10+16+1 where the middle set have their origin in the R–R sector and the rest are NS–NS sector fields. The $O(5,5;\mathbb{Z})$ representations here correspond directly to the **10**, **16**, and **1** of $SO(10)$. There are also 42 scalars.

The crucial point here is that there is a larger symmetry group of the supergravity, which is in fact $E_{6,(6)}$. It generalises the $SL(2,\mathbb{R})$ $(SU(1,1))$ S-duality group of the type IIB string in ten dimensions. In that case there are two scalars, the dilaton Φ and the R–R scalar $C_{(0)}$, and they take values on the coset space

$$\frac{SL(2,\mathbb{R})}{U(1)} \simeq \frac{SU(1,1)}{SO(2)}.$$

The low energy supergravity theory for this compactification has a continuous symmetry, $E_{6(6)}$ which is a non-compact version[176] of E_6. (See insert 12.2.)

The gauge bosons are in the **27** of $E_{6(6)}(\mathbb{Z})$, which is the same as the **27** of $E_{6(6)}$. The decomposition under $SO(10) \sim O(5,5;\mathbb{Z})$ is familiar from grand unified model building: $\mathbf{27} \rightarrow \mathbf{10} + \mathbf{16} + \mathbf{1}$. Another generalisation is that the 42 scalars live on the coset

$$\frac{E_{6,(6)}}{USp(8)}.$$

In the light of string duality, just as the various branes in type IIB string theory formed physical realisations of multiplets of $SL(2,\mathbb{Z})$, so do the branes here. A discrete subgroup $E_{6(6)}(\mathbb{Z})$ is the 'U-duality' symmetry. The particle excitations carrying the **10** charges are just the Kaluza–Klein and winding strings. The U-duality requires also states in the **16**. These are just the various ways of wrapping Dp-branes to give D-particles (10 for D2, 5 for D4 and 1 for D0). Finally, the state carrying the **1** charge is the NS5-brane, wrapped entirely on the T^5.

In fact, the U-duality group for the type II strings on T^d is $E_{d+1,(d+1)}$, where for $d = 4,3,2,1,0$ we have that the definition of the appropriate E-group is $SO(5,5), SL(5), SL(2) \times SL(3), SL(2) \times \mathbb{R}_+, SL(2)$. These groups can be seen with similar embedding of Dynkin diagrams to what we have done in insert 12.2.

12.7.2 U-duality and bound states

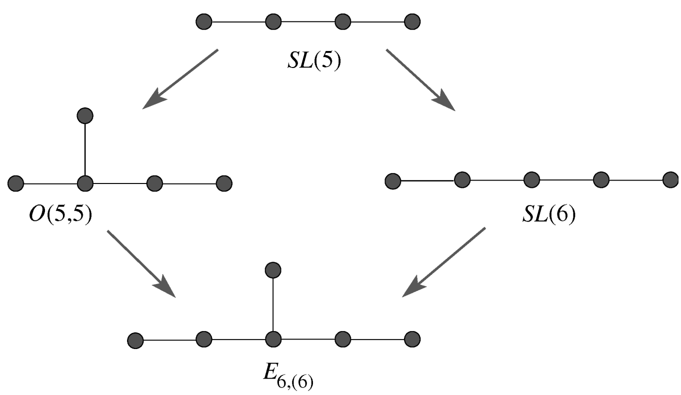
It is interesting to see how some of the bound state results from chapter 11 fit the predictions of U-duality. We will generate U-transformations as a combination of $\mathrm{T}_{mn\cdots p}$, which is a T-duality in the indicated directions, and S, the IIB strong/weak coupling transformation. The former switches between N and D boundary conditions and between momentum and winding number in the indicated directions. The latter interchanges the NS–NS and R–R 2-duals but leaves the R–R four-dual invariant, and acts correspondingly on the solitons carrying these charges. We denote by $\mathrm{D}_{mn\cdots p}$ a D-brane extended in the indicated directions, and similarly for F_m a fundamental string and p_m a momentum-carrying BPS state.

The first duality chain is

$$(\mathrm{D}_9, \mathrm{F}_9) \stackrel{\mathrm{T}_{78}}{\rightarrow} (\mathrm{D}_{789}, \mathrm{F}_9) \stackrel{\mathrm{S}}{\rightarrow} (\mathrm{D}_{789}, \mathrm{D}_9) \stackrel{\mathrm{T}_9}{\rightarrow} (\mathrm{D}_{78}, \mathrm{D}_{\emptyset}).$$

Insert 12.2. Origins of $E_{6,(6)}$ and other U-duality Groups

One way of seeing roughly where $E_{6,(6)}$ comes from is as follows: The naive symmetry resulting from a T^5 compactification would be $SL(5,\mathbb{R})$, the generalisation of the $SL(2,\mathbb{R})$ of the T^2 to the higher dimensional torus. There are two things which enlarge this somewhat. The first is an enlargement to $SL(6,\mathbb{R})$, which ought to be expected, since the type IIB string already has an $SL(2,\mathbb{R})$ in ten dimensions. This implies the existence of an an extra circle, enlarging the naive torus from T^5 to T^6. This is of course something we have already discovered in section 12.4: at strong coupling, the type IIA string sees an extra circle. Below ten dimensions, T-duality puts both type II strings on the same footing, and so it is most efficient to simply think of the problem as M-theory (at least in its eleven dimensional supergravity limit) compactified on a T^6. Another enlargement is due to T-duality. As we have learned, the full T-duality group is $O(5,5,\mathbb{Z})$, and so we should expect a classical enlargement of the naive $SL(5,\mathbb{R})$ to $O(5,5)$. That $E_{6,(6)}$ contains these two enlargements can be seen quite efficiently in the following Dynkin diagrams[181].



(Actually, the above embedding is not unique, but we are not attempting a proof here; we are simply showing that $E_{6,(6)}$ as is not unreasonable, given what we already know.) The notation $E_{6,(6)}$ means that by analytic continuation of some of the generators, we make a non-compact version of the group (much as in the same way as we get $SL(2,\mathbb{R})$ from $SU(2)$). The maximal number of generators for which this is possible is the relevant case here.

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(The last symbol denotes a D0-brane, which is of course not extended anywhere.) Thus the D-string–F-string bound state is U-dual to the 0–2 bound state, as previously indicated in sections 11.3 and 11.5.

The second chain is

$$(\mathrm{D}_{6789}, \mathrm{D}_{\emptyset}) \xrightarrow{\mathrm{T}_6} (\mathrm{D}_{789}, \mathrm{D}_6) \xrightarrow{\mathrm{S}} (\mathrm{D}_{789}, \mathrm{F}_6) \xrightarrow{\mathrm{T}_{6789}} (\mathrm{D}_6, p_6) \xrightarrow{\mathrm{S}} (\mathrm{F}_6, p_6).$$

The bound states of n D0-branes and m D4-branes are thus U-dual to fundamental string states with momentum n and winding number m. The bound state degeneracy (11.21) for $m = 1$ precisely matches the fundamental string degeneracy[177, 144, 178].

For $m > 1$ the same form (11.21) should hold but with $n \to mn$. This is believed to be the case, but the analysis (which requires the instanton picture described in the next section) does not seem to be complete[178].

A related issue is the question of branes ending on other branes[179], and we shall see more of this later. An F-string can of course end on a D-string, so from the first duality chain it follows that a Dp-brane can end on a D$(p+2)$-brane. The key issue is whether the coupling between spacetime forms and world-brane fields allows the source to be conserved, as with the NS–NS two-dual source in figure 11.1. Similar arguments can be applied to the extended objects in M-theory[179, 143].

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