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IMBEDDING ELEMENTS WHOSE NUMERICAL RANGE HAS A VERTEX AT ZERO IN HOLOMORPHIC SEMIGROUPS

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If a is an element of a complex unital Banach algebra whose numerical range is confined to a closed angular region with vertex at zero and angle strictly less than π, we imbed a in a holomorphic semigroup with parameter in the open right half plane.

There has been recently a great development in the theory of semigroups in Banach algebras (see [**6**]), with attention focused on the relation between the structure of a given Banach algebra and the existence of continuous or holomorphic non-trivial semigroups with certain properties with range in this algebra. The interest of this paper arises from the fact that we relate in it, we think for the first time, this new point of view in the theory of Banach algebras with the already classic one of numerical ranges [**2**, **3**]. The proofs of our results use, in addition to some basic ideas from numerical ranges in Banach algebras, the concept of extremal algebra $E_a(K)$ of a compact convex set K in $\mathbb{C}$ due to Bollobas [**1**] and concretely the realization of $E_a(K)$ achieved by Crabb, Duncan and McGregor [**4**].

If a is an element of a unital Banach algebra, the numerical range of a is defined as:

$$V(a)=\{\phi(a):\phi\in A',\|\phi\|=\phi(I)=1\}$$

Let α be a real number with $0\leqq\alpha<\pi/2$. In what follows K_α will denote the closed angular region with vertex at zero, angle 2α and bisected by the positive part of the real axis.

Lemma 1. *Let A be a complex unital Banach algebra, $0\leqq\alpha<\pi/2$ and $a\in A$ such that $V(a)\subset K_\alpha$. Then for all $t\in\mathbb{R}^+$ and $n\in\mathbb{N}$ we have*

$$\|a^n\exp(-ta)\|\leqq\frac{n!}{t^n\cos^n\alpha}.$$

Proof. Since $V(za)=zV(a)$, it follows that $\operatorname{Re} V(za)\geqq 0$ for $|\arg z|\leqq(\pi/2)-\alpha$ and thus that $\|\exp(-za)\|\leqq 1$ for $z\in\mathbb{C}$ with $|\arg z|\leqq(\pi/2)-\alpha$.

Let $t>0$ and consider the closed disc with centre at t and radius $t\cos\alpha$, which is contained in the set $\{z\in\mathbb{C}:|\arg z|\leqq(\pi/2)-\alpha\}$ where the boundedness given above is valid. The proof is concluded by using the Cauchy inequalities for the holomorphic function $z\to\exp(-za)$.

Proposition 2. *Let A be a complex unital Banach algebra, $0 \leqq \alpha < \pi/2$ and $a \in A$ such that $V(a) \subset K_\alpha$. Given $n \in \mathbb{N}$ take $z \in \mathbb{C}$ with $0 < \operatorname{Re} z < n$. Then the integral*

$$\int_0^\infty t^{n-z-1} a^n \exp(-ta)\, dt$$

is absolutely convergent and

$$g(z) = \int_0^\infty t^{n-z-1} a^n \exp(-ta)\, dt$$

is a holomorphic function in $D = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < n\}$.

Proof.

$$\|t^{n-z-1} a^n \exp(-ta)\| \leqq \|a^n\| t^{n-\operatorname{Re} z-1} \quad t \in \mathbb{R}^+$$

Now the convergence of the integral

$$\int_0^r t^{n-\operatorname{Re} z-1}\, dt$$

for all r in $\mathbb{R}^+$ forces the absolute convergence of

$$\int_0^r t^{n-z-1} a^n \exp(-ta)\, dt$$

for all r in $\mathbb{R}^+$.

On the other hand, the above lemma gives:

$$\|t^{n-z-1} a^n \exp(-ta)\| \leqq t^{-(\operatorname{Re} z+1)} \frac{n!}{\cos^n \alpha} \quad t \in \mathbb{R}^+$$

Now the absolute convergence of the integral

$$\int_r^\infty t^{n-z-1} a^n \exp(-ta)\, dt$$

for all r in $\mathbb{R}^+$ follows by an analogous argument to the preceding one. Hence the first part of the proposition is proved.

Taking into account Dunford's theorem [5, Th. 3.10.1], the function g will be holomorphic in D if and only if the same is true for $\phi \cdot g$ for any continuous linear functional ϕ on A. To show that $\phi \cdot g$ is holomorphic in D we may consider

$$\int_\gamma \phi(g(z))\,dz = \int_\gamma \left(\int_0^\infty t^{n-z-1} \phi(a^n \exp(-ta))\,dt \right) dz$$

where γ is the boundary of some closed triangle in D.

Now we can interchange the order of integration; hence

$$\int_\gamma \phi(g(z))\,dz = \int_0^\infty \left(\int_\gamma t^{n-z-1} \phi(a^n \exp(-ta))\,dz \right) dt = 0$$

by the analyticity of $t^{n-z-1}\ \phi(a^n \exp(-ta))$ as a function of z. By Morera's Theorem, $\phi \cdot g$ is holomorphic in D.

The concept of extremal algebra has its origin in the following well known result of Bollobas [1].

Theorem 3. *Let K be a non empty compact convex set in $\mathbb{C}$. There exists a couple (B,u) where B is a complex unital Banach algebra and u is an element of B such that the closed subalgebra generated by $\{I,u\}$ is B, and*

(a) $V(u)=K$,

(b) *if A is a complex unital Banach algebra and a is an element in A such that $V(a)\subseteq K$, the mapping $\{I,u\}\to\{I,a\}$ extends to a norm-one continuous homomorphism (obviously unique) from B into A.*

A consequence of this theorem is that the couple (B,u) is unique up to isometric isomorphisms. This is the reason why such a couple, or merely its first component, it is known as *the extremal algebra of* K and it is denoted by $E_a(K)$. The element u is called *the generator* of $E_a(K)$. A concrete materialization of $E_a(K)$ as an algebra of complex-valued continuous functions on K is due to Crabb, Duncan and McGregor [4] (see also [3, Section 24]). In this materialization the unit is the constant function equal 1 on K and the generator u is the function $z\to z$ from K into $\mathbb{C}$. Henceforth we shall always refer to this concrete realization of $E_a(K)$.

Let $w,z\in\mathbb{C}$ with w different from any negative real number and let us define w^z as $\exp(z\log w)$ if $w\neq 0$ (where log denotes the principal value of the complex logarithm) and $w^z=0$ if $w=0$.

Lemma 4. *Let K be a non empty compact convex set in $\mathbb{C}$ such that $K\subset K_\alpha$ where $0\leqq\alpha<\pi/2$ and let $z\in\mathbb{C}$ with $0<\operatorname{Re} z$. Then the function $f_z\colon K\to\mathbb{C}$ defined by $f_z(w)=w^z$ belongs to the algebra $E_a(K)$ and the mapping $z\to f_z$ is a holomorphic semigroup with parameter z in the right open half plane.*

Proof. For each $w\in K$ consider the mapping $F_w\colon E_a(K)\to\mathbb{C}$ given by $F_w(f)=f(w)$. Clearly F_w is a multiplicative linear functional on $E_a(K)$, so it is continuous. Let $n\in\mathbb{N}$ with $\operatorname{Re} z<n$. By Theorem 3 and Proposition 2, the integral

$$\int_0^\infty t^{-n-z-1} u^n \exp(-tu)\,dt$$

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defines an element, say f, of $E_a(K)$. Now $f(0)=0$ and for $w\neq 0$

$$f(w)=F_w(f)=F_w\left(\int_0^\infty t^{n-z-1}u^n\exp(-tu)\,dt\right)$$

$$=\int_0^\infty F_w(t^{n-z-1}u^n\exp(-tu))\,dt=\int_0^\infty t^{n-z-1}w^n\exp(-tw)\,dt$$

$$=w^z\int_0^\infty (wt)^{n-z-1}\exp(-tw)w\,dt=w^z\Gamma(n-z)$$

where Γ denotes Euler's gamma function. So

$$w^z=\frac{1}{\Gamma(n-z)}f(w)$$

for all $w\in K$, and therefore

$$f_z=\frac{1}{\Gamma(n-z)}f\in E_a(K)$$

The mapping F from the right open half plane into $E_a(K)$ given by $F(z)=f_z$ satisfies $F(z+w)=F(z)F(w)$ and $F(1)=u$, hence this mapping is a semigroup that we shall denote by $z\to u^z$.

Since

$$u^z=\frac{1}{\Gamma(n-z)}\int_0^\infty t^{n-z-1}u^n\exp(-tu)\,dt$$

for any natural number n, with the only restriction $\operatorname{Re} z<n$ and since $(1/\Gamma)$ is an entire function, Proposition 2 implies that the semigroup $z\to u^z$ is holomorphic.

Theorem 5. *Let A be a complex unital Banach algebra, $a\in A$ such that $V(a)\subset K_\alpha$ where $0\leqq\alpha<\pi/2$. Then there exist a holomorphic semigroup $z\to a^z$ with parameter in the right open half plane and valuated in the algebra A, satisfying $a^1=a$.*

Proof. Lemma 4 with $K=V(a)$ gives the existence of a holomorphic semigroup, $F(z)=u^z$, with parameter in the right open half plane and values in the extremal algebra $E_a(K)$. Composing F with the continuous homomorphism from $E_a(K)$ into A given by Theorem 3 we get the desired semigroup.

We obtain as a consequence of the above theorem the following corollary that improves the well known result on the existence of square roots for positive elements in a C^*-algebra.

Corollary 6. *Let A, a and α be as in the above theorem. Then a has a square root.*

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