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HILBERT C^*-MODULES AND CONDITIONAL EXPECTATIONS ON CROSSED PRODUCTS

MAHMOOD KHOSHKAM

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Abstract

In this paper, we study the structure of certain conditional expectation on crossed product C^*-algebra. In particular, we prove that the index of a conditional expectation $E : B \to A$ is finite if and only if the index of the induced expectation from $B \rtimes G$ onto $A \rtimes G$ is finite where G is a discrete group acting on B.

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Introduction

In this paper we study conditional expectations defined on certain C^*-algebras given as crossed products. Consider a pair $A \subset B$ of C^*-algebras, $E : B \to A$ a conditional expectation, and an action of a discrete group G on B commuting with E. Then, there are conditional expectations $\tilde{E}$ (respectively $\tilde{E}_r$) from $B \rtimes G$ (respectively $B \rtimes_r G$) onto $A \rtimes G$ (respectively $A \rtimes_r G$). Many properties of $\tilde{E}$ (and $\tilde{E}_r$) are realized by studying the Hilbert C^*-modules obtained by a Jones-type basic construction method. Consequently, a large portion of this paper is concerned with Hilbert C^*-modules and the C^*-algebra of the so-called compact operators on a Hilbert C^*-module. In Section 2 we consider a Hilbert C^*-module $\mathscr{E}$ equipped with an action of a group G. Then, G acts on $\mathscr{K}(\mathscr{E})$ and the main theorem of this section states that if G is discrete, then $\mathscr{K}(\mathscr{E}) \rtimes G$ (respectively $\mathscr{K}(\mathscr{E}) \rtimes_r G$) is $*$-isomorphic to $\mathscr{K}(\mathscr{E} \rtimes G)$ (respectively $\mathscr{K}(\mathscr{E} \rtimes_r G)$). In Section 3, we prove that $\tilde{E}$ (and $\tilde{E}_r$) has finite index if and only if E has finite index. We also show that the canonical conditional expectations from $B \rtimes_r G$ onto $B \rtimes_r H$ and from $B \rtimes G$ onto $B \rtimes H$ for a subgroup H of G have finite indices if and only if $[G : H] < \infty$. The notion of index considered here was introduced by Watatani [14] who was inspired by Jones' index theory for subfactors [7]. The index of a conditional expectation $E : B \to A$ is a positive element of B. When the index is scalar (for example B simple) it belongs to the set $\{4\cos^2 \pi/n : n \geq 3\} \cup [4, \infty)$. One hopes that if $E : B \to A$ has finite index, then A and B cannot be structurally very different. For example, it is known that a C*-subalgebra A of a nuclear C^*-algebra B need not be nuclear [1, 3]. However, when $E : B \to A$ has finite index, then B is nuclear if and only if A is so. Throughout this paper all C^*-algebras (except for ideals) are assumed to be unital and we deal with actions of discrete groups only. If a group G acts on a C^*-algebra A as a group of automorphisms, then $A \rtimes G$ and $A \rtimes_r G$ respectively denote the full and the reduced crossed product C^*-algebras [10].

1. Finitely-generated Hilbert C^*-modules

In this section we prove a series of technical lemmas on Hilbert C^*-modules. Let A be a C^*-algebra and $\mathscr{E}$ a Hilbert A-module. Then $\mathscr{L}(\mathscr{E})$ denotes the C^*-albegra of adjointable operators and $\mathscr{K}(\mathscr{E})$ the closed ideal in $\mathscr{L}(\mathscr{E})$ generated by the elements $\theta_{\xi,\eta}$ where $\xi, \eta \in \mathscr{E}$ (cf. [8]). If $\mathscr{E}_1$ is a right Hilbert A-module, $\mathscr{E}_2$ a right Hilbert B-module, and $\pi : \mathscr{E} \to \mathscr{L}(\mathscr{E}_2)$ a $*$-representation, then the algebraic tensor product $\mathscr{E}_1 \odot \mathscr{E}_2$ has a natural B-valued inner product. Namely, $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_2, \pi(\langle x_1, y_1 \rangle) y_2 \rangle$ with $x_1, y_1 \in \mathscr{E}_1$ and $x_2, y_2 \in \mathscr{E}_2$. Let $\mathscr{E}_1 \otimes_A \mathscr{E}_2$ denote the completion of $\mathscr{E}_1 \odot \mathscr{E}_2$ after vectors of length zero have been factored out. For a Hilbert module $\mathscr{E}$, $1_{\mathscr{E}}$ denotes the identity operator on $\mathscr{E}$.

LEMMA 1.1. *Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be Hilbert modules over C^*-algebras A and B respectively, and $\pi : A \to \mathscr{L}(\mathscr{E}_2)$ a $*$-representation. If π is faithful, then the mapping $\mathscr{L}(\mathscr{E}_1) \to \mathscr{L}(\mathscr{E}_1 \otimes_A \mathscr{E}_2)$ defined by $T \to T \otimes 1_{\mathscr{E}_2}$ is faithful.*

PROOF. Let $T \in \mathscr{L}(\mathscr{E}_1)$, and $T \neq 0$. Then, there exists $\xi \in \mathscr{E}_1$ such that $T\xi \neq 0$. Since π is faithful $\pi(\langle T\xi, T\xi \rangle) \neq 0$. Hence, there exists $\eta \in \mathscr{E}_2$ such that $\pi(\langle T\xi, T\xi \rangle)\eta \neq 0$. Therefore, $\langle \pi(\langle T\xi, T\xi \rangle)\eta, \eta_1 \rangle \neq 0$ for some $\eta_1 \in \mathscr{E}_2$. Hence,

$$\langle (T \otimes 1)(\xi \otimes \eta_1), (T \otimes 1)(\xi \otimes \eta) \rangle = \langle \eta_1, \pi(\langle T\xi, T\xi \rangle)\eta \rangle \neq 0$$

and $T \otimes 1 \neq 0$.

LEMMA 1.2. *Let A be a unital C^*-algebra and $\mathscr{E}$ a Hilbert A-module. If $1_{\mathscr{E}} \in \mathscr{K}(\mathscr{E})$, then there exist $u_1, \dots, u_n \in \mathscr{E}$ such that $1_{\mathscr{E}} = \sum_{i=1}^n \theta_{u_i,u_i}$.*

PROOF. Choose $y_1, y_2, \dots, y_m; x_1, x_2, \dots, x_m \in \mathscr{E}$ such that $T = \sum_{i=1}^m \theta_{x_i,y_i}$ and $\|1_{\mathscr{E}} - T\| < 1$. Then, $T + T^* = \sum_{i=1}^m \theta_{x_i,y_i} + \theta_{y_i,x_i}$ is invertible. For every $\xi, \eta \in \mathscr{E}$, we have that $\langle \theta_{\xi,\xi}(\eta), \eta \rangle = \langle \xi, \eta \rangle \langle \xi, \eta \rangle^*$. Since $\langle \xi, \eta \rangle \langle \xi, \eta \rangle^*$ is positive in A by

[12, Corollary 2.7], $\theta_{\xi,\xi}$ is a positive element of $\mathscr{L}(\mathscr{E})$. Using this and the equation $\theta_{x+y,x+y} = \theta_{x,y} + \theta_{y,x} + \theta_{x,x} + \theta_{y,y}$ we have

$$\theta_{x,y} + \theta_{y,x} \leq \theta_{x+y,x+y} \leq \theta_{x+y,x+y} + \theta_{x-y,x-y} = 2\theta_{x,x} + 2\theta_{y,y}.$$

Apply this inequality to each term of $\sum_{i=1}^{m} \theta_{x_i,y_i} + \theta_{y_i,x_i}$ to conclude that the operator $S = \sum_{i=1}^{m} \theta_{x_i,x_i} + \theta_{y_i,y_i}$ is positive and invertible. Then,

$$1_{\mathscr{E}} = S^{-1/2} S S^{-1/2} = \sum_{i=1}^{m} \theta_{S^{-1/2}x_i, S^{-1/2}x_i} + \theta_{S^{-1/2}y_i, S^{-1/2}y_i}$$

which is the desired result.

LEMMA 1.3. *Let $\mathscr{E} = \mathscr{E}_1 \otimes_A \mathscr{E}_2$ and $1_{\mathscr{E}} \in \mathscr{K}(\mathscr{E})$. Then, there exist $x_1, x_2, \ldots, x_m \in \mathscr{E}_1$ and $y_1, \ldots, y_m \in \mathscr{E}_2$ such that $\sum_{i=1}^{n} \theta_{x_i \otimes y_i, x_i \otimes y_i}$ is positive and invertible.*

PROOF. By Lemma 1.2 there exist $z_1, \ldots, z_n \in \mathscr{E}$ such that $1_{\mathscr{E}} = \sum_{i=1}^{n} \theta_{z_i,z_i}$. Without loss of generality we assume that $\|z_i\| \leq 1$ for $i = 1, \ldots, n$. Given $\epsilon > 0$, choose $x_{ij} \in \mathscr{E}_1$ and $y_{ij} \in \mathscr{E}_2$, $j = 1, 2, \ldots, n_i$ such that $\|z_i - \sum_{j=1}^{n_i} x_{ij} \otimes y_{ij}\| < \epsilon/n$. Let $w_i = \sum_{j=1}^{n_i} x_{ij} \otimes y_{ij}$. Then $\|w_i\| \leq 1 + \epsilon/n$, and $\theta_{w_i,w_i} = \sum_{j,k=1}^{n_i} \theta_{x_{ij}\otimes y_{ij}, x_{ik} \otimes y_{ik}}$.
But

$$\begin{aligned}\|\sum_{i=1}^{n} \theta_{w_i,w_i} - 1_{\mathscr{E}}\| &= \|\sum_{i=1}^{n} \theta_{w_i,w_i} - \sum_{i=1}^{n} \theta_{z_i,z_i}\| \leq \sum_{i=1}^{n} \|\theta_{w_i,w_i} - \theta_{z_i,z_i}\| \\ &\leq \sum_{i=1}^{n} \|w_i - z_i\|(\|z_i\| + \|w_i\|) < n\frac{\epsilon}{n}\left(2 + \frac{\epsilon}{n}\right) = \epsilon\left(2 + \frac{\epsilon}{2n}\right).\end{aligned}$$

This shows that $\sum_{i=1}^{n} \theta_{w_i,w_i}$ is invertible if ϵ is sufficiently small. This, together with the inequality in the proof of 1.3 implies that the element $T = \sum_{i=1}^{n} \sum_{j=1}^{n_j} \theta_{x_{ij}\otimes y_{ij}, x_{ij} \otimes y_{ij}}$ is positive and invertible once ϵ is chosen sufficiently small.

LEMMA 1.4. *Let $\mathscr{E}$ be a Hilbert B-module, $\mathscr{E}'$ a Hilbert A-module, and $\pi : A \to \mathscr{L}(\mathscr{E})$ a faithful $*$-representation. If $1_{\mathscr{E}'\otimes_A \mathscr{E}} = \sum_{i=1}^{n} \theta_{x_i,x_i}$ with $x_i \in \mathscr{E}' \otimes_A \mathscr{E}$, then there exist $u_1, u_2, \ldots, u_n \in \mathscr{E}'$ such that $1_{\mathscr{E}'} = \sum_{i=1}^{n} \theta_{u_i,u_i}$.*

PROOF. By 1.3 there exist $y_1, \ldots, y_n \in \mathscr{E}'$ and $z_1, \ldots z_n \in \mathscr{E}$ such that $T = \sum_{i=1}^{n} \theta_{y_i \otimes z_i, y_i \otimes z_i}$ is positive and invertible. For each $x \in \mathscr{E}'$ define $T_x : \mathscr{E} \to \mathscr{E}' \otimes_A \mathscr{E}$ by $T_x(y) = x \otimes_A y$.

Then $T_x^* : \mathscr{E}' \otimes_A \mathscr{E} \to \mathscr{E}$ is given by $T_x^*(\xi \otimes \eta) = \pi(\langle x, \xi \rangle)\eta$. Also, for $z \in \mathscr{E}$, let $S_z : B \to \mathscr{E}$ be defined by $S_z(c) = zc$. Then $S_z^*(x) = \langle z, x \rangle$, and $\theta_{y_i \otimes z_i, y_i \otimes z_i} = T_{y_i} S_{z_i} S_{z_i}^* T_{y_i}^*$.

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Hence

$$\sum_{i=1}^{n} \theta_{y_i \otimes z_i, y_i \otimes z_i} = \sum_{i=1}^{n} T_{y_i} S_{z_i} S_{z_i}^* T_{y_i}^* \leq \sum_{i=1}^{n} \|S_{z_i} S_{z_i}^*\| T_{y_i} T_{y_i}^*.$$

Therefore

$$\sum_{i=1}^{n} T_{y_i} T_{y_i}^* \geq \frac{1}{M} \sum \theta_{y_i \otimes z_i, y_i \otimes z_i}$$

where $M = \max\{\|S_{z_i} S_{z_i}^*\| : i = 1, \dots, n\} > 0$. Then $\sum_{i=1}^{n} T_{y_i} T_{y_i}^* = \sum_{i=1}^{n} \theta_{y_i, y_i} \otimes 1_{\mathscr{E}}$, and hence $S = \sum_{i=1}^{n} \theta_{y_i, y_i}$ is positive and invertible. Let $u_i = S^{-1/2} y_i$ to get $1_{\mathscr{E}'} = \sum_{i=1}^{n} \theta_{S^{-1/2} y_i, S^{-1/2} y_i}$.

COROLLARY 1.5. *Let $\mathscr{E}'$, $\mathscr{E}$, and π be as in Lemma* 1.4. *If $\mathscr{E}' \otimes_A \mathscr{E}$ is a finitely-generated projective C^*-module, then $\mathscr{E}'$ is a finitely generated projective A-module.*

PROOF. Since $\mathscr{E}' \otimes_A \mathscr{E}$ is finitely-generated and projective, it follows that $1_{\mathscr{E}' \otimes_A \mathscr{E}}$ satisfies the hypothesis of Lemma 1.4. Let $u_1, \dots, u_n \in \mathscr{E}'$ be as in Lemma 1.4. Then $f_i(x) = \langle u_i, x \rangle$ is an element of $\mathrm{Hom}_A(\mathscr{E}', A)$ and $\{(u_i, f_i) : i = 1, \dots, n\}$ is a projective system. Hence $\mathscr{E}'$ is a finitely generated projective A-module.

2. Hilbert G-modules

Let $\mathscr{E}$ be a Hilbert A-module equipped with an action of a discrete group G such that:

(i) $t(xa) = (tx)(ta)$, $x \in \mathscr{E}$, $a \in A$, $t \in G$,

(ii) $t\langle x, y \rangle = \langle tx, ty \rangle$, $x, y \in \mathscr{E}$, $t \in G$.

The induced action of G on $\mathscr{K}(\mathscr{E})$ is defined by $\big(t\mathscr{S}\big)(x) = t\big(\mathscr{S}(t^{-1}x)\big)$ for $\mathscr{S} \in \mathscr{K}(\mathscr{E})$, $x \in \mathscr{E}$ and $t \in G$. Let $C_c(G, \mathscr{E})$ be the set of functions with finite support from G into $\mathscr{E}$. Define an $A \rtimes G$-valued inner product on $C_c(G, \mathscr{E})$ by $\langle e_1, e_2 \rangle(t) = \sum_{s \in G} s^{-1}\big(\langle e_1(s), e_2(st) \rangle\big)$ where $e_1, e_2 \in C_c(G, \mathscr{E})$ and $t \in G$. If $e \in C_c(G, \mathscr{E})$ and $a \in C_c(G, A)$ let $(e.a)(t) = \sum_{s \in G} e(s) s\big(a(s^{-1}t)\big)$.

Let $\mathscr{E} \rtimes G$ be the completion of $C_c(G, \mathscr{E})$ in the norm $\|e\| = \|\langle e, e \rangle\|^{1/2}$ when $\langle e, e \rangle$ is regarded as an element of $A \rtimes G$. Similarly $\mathscr{E} \rtimes_r G$ is defined to be the closure of $C_c(G, \mathscr{E})$ with respect to the norm $\|e\|_r = \|\langle e, e \rangle\|_r^{1/2}$, that is, $\langle e, e \rangle$ is regarded as an element of the reduced crossed product $A \rtimes_r G$. Then $\mathscr{E} \rtimes_r G$ is a Hilbert $A \rtimes_r G$-module. For more on this construction we refer to [4, 9]. Using the action of G on $\mathscr{K}(\mathscr{E})$ we form the full and the reduced crossed products $\mathscr{K}(\mathscr{E}) \rtimes G$ and $\mathscr{K}(\mathscr{E}) \rtimes_r G$. We have the following theorem.

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THEOREM 2.1. *Let G be a discrete group acting on a C^*-algebra A and a Hilbert A-module $\mathscr{E}$. Then*

(a) $\mathscr{K}(\mathscr{E}) \rtimes G \cong \mathscr{K}(\mathscr{E} \rtimes G)$;

(b) $\mathscr{K}(\mathscr{E}) \rtimes_r G \cong \mathscr{K}(\mathscr{E} \rtimes_r G)$.

PROOF. (a) Define a covariant representation of the pair $(\mathscr{K}(\mathscr{E}), G)$ on the $A \rtimes G$-module $\mathscr{E} \rtimes G$ by $(u_t f)(s) = t(f(t^{-1}s))$ and $(Tf)(s) = T(f(s))$ for $f \in C_c(G, \mathscr{E})$, $t, s \in G$, and $T \in \mathscr{K}(\mathscr{E})$. It is routine to check that these equations define unitary and $*$-representations. Moreover,

$$(u_t T u_t^*)(f)(s) = t\big[T(u_t^* f)(t^{-1}s)\big] = tT\big((u_t^* f)(t^{-1}s)\big)$$
$$= t\big(T(t^{-1}(f(s)))\big) = (tT)(f)(s).$$

Hence, by [10, Proposition 7.6.4] we obtain a $*$-representation $\pi : \mathscr{K}(\mathscr{E}) \rtimes G \to \mathscr{L}(\mathscr{E} \rtimes G)$.

Since $\mathscr{K}(\mathscr{E})$ is generated by the rank one elements $\theta_{\xi,\eta}$, and G is discrete, $\mathscr{K}(\mathscr{E}) \rtimes G$ is generated by the elements $\theta_{\xi,\eta} u_t$ for $\xi, \eta \in \mathscr{E}$ and $t \in G$. It is straightforward to verify that π sends these elements into $\mathscr{K}(\mathscr{E} \rtimes G)$ and that the range of π contains the generators of $\mathscr{K}(\mathscr{E} \rtimes G)$. Hence π is onto. To show that π is one-to-one define a $*$-homomorphism

$$\hat{\pi} : \mathscr{K}(\mathscr{E} \rtimes G) \to M(\mathscr{K}(\mathscr{E}) \rtimes G)$$

such that $\hat{\pi} \circ \pi$ is identity on $\mathscr{K}(\mathscr{E}) \rtimes G$. Let $\mathscr{E}^*$ be $\mathscr{E}$ with the $\mathscr{K}(\mathscr{E})$-valued inner product $\langle x^*, y^* \rangle = \theta_{x,y}$ and the module action $x^*.T = (T^*(x)^*)$ for $x, y \in \mathscr{E}$ and $T \in \mathscr{K}(\mathscr{E})$ (cf. [12, Definition 6.17]). Here x^* denotes x seen as an element of $\mathscr{E}^*$. Since $\|\theta_{x,x}\| = \|x\|^2$, it follows that $\mathscr{E}^*$ is closed in the norm induced by the above inner-product. Define $\varphi : A \to \mathscr{L}(\mathscr{E}^*)$ by $\varphi(a)(x^*) = (xa^*)^*$. Then it is easy to verify that φ is a $*$-representation, and we can form the tensor product $\mathscr{E} \otimes_A \mathscr{E}^*$ equipped with the diagonal action of G. Furthermore, $\mathscr{E} \otimes_A \mathscr{E}^*$ is naturally and equivariantly isomorphic to $\mathscr{K}(\mathscr{E})$ as Hilbert $\mathscr{K}(\mathscr{E})$-modules (cf. [12, Lemma 6.22]). Using this we conclude that $\mathscr{K}(\mathscr{E}) \rtimes G$ and $\mathscr{E} \otimes_A \mathscr{E}^* \rtimes G$ are isomorphic as Hilbert $\mathscr{K}(\mathscr{E}) \rtimes G$-modules. Then by [9, Lemma 3.10] we have

$$(\mathscr{E} \rtimes G) \otimes_{A \rtimes G} (\mathscr{E}^* \rtimes G) \cong (\mathscr{E} \otimes_A \mathscr{E}^*) \rtimes G.$$

Hence

$$\mathscr{L}\big((\mathscr{E} \rtimes G) \otimes_{A \rtimes G} (\mathscr{E}^* \rtimes G)\big) \cong \mathscr{L}\big((\mathscr{E} \otimes_A \mathscr{E}^*) \rtimes G\big),$$
$$\cong \mathscr{L}\big(\mathscr{K}(\mathscr{E}) \rtimes G\big)$$
$$\cong \mathscr{M}\big(\mathscr{K}(\mathscr{E}) \rtimes G\big)$$

(cf. [8]). Using the above isomorphisms and the mapping $T \to T \otimes 1$ of $\mathcal{K}(\mathcal{E} \rtimes G)$ into $\mathcal{L}\big((\mathcal{E} \rtimes G) \otimes_{A \rtimes G} (\mathcal{E}^* \rtimes G)\big)$ we obtain a $*$-homomorphism $\hat{\pi} : \mathcal{K}(\mathcal{E} \rtimes G) \to M(\mathcal{K}(\mathcal{E}) \rtimes G)$. It is routine to show that $\hat{\pi} \circ \pi$ is the identity on $\mathcal{K}(\mathcal{E}) \rtimes G$.

(b) Define a covariant representation of the pair $(\mathcal{K}(\mathcal{E}), G)$ on the space $\mathcal{E} \otimes l^2(G)$ by $(u_t f)(s) = t f(t^{-1}s)$ and $(Tf)(s) = T(f(s))$ for $f \in l^2(G, \mathcal{E})$, $T \in \mathcal{K}(\mathcal{E})$, and $s, t \in G$. Since this is a faithful representation of $\mathcal{K}(\mathcal{E})$ by [10, Theorem 7.7.5], we obtain a faithful representation

$$\psi : \mathcal{K}(\mathcal{E}) \rtimes_r G \to \mathcal{L}\big(\mathcal{E} \otimes \mathcal{L}^2(G)\big).$$

By Lemma 1.1 the mapping $T \to T \otimes 1$ from $\mathcal{K}(\mathcal{E} \rtimes_r G)$ into $\mathcal{L}((\mathcal{E} \rtimes_r G) \otimes (A \otimes l^2(G)))$ is faithful. Since $(\mathcal{E} \rtimes_r G) \otimes_{A \rtimes_r G} A \otimes l^2(G)$ and $\mathcal{E} \otimes l^2(G)$ are naturally isomorphic we get a faithful representation $\wedge$ of $\mathcal{K}(\mathcal{E} \rtimes_r G)$ on the space $\mathcal{E} \otimes l^2(G)$. Thus

$$\psi : \mathcal{K}(\mathcal{E}) \rtimes_r G \to \mathcal{L}\big(\mathcal{E} \otimes l^2(G)\big),$$
$$\wedge : \mathcal{K}(\mathcal{E} \rtimes_r G) \to \mathcal{L}\big(\mathcal{E} \otimes l^2(G)\big)$$

are faithful $*$-representations. Let $\pi : \mathcal{K}(\mathcal{E}) \rtimes G \to \mathcal{K}(\mathcal{E} \rtimes G)$ be the $*$-isomorphism given by part (a). Let $q : \mathcal{K}(\mathcal{E}) \rtimes G \to \mathcal{K}(\mathcal{E}) \rtimes_r G$ and $q' : \mathcal{K}(\mathcal{E} \rtimes G) \to \mathcal{K}(\mathcal{E} \rtimes_r G)$ be the natural surjections. Then one can check that $\varphi \circ q = \wedge \circ q' \circ \pi$. Clearly this shows that the ranges of φ and $\wedge$ coincide. Hence $\mathcal{K}(\mathcal{E}) \rtimes_r G$ and $\mathcal{K}(\mathcal{E} \rtimes_r G)$ are $*$-isomorphic.

REMARK 2.2. Consider a pair $A \subset B$ of C^*-algebras with a common identity and a faithful conditional expectation $E : B \to A$. Moreover, assume that B is equipped with the action α of a discrete group G such that α commutes with E. Then, we show that there are induced conditional expectations from $B \rtimes G$ (respectively $B \rtimes_r G$) onto $A \rtimes G$ (respectively $A \rtimes_r G$). We prove that E is of finite index type in the sense of [14] if and only if the induced conditional expectations on the crossed products are so. Recall that if A is a C*-subalgebra of a C^*-algebra B, then a positive norm one projection $E : B \to A$ is said to be a conditional expectation from B onto A if $E(axb) = aE(x)b$ for $a, b \in A$ and $x \in B$. We say E is faithful if $x = 0$ whenever $E(x^*x) = 0$ (cf. [14]).

DEFINITION 2.3. ([15]). A conditional expectation $E : B \to A$ is said to have *finite index* if there exists $u_1, \dots, u_n \in B$ such that $x = \sum_{i=1}^n u_i E(u_i^* x)$, $x \in B$. The set $u_1, u_2, \dots, u_n$ is called a *basis* for E and the *index* of E is defined to be $\operatorname{ind} E = \sum_{i=1}^n u_i u_i^*$.

REMARK 2.4. It follows directly from the above definition and the A-linearity of E that $\operatorname{ind} E$ is independent of basis. By [15, Proposition 1.2.8], $\operatorname{ind} E$ belongs to the center of B. In particular, when B is simple $\operatorname{ind} E$ is a scalar and belongs to the familiar set $\{4\cos^2 \pi/n : n \geq 3\} \cup [4, \infty)$ discovered by Jones for the index of subfactors of type II_1 factors (cf. [7]).

REMARK 2.5. Given a faithful conditional expectation $E : B \to A$ we denote by $\mathscr{E}_E$ the completion of B with respect to the norm $\|x\|_E^2 = \|E(x^*x)\|$, that is, the norm induced by the inner product $\langle x, y\rangle = E(x^*y)$. Note that $\|.\|_E$ is a norm because E is assumed to be faithful. Since E is A-linear and of norm one it extends to a projection $e_A : \mathscr{E}_E \to \mathscr{E}_E$. Also regard B as a subalgebra of $\mathscr{L}(\mathscr{E}_E)$ through left multiplication. The C^*-subalgebra of $\mathscr{L}(\mathscr{E}_E)$ generated by Be_AB is just $\mathscr{K}(\mathscr{E}_E)$. In fact, we have that $(xe_Ay)(x'e_Ay') = xE(yx')e_Ay'$ and xe_Ay is just θ_{x,y^*}.

REMARK 2.6. Let $\alpha : G \to \operatorname{Aut}(B)$ be an action of a discrete group G and $E : B \to A$ a conditional expectation satisfying $E(\alpha_t(x)) = \alpha_t(E(x))$. This implies that A is G-invariant. By [10, Proposition 7.7.9] $A \rtimes_r G$ is $*$-isomorphic to a subalgebra of $B \rtimes_r G$. In our situation, $A \rtimes G$ may also be regarded as a C^*-subalgebra of $B \rtimes G$. To see this, we only need to show that every covariant representation of the pair (G, A) on a Hilbert space H extends to a covariant pair $(\hat{\pi}, \hat{u})$ of the pair (B, G) on a space K containing H. Let K be the completion of the algebraic tensor product $B \odot H$ with respect to the inner product $\langle b \odot \xi, c \odot \eta\rangle = \langle \xi, \pi(E(b^*c))\eta\rangle$ after the vectors of norm zero are factored out. Let $\hat{\pi}(b)(c \otimes \xi) = bc \otimes \xi$ and $\hat{u}_t(d \otimes \eta) = \alpha_t(d) \otimes u_t\eta$. It is easy to check that $(\hat{\pi}, \hat{u})$ is a covariant pair. Hence, the crossed product $A \rtimes G$ is viewed as a subalgebra of $B \rtimes G$. Now one may use the proof of [12, Lemma 1.1] with appropriate modifications to prove that the obvious projection of $l^1(G, B)$ onto $l^1(G, A)$ defined by $f \to E \circ f$ extends to a conditional expectation $\tilde{E}$ (respectively $\tilde{E}_r$) from $B \rtimes G$ (respectively $B \rtimes_r G$) onto $A \rtimes G$ (respectively $A \rtimes_r G$). Furthermore, since the conditional expectations $E_\circ : B \rtimes_r G \to B$ and $E'_\circ : A \rtimes_r G \to A$ given by the evaluation at the identity of G are faithful (cf. [2]) and $E'_\circ \tilde{E}_r = E \circ E_\circ$, it follows that $\tilde{E}_r$ is faithful if and only if E is so.

PROPOSITION 2.7. *Let $E : B \to A$ be a faithful conditional expectation. Let $\tilde{E}$ and $\tilde{E}_r$ be the conditional expectations induced by E on $B \rtimes G$ and $B \rtimes_r G$ respectively (see Remark 2.5). Then $\mathscr{E}_E \rtimes_r G$ and $\mathscr{E}_{\tilde{E}_r}$ are isomorphic as Hilbert modules over $A \rtimes_r G$. If $\tilde{E}$ is faithful, then $\mathscr{E}_E \rtimes_r G$ and $\mathscr{E}_{\tilde{E}}$ are isomorphic as Hilbert modules over $A \rtimes G$.*

PROOF. As pointed out in Remark 2.6, if E is faithful, then $\tilde{E}_r$ is also faithful. Then $\mathscr{E}_{\tilde{E}_r}$ is the completion of $C_c(G, B)$ with respect to the norm induced by $\tilde{E}_r$. Since B is dense in $\mathscr{E}_E$ the action of G extends to $\mathscr{E}_E$ and $\mathscr{E}_E \rtimes_r G$ can be formed. Moreover,

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$C_c(G, B)$ is also dense in $\mathscr{E} \rtimes_r G$. Hence we only need to show that the $A \rtimes_r G$-valued inner products on $\mathscr{E}_{\tilde{E}}$ and $\mathscr{E}_E \rtimes_r G$ coincide on $C_c(G, B)$. Given $x, y \in C_c(G, B)$ and $t \in G$ we have:

$$\begin{aligned}\langle x, y\rangle_{\mathscr{E}_E \rtimes_r G}(t) &= \sum_{s\in G} \alpha_{s^{-1}}\big(\langle x(s), y(st)\rangle\big) = \sum_{s\in G} E\big(\alpha_{s^{-1}}(x(s)^* y(st))\big)\\ &= E\Big(\sum_{s\in G} \alpha_{s^{-1}}(x(s)^*)\alpha_{s^{-1}}(y(st))\Big) = E\Big(\sum_{s\in G} x^*(s^{-1})\alpha_{s^{-1}}(y(st))\Big)\\ &= \tilde{E}_r(x^* * y)(t).\end{aligned}$$

But $\tilde{E}_r(x^* * y)$ is the inner product of x with y when they are seen as elements of $\mathscr{E}_{\tilde{E}_r}$. If $\tilde{E}$ is faithful, then the above argument can be repeated to get the desired result.

THEOREM 2.8. *Let G be a discrete group acting on a C^*-algebra B and let $E : B \to A$ be a conditional expectation onto a C^*-subalgebra A of B commuting with the action of G. Then the following are equivalent.*

(a) *E has finite index,*
(b) *$\tilde{E}$ has finite index,*
(c) *$\tilde{E}_r$ has finite index.*

PROOF. Suppose (a) holds. Let $\{b_1, b_2, \dots, b_n\}$ be a basis for E. For every $x \in B$ and $t \in G$ we denote by $\lambda_{t,x}$ the element of $l^1(G, B)$ which is x at t and zero elsewhere. Then denoting the identity of G by e we have

$$\begin{aligned}\sum \lambda_{e,b_i} \tilde{E}\left(\lambda_{e,b_i^*} * \lambda_{t,x}\right) &= \sum_{i=1}^{n} \lambda_{e,b_i} \tilde{E}\left(\lambda_{t,b_i^* x}\right) = \sum_{i=1}^{n} \lambda_{e,b_i} \lambda_{t,E(b_i^* x)}\\ &= \lambda_{t,\sum_{i=1}^n b_i E(b_i^* x)} = \lambda_{t,x}.\end{aligned}$$

Hence the set $\{\lambda_{e,b_i} : i = 1, 2, \dots, n\}$ is a basis for $\tilde{E}$. Moreover,

$$\operatorname{ind} \tilde{E} = \sum_{i=1}^{n} \lambda_{e,b_i} * \lambda_{e,b_i^*} = \lambda_{e,\sum_{i=1}^n b_i b_i^*} = \lambda_{e,\operatorname{ind} E}$$

Next we show that (b) implies (a). First note that if $\tilde{E}$ has finite index, then by [15, Proposition 2.1.5], it is faithful. Hence by Proposition 2.7, $\mathscr{E}_{\tilde{E}}$ is isomorphic to $\mathscr{E}_E \rtimes G$. By Theorem 2.1, $\mathscr{K}(\mathscr{E}_{\tilde{E}})$ is $*$-isomorphic to $\mathscr{K}(\mathscr{E}) \rtimes G$. If $\tilde{E}$ has finite index, then $\mathscr{K}(\mathscr{E}_{\tilde{E}})$ has an identity ([15, Proposition 2.1.5]). Since G is discrete, it follows that $\mathscr{K}(\mathscr{E})$ has an identity. Then, by Lemma 1.2, there exist elements $u_1, \dots u_n \in \mathscr{E}$ such that $1_{\mathscr{E}} = \sum_{i=1}^n \theta_{u_i,u_i}$. By [15, Proposition 2.1.5] there exists a constant $d > 0$ such that $\|\tilde{E}(x^*x)\| \geq d\|x^*x\|$. Given $b \in B$, we have

$$\begin{aligned} \|\tilde{E}(\lambda^*_{e,b} * \lambda_{e,b})\| &\geq d\|\lambda^*_{e,b} * \lambda_{e,b}\|, \\ \|\tilde{E}(\lambda_{e,b^*b})\| &\geq d\|\lambda_{e,b^*b}\|, \\ \|\lambda_{e,E(b^*b)}\| &\geq d\|\lambda_{e,b^*b}\|, \\ \|E(b^*b)\| &\geq d\|b^*b\|. \end{aligned}$$

This shows that $\mathscr{E} = B$ and $u_1, \ldots, u_n \in B$. Clearly, the set $\{u_1, \ldots, u_n\}$ is a basis for E. The equivalence of (a) and (c) is similar.

3. Conditional expectations corresponding to subgroups

Let G be a discrete group and let $\alpha : G \to \mathrm{Aut}(A)$ be a continuous action of G on a C^*-algebra A. If H is a subgroup of G, then we define conditional expectations E_H (respectively E^r_H) from $A \rtimes G$ (respectively $A \rtimes_r G$) onto $A \rtimes H$ (respectively $A \rtimes_r H$). We show that E_H and E^r_H are of finite index type if and only if $[G : H] < \infty$. Here again the ideas in [12, Proposition 1.2] are used to show that the projection of $l^1(G, A)$ onto $l^1(H, A)$ given by restriction extends to a norm one projection of $A \rtimes G$ onto $A \rtimes H$. We present a proof of this fact for completeness.

PROPOSITION 3.1. *Let G be a discrete group acting on a C^*-algebra A and let H be a subgroup of G. Then, the projection of $l^1(G, A)$ onto $l^1(H, A)$ extends to a conditional expectation of $A \rtimes G$ onto $A \rtimes H$.*

PROOF. First we show that $A \rtimes H$ is a C^*-subalgebra of $A \rtimes G$. Clearly if $f \in l^1(H, A)$, then $\|f\|_{A \rtimes G} \leq \|f\|_{A \rtimes H}$. We need to show the reverse inequality. Let ϕ be a state of $A \rtimes H$. Then by [10, Proposition 7.6.10], there exits a positive definite function $\Phi : H \to A^*$ such that for each $f \in l^1(H, A)$ we have $\phi(f) = \sum_{t \in H} \Phi(t)\big(f(t)\big)$, and $\sum_{t \in H} \Phi(t)(f^* * f)(t) \geq 0$.

Extend Φ to G by letting it to be zero off H. Let $\{x_i\}$ be a complete set of representatives of the right cosets of H. For $f \in l^1(G, A)$ we have

$$\begin{aligned} \sum_{t \in G} \Phi(t)\big(f^* * f(t)\big) &= \sum_{t \in H} \Phi(t) \left(\sum_{s \in G} \alpha_s \left(f(s^{-1})^* f(s^{-1}t) \right) \right) \\ &= \sum_{s \in H} \Phi(t) \left(\sum_i \sum_{t \in H} \alpha_s \left(f(x_i^{-1}s^{-1})^* f(x_i^{-1}st) \right) \right) \\ &= \sum_i \sum_{s \in H} \sum_{t \in H} \Phi(t) \left(\alpha_s \left(f(x_i^{-1}s^{-1})^* f(x_i^{-1}s^{-1}t) \right) \right) \\ &= \sum_i \sum_{t \in H} \Phi(t) \left(f^*_{x_i} * f_{x_i}(t) \right) \geq 0 \end{aligned}$$

where $f_{x_i}(s) = f(x_i^{-1}s)$ and is restricted to H. The inner sums in the last expression are non-negative as Φ is positive definite on H. But this shows that $\Phi : G \to A^*$ is also positive definite. Now by [10, Proposition 7.6.10], Φ defines a positive linear functional on $A \rtimes G$. Hence, every positive linear functional of $A \rtimes H$ extends to a positive linear functional on $A \rtimes G$. It follows that $\|f\|_{A\rtimes H} \le \|f\|_{A\rtimes G}$ for each $f \in l^1(H, A)$. Therefore $A \rtimes G$ contains $A \rtimes H$ as a C^*-subalgebra. Let $f \in l^1(G, A)$ be self-adjoint. Then $f|_H$ is a self-adjoint element of $A \rtimes H$. Hence there exists a state ϕ of $A \rtimes H$ such that $\|f|_H\|_{A\rtimes H} = |\phi(f|_H)|$. Then $\|f|_H\|_{A\rtimes H} = |\phi(f|_H)|$, and $|\tilde{\phi}(f)| \le \|f\|_{A\rtimes G}$ where $\tilde{\phi}$ is the extension of ϕ to $A \rtimes G$. This shows that the mapping of $l^1(G, A)$ onto $l^1(H, A)$ given by restriction extends to a norm one projection of $A \rtimes G$ onto $A \rtimes H$. Finally it is straightforward to verify that this projection is actually a conditional expectation.

REMARK 3.2. The projection of $l^1(G, A)$ onto $l^1(H, A)$ also extends to a conditional expectation of $A \rtimes_r G$ onto $A \rtimes_r H$. We refer to [2] for a proof.

Let G, H, and A be as in the statement of Proposition 3.1. Then

$$E_H : A \rtimes G \to A \rtimes H, \quad \text{and}$$
$$E_H^r : A \rtimes_r G \to A \rtimes_r H$$

denote the conditional expectations given by Proposition 3.1 and Remark 3.2. In general, E_H is not faithful. For example, if H is an amenable subgroup of a non-amenable group G, then E_H is not faithful. However, E_H^r is always faithful. This is because $E_1 : A \rtimes_r G \to A$ and $E_2 : A \rtimes_r H \to A$ evaluations at the identity of G are faithful (cf. [13]), and $E_2 \circ E_H^r = E_1$.

NOTATION 3.3. If $t \in G$ and $a \in A$, then $\lambda_{t,a}$ is the element of $l^1(G, A)$ which is a at t and zero elsewhere. Let $\mathscr{E}_H$ (respectively $\mathscr{E}_H^r$) be the Hilbert $A \rtimes H$-module (respectively the Hilbert $A \rtimes_r H$-module) associated with E_H (respectively E_H^r) as in Remark 2.5.

THEOREM 3.4. *Let G be a discrete group acting on a unital C^*-algebra A and let H be a subgroup of G. Then, the following are equivalent :*

(i) $[G : H] < \infty$,
(ii) *E_H has finite index,*
(iii) *E_H^r has finite index.*

PROOF. (i) implies (ii): If $G = g_1H \cup g_2H \cup \cdots \cup g_nH$, then it is easy to show that $\{\lambda_{g_i,I} : i = 1, 2, \ldots, n\}$ with I the identity of A is a basis for E_H and that $\operatorname{ind} E_H = \sum_i \lambda_{g_i,I}^* \lambda_{g_i,I} = [G : H]\lambda_{e,I}$.

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(ii) implies (i): If ind E_H were finite, then by [15, Proposition 1.7.2], $\mathcal{K}(\mathcal{E}_H)$ contains the identity operator. In the notation of [10, 6], $\mathcal{E}_H$ is just the rigged space Z on [6, p. 92]. Hence by [6, Theorem 2.4], $\mathcal{E}_H$ is isomorphic to $(A \rtimes H) \otimes l^2(G/H)$ as Hilbert $A \rtimes H$-modules. Hence $\mathcal{K}(\mathcal{E}_H)$ is $*$-isomorphic to $(A \rtimes H) \otimes K(l^2(G/H)$. If $K(\mathcal{E}_H)$ were unital, then it follows that $l^2(G/H)$ must be finite dimensional. Hence G/H is a finite set.

(iii) implies (i): In this case the proofs of [6, Lemma 2.3] and [6, Theorem 2.4] can be used to prove that $\mathcal{E}_H^r$ is isomorphic to $(A \rtimes_r H) \otimes l^2(G/H)$ as Hilbert modules over $A \rtimes_r H$. Now the argument given in the non-reduced case can be repeated. This completes the proof of the theorem.

PROPOSITION 3.5. *Let A, B and C be C^*-algebras with the same unit. Let $E : B \to A$ and $F : C \to B$ be faithful conditional expectations. Then $E \circ F : C \to A$ has finite index if and only if E and F have finite indices.*

PROOF. If $\{u_1, u_2, \ldots, u_n\}$, and $\{v_1, \ldots, v_m\}$ are respectively bases for F and E, then the set $\{u_i v_j : i = 1, \ldots, n; j = 1, \ldots, m\}$ is a bases for $E \circ F$. Conversely, suppose that $E \circ F$ has finite index. Then, $E \circ F$ is faithful and by [15, Proposition 1.7.2], E has finite index. It remains to show that F has finite index. Let $\pi : B \to \mathcal{L}(\mathcal{E}_E)$ be the $*$-representation given by left multiplication. Form the tensor product $\mathcal{E}_F \otimes_B \mathcal{E}_E$. Then $\mathcal{E}_F \otimes_B \mathcal{E}_E$, and $\mathcal{E}_{E\circ F}$ are Hilbert A-modules. We show that the A-valued inner products on the dense subset $C \otimes_B B (= C)$ of $\mathcal{E}_F \otimes \mathcal{E}_F$ and the dense subset C of $\mathcal{E}_{E\circ F}$ coincide. Let $b_1, b_2 \in B$ and $c_1, c_2 \in C$. Then,

$$\begin{aligned}\langle c_1 \otimes b_1, c_2 \otimes b_2\rangle &= \langle b_1, \pi(\langle c_1, c_2\rangle) b_2\rangle = \langle b_1, F(c_1^*, c_2) b_2\rangle \\ &= E\big(b_1^* F(c_1^*, c_2) b_2\big) = E\big(F(b_1^* c_1^* c_2 b_2)\big) \quad \text{as} \quad b_1^* b_2 \in B \\ &= E \circ F\big((c_1, b_1)^* (c_2 b_2)\big) = \langle c_1 b_1, c_2 b_2\rangle.\end{aligned}$$

The above computation shows that the mapping $c \otimes_B b \to cb$ from $C \otimes_B B$ to C extends to an isomorphism of $\mathcal{E}_F \otimes_B \mathcal{E}_E$ onto $\mathcal{E}_{E\circ F}$ as Hilbert A-modules. If $E \circ F$ has finite index, then $\mathcal{E}_{E\circ F}$ and hence $\mathcal{E}_F \otimes_B \mathcal{E}_E$ is a finitely-generated projective A-module (cf. [15, Proposition 1.3.4]). Hence we are in the situation of Proposition 1.5. Since π is faithful (cf. 2.5) there exist $u_1, u_2, \ldots, u_n \in \mathcal{E}_F$ such that $1_{\mathcal{E}_F} = \sum_{i=1}^n \theta_{u_i, u_i}$. As $E \circ F$ has finite index, C is closed in $\|.\|_{E\circ F}$ and clearly $\|x\|_{E\circ F} \le \|x\|_F$, $x \in C$. Hence C is closed in $\|.\|_F$ and $\mathcal{E}_F = C$. This means that $u_1, \ldots, u_n \in C$ and hence F has finite index with basis $\{u_1, \ldots, u_n\}$.

THEOREM 3.6. *Let H be a subgroup of the discrete group G and let*

$$\tilde{E} : B \rtimes G \to A \rtimes G,$$
$$E_H : A \rtimes G \to A \rtimes H$$

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be as defined in Remark 2.6 *and Remark* 3.2. *Then* $E_H \circ \tilde{E}$ *has finite index if and only if* E *has finite index, and* $[G : H] < \infty$. *Moreover,* $\operatorname{ind} E_H \circ \tilde{E} = [G : H] \operatorname{ind} E$. *The same results hold in the reduced case.*

PROOF. If E has finite index, then by Theorem 2.8, $\tilde{E}$ has finite index and $\operatorname{ind} \tilde{E} = \operatorname{ind} E$. If $[G : H]$ is also finite, then by Theorem 3.4, E_H has finite index with $\operatorname{ind} E_H = [G : H]\lambda_{e,I}$, which is an element of the center of $B \rtimes G$. Hence, by [15, 1.7.1] $E_H \circ \tilde{E}$ has finite index and we have:

$$\operatorname{ind} E_H \circ \tilde{E} = (\operatorname{ind} E_H)(\operatorname{ind} \tilde{E}) = [G : H](\operatorname{ind} E).$$

Conversely, suppose that $E_H \circ \tilde{E}$ has finite index. Then by Proposition 3.6, E_H and $\tilde{E}$ have finite index. Hence, Theorem 3.4 and Theorem 2.8 imply that $[G : H] < \infty$ and E has finite index.

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Department of Mathematics and Statistics
University of Saskatchewan
106 Wiggins Road
Saskatoon, SK S7N 5E6
Canada
e-mail: khoshkam@math.usask.ca