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# ON THE RING OF QUOTIENTS AT A PRIME IDEAL OF A RIGHT NOETHERIAN RING

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**1. Introduction.** J. Lambek and G. Michler [**3**] have initiated the study of a ring of quotients $R_P$ associated with a two-sided prime ideal $P$ in a right noetherian ring $R$. The ring $R_P$ is the quotient ring (in the sense of [**1**]) associated with the hereditary torsion class $\tau$ consisting of all right $R$-modules $M$ for which $\mathrm{Hom}_R(M, E_R(R/P)) = 0$, where $E_R(X)$ is the injective hull of the $R$-module $X$.

In the present paper, we shall study further the properties of the ring $R_P$. The main results are Theorems 4.3 and 4.6. Theorem 4.3 gives necessary and sufficient conditions for the torsion class associated with $P$ to have property $(T)$, as well as some properties of $R_P$ when these conditions are indeed satisfied, while Theorem 4.6 gives necessary and sufficient conditions for $R$ to satisfy the right Ore condition with respect to $\mathscr{C}(P)$.

**2. Background.** Throughout, $R$ will denote a right noetherian ring with unity, and "module" will mean a unital right module. Also, $P$ will denote a two-sided prime ideal of $R$. The hereditary torsion class associated with $P$ will be denoted by $\tau$, and the corresponding idempotent topologizing filter of right ideals will be denoted by $\mathscr{D}_P$. One of the results of [**3**] is that $\mathscr{D}_P$ consists of those right ideals $I$ of $R$ for which $r^{-1}I \cap \mathscr{C}(P) \neq \emptyset$, where

$$r^{-1}I = \{x \in R | rx \in I\}.$$

We will denote by $Q(M)$ the "module of quotients" of an $R$-module $M$ with respect to $\tau$, and by $\eta_M$ the associated $R$-homomorphism from $M$ to $Q(M)$. (See [**1**] for the construction of $Q$.) $Q$ can be regarded as a covariant left exact functor from Mod-$R$ to Mod-$R$, and $\eta$ as a natural transformation from the identity functor on Mod-$R$ to $Q$. Also, $Q(R)$ is a ring, $\eta_R$ is a ring homomorphism, and $Q(M)$ has a $Q(R)$-module structure extending the $R$-module structure. Denoting the ring $Q(R)$ by $R_P$, $Q$ can be regarded as a left exact functor from Mod-$R$ to Mod-$R_P$. Theorem 4.1 of [**1**] shows that the following are equivalent:

(i) Every $R_P$-module is $\tau$-torsion-free as an $R$-module.

(ii) Every $R_P$-module is $\tau$-torsion-free and $\tau$-injective as an $R$-module.

(iii) $Q$ is right exact and commutes with direct sums.

(iv) $Q$ is equivalent to the functor $-\otimes_R R_P$.

(v) For each $I$ in $\mathscr{D}_P$, $\eta_R(I)R_P = R_P$.

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We say that $\tau$ has property $(T)$ if these conditions are in fact satisfied. (In [**3**], condition (v) is called the "Walkers' Condition".) If $\tau$ has property $(T)$, then $\eta_R$ is a right localization (as defined in [**5**]), and so $R_P$ will also be right noetherian.

It was shown in [**3**] that $\tau(R)$ is contained in $P$, and that the quotient ring $R_P$ is the same as the ring $R'_{P'}$ obtained from the ring $R' = R/\tau(R)$ by "localizing" at the prime $P' = P/\tau(R)$. It is easy to see that the corresponding filter $\mathscr{D}'_{P'}$ of right $R'$ ideals consists of those ideals $I/\tau(R)$ for which $I \in \mathscr{D}_P$ and (using (v)) that the torsion class $\tau$ in Mod-$R$ has property $(T)$ if and only if the torsion class $\tau'$ in Mod-$R'$ has property $(T)$. Similarly, Proposition 5.5 of [**3**] shows that $R$ has the right Ore condition with respect to $\mathscr{C}(P)$ if and only if $R'$ has the right Ore condition with respect to $\mathscr{C}(P')$.

*In each proof that follows, we shall assume that* $\tau(R) = 0$. Bearing the remarks of the previous paragraph in mind, the reader should have no difficulty in verifying that the statements of the results presented here are valid whether $\tau(R) = 0$ or not. The sole exception is Proposition 3.1, for which the modification required is clear.

**3. General results.** From what has been said above, we see that we have the following commutative diagram of right $R$-modules:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{\ i\ } & R & \xrightarrow{\ \pi\ } & R/P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_P} & & \downarrow{\scriptstyle \eta_R} & & \downarrow{\scriptstyle \eta_{R/P}} & & \\
0 & \longrightarrow & Q(P) & \xrightarrow{Q(i)} & R_P & \xrightarrow{Q(\pi)} & Q(R/P) & &
\end{array}$$

and all rows and columns are exact. In what follows, we shall regard the monomorphisms as inclusions.

PROPOSITION 3.1. *Let $R$ be a right noetherian ring, $P$ a two-sided prime ideal in $R$, and let $Q$ and $\mathscr{D}_P$ have the meanings defined above. Then, if* $\tau(R) = 0$,

(1) $Q(P) \cap R = P$;

(2) $Q(P) = \{q \in R_P | qI \subseteq P$ *for some* $I \in \mathscr{D}_P\}$;

(3) $Q(P)$ *is the largest $R$-submodule $X$ of $R_P$ for which* $X \cap R \subseteq P$.

*Proof.* (1): Clearly $P \subseteq Q(P) \cap R$. Now, if $x \in Q(P) \cap R$, then, from the construction of $Q(P)$ as described in [**1**], there is an $I$ in $\mathscr{D}_P$ such that $xI \subseteq P$. But $I \cap \mathscr{C}(P) \neq \emptyset$, so $xc \in P$ for some $c \in \mathscr{C}(P)$, and this implies that $x \in P$.

(2): Denote the right hand side of (2) by $\mathrm{Cl}(P)$. The construction of $Q(P)$ shows that $Q(P) \subseteq \mathrm{Cl}(P)$. Now, if $y \in \mathrm{Cl}(P)$, $yI \subseteq P$ for some $I \in \mathscr{D}_P$.

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Then $Q(\pi)[y]I = 0$, so $Q(\pi)[y] \in \tau(Q(R/P)) = 0$. Therefore

$$y \in \text{Kernel}(Q(\pi)) = Q(P).$$

(3): If $X$ is an $R$-submodule of $R_P$, and $X \cap R \subseteq P$, then for each $x$ in $X$, $x(x^{-1}R) \subseteq X \cap R \subseteq P$. But $x^{-1}R \in \mathscr{D}_P$, so $x \in \text{Cl}(P) = Q(P)$.

Since $Q(P)$ is the kernel of $Q(\pi)$, there is a monomorphism from $R_P/Q(P)$ into $Q(R/P)$. The next result shows that this can be regarded as the imbedding of a $\tau$-torsion free module into its module of quotients.

Proposition 3.2. *With assumptions and notation as above, $Q(R/P)$ is isomorphic to $Q(R_P/Q(P))$.*

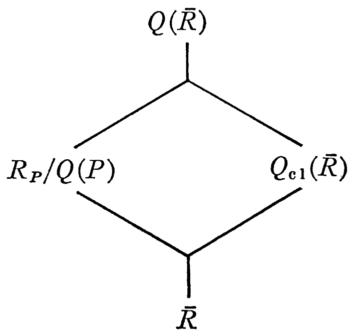
*Proof.* From [**1**, § 3], it suffices to show that $Q(R/P)/\text{Image}(Q(\pi))$ is a $\tau$-torsion module. But, for any $x$ in $Q(R/P)$, there is an $I$ in $\mathscr{D}_P$ such that $xI \subseteq R/P$. Then $xI \subseteq \text{Image}(\eta_{R/P}\pi) = \text{Image}(Q(\pi)\eta_R) \subseteq \text{Image}(Q(\pi))$, as desired.

In what follows we shall denote $R/P$ by $\bar{R}$, and the coset $r + P$ will be denoted $\bar{r}$ for any $r$ in $R$. Any $\bar{R}$-module $M$ can be given a natural $R$-module structure by defining $m*r$ to be $m\bar{r}$. Let $Q_{cl}(\bar{R})$ be the classical quotient ring of $\bar{R}$. Then $Q_{cl}(\bar{R})$ will be an essential extension (as an $R$-module) of $\bar{R}$, so we have $\bar{R} \subseteq Q_{cl}(\bar{R}) \subseteq E_R(\bar{R})$. For any $q$ in $Q_{cl}(\bar{R})$ there is a $c \in \mathscr{C}(P)$ such that $q\bar{c} \in \bar{R}$. Therefore $q*(cR + P) \subseteq \bar{R}$, and, as is shown in [**3**], $cR + P \in \mathscr{D}_P$. Therefore $Q_{cl}(\bar{R})/\bar{R}$ is a $\tau$-torsion module, and is contained in $\tau(E_R(\bar{R})/\bar{R}) = Q(\bar{R})/\bar{R}$. Thus $Q_{cl}(\bar{R})$ is contained in $Q(\bar{R})$.

Proposition 3.3. $Q_{cl}(\bar{R}) = \text{Ann}_{Q(\bar{R})}(P)$.

*Proof.* Clearly the $R$-module $Q_{cl}(\bar{R})$ is annihilated by $P$. Conversely, $\text{Ann}_{Q(\bar{R})}(P)$ has a natural $\bar{R}$-module structure. As such, it is an essential extension of $\bar{R}$, since $Q(\bar{R})$ is an essential ($R$-module) extension of $\bar{R}$. Therefore $\text{Ann}_{Q(\bar{R})}(P)$ is contained in $E_{\bar{R}}(\bar{R}) = Q_{cl}(\bar{R})$.

If we regard $R_P/Q(P)$ as a submodule of $Q(\bar{R})$, we see that $Q(\bar{R})$ has $R$-submodules as represented in the following diagram:



We identify the submodule $(R + Q(P))/Q(P)$ of $R_P/Q(P)$ with

$$R/(R \cap Q(P)) = R/P = \bar{R}.$$

Recall that, for any right ideal $B$ in a ring $S$, the *idealizer* of $B$ in $S$ (denoted $\mathfrak{J}(B)$) is $\{s \in S | sB \subseteq S\}$. This is the largest subring of $S$ which contains $B$ as a two-sided ideal.

PROPOSITION 3.4. *With definitions and notations as above, $\mathfrak{J}(Q(P))$, the idealizer of $Q(P)$ in $R_P$, satisfies*

$$\mathfrak{J}(Q(P))/Q(P) = Q_{c1}(\bar{R}) \cap (R_P/Q(P))$$

*Proof.* First of all, note that, for any $q \in R_P$, $qQ(P) \subseteq Q(P)$ if and only if $qP \subseteq Q(P)$. The proof one way is trivial, so suppose that $q \in R_P$ and $qP \subseteq Q(P)$. For any $\rho$ in $Q(P)$, $\rho^{-1}P$ is in $\mathscr{D}_P$ (Proposition 3.1) and $q\rho(\rho^{-1}P) \subseteq qP \subseteq Q(P)$. Thus $Q(\pi)[q\rho] \in \tau(Q(\bar{R})) = 0$, so $q\rho \in \text{Kernel}(Q(\pi)) = Q(P)$.

Now we turn to the result in question. For $q \in R_P$, denote by $\bar{q}$ the coset $q + Q(P)$, an element of $R_P/Q(P) \subseteq Q(\bar{R})$. For any such $q$, we have

$$\begin{aligned}\bar{q} \in Q_{c1}(\bar{R}) \cap (R_P/Q(P)) &\Leftrightarrow \bar{q}P = 0 \text{ (by Proposition 3.3)}\\ &\Leftrightarrow qP \subseteq Q(P)\\ &\Leftrightarrow qQ(P) \subseteq Q(P)\\ &\Leftrightarrow q \in \mathfrak{J}(Q(P))\\ &\Leftrightarrow \bar{q} \in \mathfrak{J}(Q(P))/Q(P).\end{aligned}$$

This completes the proof of the proposition.

We wish to examine the conditions under which $Q(P)$ is a two-sided ideal of $R_P$. In order to do this, it is useful to examine the concept of primeness. In [**3**], a right ideal $B$ of a ring $S$ was said to be a *prime right ideal* if, whenever $sSt \subseteq B$, for $s$ and $t$ in $S$, then one of $s$ or $t$ is in $B$. On the other hand, in [**5**], a right $S$-module $M$ is said to be a *prime module* if $\text{Ann}_S(M) = \text{Ann}_S(M')$ for each nonzero submodule $M'$ of $M$. These two notions are compatible, for it is not difficult to verify (when $S$ has a unity) that $B$ is a prime right ideal of $S$ if and only if $S/B$ is a prime module.

PROPOSITION 3.5. *If $P$ is a two-sided ideal in a right noetherian ring $R$, $Q(P)$ is a prime right ideal in $R_P$, whence $R_P/Q(P)$ is a prime right $R_P$-module.*

*Proof.* Suppose that $q_1R_Pq_2 \subseteq Q(P)$ for $q_1$, $q_2$ in $R_P$. Then

$$q_1(q_1^{-1}R) \cdot q_2(q_2^{-1}R) \subseteq Q(P) \cap R = P,$$

and each $q_i(q_i^{-1}R)$ is a right ideal of $R$. Since $P$ is prime, either $q_1(q_1^{-1}R)$ or $q_2(q_2^{-1}R)$ is in $P$, and so, by Proposition 3.1, either $q_1$ or $q_2$ is in $Q(P)$.

*Remark.* The annihilator of any prime module is a two-sided prime ideal. In particular, $\text{Ann}_{R_P}(R_P/Q(P))$ is a prime ideal in $R_P$. This ideal, which we will denote by $\tilde{P}$, is the largest two-sided ideal of $R_P$ conained in the right ideal $Q(P)$.

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The next result tells us when $Q(P)$ is itself a two-sided ideal of $R_P$.

THEOREM 3.6. *Let $P$ be a two-sided prime ideal in the right noetherian ring $F$. Then the following are equivalent:*

(1) $Q(P) = \mathrm{Ann}_{R_P}(R_P/Q(P))$;

(2) *$Q(P)$ is a two-sided ideal of $R_P$*;

(3) $R_P P \subseteq Q(P)$;

(4) $R_P/Q(P) \subseteq Q_{\mathrm{cl}}(R/P)$;

(5) *$R_P/Q(P)$ is a prime $R$-module.*

*Proof.* The implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) are trivial, and the first paragraph of the proof of Proposition 3.4 shows (3) implies (2). Clearly (2) and (4) are equivalent by Proposition 3.4. Since $P$ is the annihilator of any non-zero $R$-submodule of $Q_{\mathrm{cl}}(R/P)$ we see that (4) implies (5). The fact that $P$ annihilates the $R$-submodule $(R + Q(P))/Q(P)$ of $R_P/Q(P)$ guarantees that, when (5) holds, $P$ annihilates all of $R_P/Q(P)$ so (3) follows.

In the situation described by the following commutative diagram with exact rows and columns,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{i} & R & \xrightarrow{\pi} & \bar{R} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_P} & & \downarrow{\scriptstyle \eta_R} & & \downarrow{\scriptstyle \eta_{\bar{R}}} & & \\
0 & \longrightarrow & Q(P) & \xrightarrow{Q(i)} & R_P & \xrightarrow{Q(\pi)} & Q(\bar{R}) & &
\end{array}$$

it is tempting to ask when $Q(\bar{R})$ can be made into a ring so that $\eta_{\bar{R}}$ is a ring homomorphism. The next result answers this question.

THEOREM 3.7. *Let $P$ be a two-sided prime ideal in the right noetherian ring $R$. Then the following are equivalent:*

(1) *$Q(\bar{R})$ is a prime $R$-module*;

(2) $P \subseteq \mathrm{Ann}_R(Q(\bar{R}))$;

(3) $Q(\bar{R}) = Q_{\mathrm{cl}}(\bar{R})$;

(4) *$Q(\bar{R})$ has a ring structure for which $\eta_{\bar{R}}$ is a ring homomorphism*;

(5) *$Q(\bar{R})$ has a ring structure for which $Q(\pi)$ is a ring homomorphism*;

(6) $Q(P) = \mathrm{Ann}_{R_P}(Q(\bar{R}))$;

(7) *$Q(\bar{R})$ is a prime right $R_P$-module, and $Q(P)$ is a two-sided ideal of $R_P$.*

*Proof.* Since $P$ annihilates the $R$-submodule $\bar{R}$ of $Q(\bar{R})$, (1) implies (2). Conversely, if (2) holds, then, since $Q(\bar{R})$ is an essential extension of the $R$-module $\bar{R}$, for $W$ a nonzero $R$-submodule of $Q(\bar{R})$,

$$P \subseteq \mathrm{Ann}_R(Q(\bar{R})) \subseteq \mathrm{Ann}_R(W) \subseteq \mathrm{Ann}_R(W \cap \bar{R}) = \mathrm{Ann}_R(\bar{R}) = P,$$

so $\mathrm{Ann}_R(W) = \mathrm{Ann}_R(Q(\bar{R})) = P$, and $Q(\bar{R})$ is indeed a prime $R$-module.

We now give a cyclical proof of the equivalence of (2) through (7).

If (2) holds, then $Q(\bar{R}) = \mathrm{Ann}_{Q(\bar{R})}(P) = Q_{cl}(\bar{R})$, by Proposition 3.3, proving (3). Also, (3) clearly implies (4).

Suppose (4) holds, and that $*$ is the multiplication in $Q(\bar{R})$. For $q_1$ and $q_2$ in $R_P$, denote by $w$ the element $Q(\pi)[q_1] * Q(\pi)[q_2] - Q(\pi)[q_1q_2]$. Then, setting $I = q_2^{-1}R$ (an element of $\mathscr{D}_P$), we see $wI = 0$, so $w \in \tau(Q(\bar{R})) = 0$. Thus $Q(\pi)$ will be a ring homomorphism, and (4) implies (5).

If (5) holds, then $Q(P) = \mathrm{Kernel}(Q(\pi))$ is a two-sided ideal of $R_P$. Since $Q(\pi)$ is always a homomorphism of $R_P$ modules, the fact that it is in this case a ring homomorphism as well guarantees that $Q(P)$ will annihilate $Q(\bar{R})$. On the other hand, $\mathrm{Ann}_{R_P}Q((\bar{R})) \subseteq \mathrm{Ann}_{R_P}(R_P/Q(P)) = \tilde{P} \subseteq Q(P)$. Thus (5) implies (6).

To show that (6) implies (7), note that (6) certainly implies that $Q(P)$ is a two-sided ideal of $R_P$. From Proposition 3.2, $Q(\bar{R})$ is essential over $R_P/Q(P)$ as an $R$-module, and, *a fortiori*, as an $R_P$-module. Thus, for any nonzero $R_P$-submodule $Y$ of $Q(\bar{R})$, $Q(P) = \mathrm{Ann}_{R_P}(Q(\bar{R})) \subseteq \mathrm{Ann}_{R_P}(Y) \subseteq \mathrm{Ann}_{R_P}(Y \cap (R_P/Q(P))) =$ (Proposition 3.5) $\mathrm{Ann}_{R_P}(R_P/Q(P)) = \tilde{P} \subseteq Q(P)$. Thus $\mathrm{Ann}_{R_P}(Y) = Q(P)$, so $Q(\bar{R})$ is a prime $R_P$-module.

Finally, assume (7). Then $\mathrm{Ann}_{R_P}(Q(\bar{R})) = \mathrm{Ann}_{R_P}(R_P/Q(P)) = Q(P)$, so $\mathrm{Ann}_R(Q(\bar{R})) = R \cap \mathrm{Ann}_{R_P}(Q(\bar{R})) = R \cap Q(P) = P$, which proves (2).

## 4. Property (T).

LEMMA 4.1. *Let $P$ be a two-sided prime ideal in the right noetherian ring $R$, and let $\tau$, $\mathscr{D}_P$, and $R_P$ be the hereditary torsion class, idempotent filter, and ring of quotients associated with $P$ respectively. Denote $R/P$ by $\bar{R}$, and suppose that $X$ and $Y$ are $\bar{R}$-modules such that $X$ is an essential submodule of $Y$. Then $X$ and $Y$ are, in a natural way, $R$-modules, and the $R$-module $Y/X$ is* $\tau$-torsion.

*Proof.* For any $y \in Y$, $J = \{\bar{r} \in \bar{R} | y\bar{r} \in X\}$ is an essential right ideal of the prime noetherian ring $\bar{R}$, and so, by a well-known result of Goldie, $J$ contains a regular element of $\bar{R}$. Thus, regarding $X$ and $Y$ as $R$-modules, $y^{-1}X = \{r \in R | yr \in X\}$ contains an element $c$ of $\mathscr{C}(P)$. Therefore $y^{-1}X \supseteq cR + P$, and (see [**3**]) $cR + P$ is in $\mathscr{D}_P$.

LEMMA 4.2. *With the same notation as above, $Q(\bar{R})$ is isomorphic to $\bigoplus_{i=1}^n X_i$, where the $X_i$'s are mutually isomorphic uniform $R$-modules (and $R_P$-modules), and $n$ = Goldie dimension of the ring $\bar{R}$.*

*Furthermore, if $\tau$ has property (T), each $X_i$ is an irreducible $R_P$-module.*

*Proof.* The ring $\bar{R}$ is prime and right noetherian. As is well known, for any non-zero uniform right ideal $U$ of $\bar{R}$, there are uniform right ideals $W_1 = U$, $W_2$, $W_3$, ... $W_n$ of $\bar{R}$ such that $W = \sum_{i=1}^n W_i$ is essential, and the sum is direct. By Lemma 4.1, $\bar{R}/W$ is $\tau$-torsion, so $Q(\bar{R}) \cong Q(W) \cong \bigoplus_{i=1}^n Q(W_i)$.

For any $i$ and $j$, $W_iW_j \neq 0$ (since $\bar{R}$ is prime) so $wW_j \neq 0$ for some $w \in W_i$. The $\bar{R}$-map $\lambda_w: W_j \to W_i$ defined by $\lambda_w(x) = wx$ is one-to-one, as a well-

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known argument of Goldie shows. Applying the left exact functor $Q$ to the exact sequence

$$0 \to W_j \xrightarrow{\lambda_w} W_i \to W_i/wW_j \to 0$$

of $R$-modules, and noting that $W_i/wW_j$ is $\tau$-torsion, we see that $Q(\lambda_w)$ is an $R_P$-isomorphism. Each $Q(W_i)$ is uniform since it is an essential extension of the uniform $R$-module $W_i$.

Finally, suppose $\tau$ has property $(T)$, and let $X$ be any nonzero $R_P$-submodule of $Q(W_1)$. Then $Q(W_1)/X$ is an $R$-module homomorphic image of $Q(W_1)/(X \cap W_1)$, and the latter is an extension of $W_1/(X \cap W_1)$ by $Q(W_1)/W_1$. Since $Q(W_1)/W_1$ is always $\tau$-torsion, and $W_1/(X \cap W_1)$ is $\tau$-torsion (by Lemma 4.1), it follows that $Q(W_1)/(X \cap W_1)$ and $Q(W_1)/X$ are $\tau$-torsion. But $Q(W_1)/X$ is an $R_P$-module, and property $(T)$ guarantees that it is also $\tau$-torsion free. Therefore $Q(W_1)/X = 0$, and $Q(W_1)$ is irreducible.

Suppose that $Q(\bar{R})$ is a semisimple $R_P$ module. (This is true, for example, when $\tau$ has property $(T)$, as Lemma 4.2 shows.) From Proposition 3.2, $Q(\bar{R})$ is an essential $R_P$-extension of $R_P/Q(P)$, and therefore $Q(\bar{R}) = R_P/Q(P)$. Therefore $Q(P)$ is a *semimaximal* right ideal of $R_P$, that is, $Q(P)$ is a finite intersection $\bigcap_{i=1}^{n} M_i$ of maximal right ideals of $R_P$. From Lemma 4.2, it follows that $R_P/M_i \cong R_P/M_j$ for all $i$ and $j$. Denote by $\tilde{P}$ the two-sided $R_P$-ideal $\mathrm{Ann}_{R_P}(R_P/M_i) = \mathrm{Ann}_{R_P}(R_P/Q(P))$, the two-sided prime associated with the prime right ideal $Q(P)$ of $R_P$. One can form the hereditary torsion class $\sigma$ in Mod-$R_P$ determined by $R_P/M_i$ (equivalently, by $R_P/Q(P) = Q(\bar{R})$). This consists of all $R_P$-modules $X$ for which $\mathrm{Hom}_{R_P}(X, E_{R_P}(Q(\bar{R}))) = 0$. Since the $R$-module $Q(\bar{R})$ is $\tau$-torsion free, so is its $R$-module injective hull $I = E_R(Q(\bar{R}))$. Then $I = Q(I)$, so $I$ has an $R_P$-module structure, and we have $Q(\bar{R}) \subseteq I \subseteq E_{R_P}(Q(\bar{R}))$. Furthermore, since $I$ is an $R_P$-module which is $\tau$-torsion free as an $R$-module, $\mathrm{Hom}_{R_P}(X, I) = \mathrm{Hom}_R(X, I)$ for all $X$ in Mod-$R_P$. (For if $\alpha: X \to I$ is an $R$-map and $q \in R_P$, then $w = \alpha(x)q - \alpha(xq)$ is annihilated by the right $R$ ideal $q^{-1}R$, a member of $\mathscr{D}_P$, so $w \in \tau(I) = 0$.) In particular, any $R_P$-module $X$ in $\sigma$ satisfies $\mathrm{Hom}_R(X, I) = \mathrm{Hom}_{R_P}(X, I) \subseteq \mathrm{Hom}_{R_P}(X, E_{R_P}(Q(\bar{R}))) = 0$. Thus such an $X$ satisfies $\mathrm{Hom}_R(X, I) = 0$, where $I = E_R(Q(\bar{R})) = E_R(\bar{R})$, and $X$, as an $R$-module, is in $\tau$. In particular, if $\tau$ has property $(T)$, then $X$, being both $\tau$-torsion and $\tau$-torsion free, would be 0, so $\sigma$ would be the zero torsion class. These observations lead us to the next result.

THEOREM 4.3. *Let $R$ be a right noetherian ring, and $P$ a two-sided prime ideal in $R$. Then the torsion class $\tau$ determined by $P$ has property $(T)$ if and only if the following are true:*

(i) *$R_P$ has only one isomorphism class of irreducible module, and*

(ii) *The socle of the $R_P$ module $Q(R/P)$ is nonzero.*

*Furthermore, if $\tau(R) = 0$ and $\tau$ does have property $(T)$, then $Q(P) = PR_P$, and this is a semimaximal right ideal of $R_P$. The ring $R_P$ is right noetherian and its*

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*Jacobson radical is $\tilde{P}$, the two-sided prime associated with $PR_P$. This is a unique maximal two-sided ideal of $R_P$. In particular, either $PR_P = \tilde{P}$ or $R_P PR_P = R_P$.*

*Proof.* Suppose first that (i) and (ii) are satisfied. Then there is a simple $R_P$-module $V$ which is $\tau$-torsion free as an $R$-module. By (i). $E_{R_P}(V)$ is a cogenerator for Mod-$R_P$, so any $R_P$-module can be embedded into a product of copies of $E_{R_P}(V)$. We claim that $E_{R_P}(V)$ is $\tau$-torsion free as an $R$-module. For if $T = \tau(E_{R_P}(V))$ is not zero, then $TR_P \supseteq V$, and any $v$ in $V$ can be written $v = t_1q_1 + t_2q_2 + \ldots + t_kq_k$, where each $t_i \in T$ and each $q_i \in R_P$. For $I = \bigcap_{i=1}^{k} q_i^{-1}R \in \mathscr{D}_P$, $vI \subseteq T$ so $vI$ is $\tau$-torsion. Therefore $vI = 0$, and $v \in \tau(V) = 0$, a contradiction. Thus $E_{R_P}(V)$ is $\tau$-torsion free as an $R$-module.

Since any $R_P$-module can be embedded into a product of $\tau$-torsion free modules, any $R_P$-module is $\tau$-torsion free as an $R$-module, and property $(T)$ holds.

Conversely, suppose $\tau$ has property $(T)$. Then $Q(P) = PR_P$, and it follows from Lemma 4.2 that this is semimaximal. Since in this case $R_P/Q(P) = Q(R/P)$, (ii) follows. Suppose that $PR_P = \bigcap_{i=1}^{n} M_i$, where each $M_i$ is maximal in $R_P$. From the remarks preceding this theorem, the torsion class in Mod-$R_P$ determined by each $R_P/M_i$ (equivalently, by $R/PR_P$) is the zero torsion class. Thus (see [**2**]) any $R_P$-module can be embedded into a product of copies of $E_{R_P}(R_P/M_1)$. In particular, any irreducible $R_P$-module is isomorphic to $R_P/M_1$, and $(i)$ is established. Also $J(R_P)$, the intersection of the annihilators of all irreducible $R_P$-modules, coincides with

$$\mathrm{Ann}_{R_P}(R_P/M_1) = \mathrm{Ann}_{R_P}(R_P/PR_P) = \tilde{P}.$$

Any proper two-sided ideal $I$ of $R_P$ is contained in $\tilde{P}$. To see this, note that $I$ is contained in some maximal right $R_P$-ideal, say $I \subseteq U$. Then $R_P/U$ is irreducible, and $I \subseteq \mathrm{Ann}_{R_P}(R_P/U) = J(R_P) = \tilde{P}$.

The last statement is evident.

When $\tau$ has property $(T)$, the remarks preceding Theorem 4.3 show that the zero torsion class in Mod-$R_P$ is, in one sense, a torsion class determined by the ideal $P$ of $R$. Since, in this case, $R_P$ is right noetherian, there is another torsion class in Mod-$R_P$ related to $P$, namely the torsion class $\mu$ determined by the two-sided prime $\tilde{P}$ of $R_P$. The next result determines when these two torsion classes coincide. This occurs, for example, if $PR_P = \tilde{P}$, since Lemma 4.2 shows condition (2) below is satisfied. However, Example 4.5 below shows these classes do not always coincide.

It was Arthur Chatters who pointed out to the author that condition (4) can be included in this result.

PROPOSITION 4.4. *Let $P$ be a two-sided prime ideal in the right noetherian ring $R$, and suppose $\tau$ has property $(T)$. Let $\tilde{P}(= J(R_P))$ be the two-sided prime in $R_P$ associated with the prime right ideal $PR_P$, and let $\mu$ be the torsion class in* Mod-$R_P$ *determined by $\tilde{P}$. Then the following are equivalent:*

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(1) *$\mu$ is the zero torsion class;*

(2) *$R_P/\tilde{P}$ has non-zero socle (and hence is a simple artinian ring);*

(3) *$RP_P \cap \mathscr{C}(\tilde{P}) = \emptyset$, where $\mathscr{C}(\tilde{P})$ is the set of elements of $R_P$ which are regular modulo $\tilde{P}$;*

(4) *The elements of $\mathscr{C}(\tilde{P})$ are units in $R_P$.*

*Proof.* To see that (1) implies (2), let $I$ be any right ideal of $R_P$ containing $\tilde{P}$. Then $I = R_P$ or $I \cap \mathscr{C}(\tilde{P}) = \emptyset$, for if $c \in I \cap \mathscr{C}(\tilde{P})$, then $I \supseteq cR_P + \tilde{P} = R_P$, the latter equality due to the fact that $cR_P + \tilde{P}$ is in the filter of ideals associated with $\mu$. Thus no proper right ideal $I/\tilde{P}$ of $R_P/\tilde{P}$ contains a regular element. By a theorem of Goldie, since $R_P/\tilde{P}$ is prime and noetherian, $R_P/\tilde{P}$ has no proper essential ideals, so it is simple artinian.

That (2) implies (4) follows easily from the fact that, if (2) holds, elements of $\mathscr{C}(\tilde{P})$ are units modulo $\tilde{P}$, and thus (since $\tilde{P} = J(R_P)$) are in fact units in $R_P$.

Since every member of the filter $\mathscr{D}_{\tilde{P}}$ associated with $\mu$ contains an element of $\mathscr{C}(\tilde{P})$, (4) implies that $\mathscr{D}_{\tilde{P}} = \{R_P\}$, so (1) follows.

(4) implies (3), for otherwise we would have $PR_P = R_P$. If (3) holds, then $PR_P/\tilde{P}$ is not an essential right ideal of $R_P/\tilde{P}$. If $X$ is a right ideal of $R_P/\tilde{P}$, intersecting $PR_P/\tilde{P}$ in 0, then $X$ lies in the socle of $R_P/\tilde{P}$, since $X$ is isomorphic to a submodule of the completely reducible module $R_P/PR_P$. Thus (3) implies (2).

Example 5.9 of [**3**] gives an instance where $\tau$ has property $(T)$, yet $PR_P$ is not an ideal in $R_P$. In that example, $R_P/J(R_P)$ is a simple artinian ring. We give now an example of a case where $\tau$ has property $(T)$, $PR_P$ is not an ideal of $R_P$, and $R_P/J(R_P)$ is *not* simple artinian. This example is taken from § 7 of [**4**].

*Example* 4.5. Let $F$ be a field of characteristic zero, and let $S$ be the ring $F[y][x]$, where $xy - yx = 1$. Any element $\sigma$ in $S$ can be written in the form $\sigma_0(y) + x\sigma_1(y) + \ldots x^k\sigma_k(y)$, where each $\sigma_i(y)$ is a polynomial in $y$. For $\sigma \in S$, $x\sigma - \sigma x = \partial\sigma/\partial y$. The ring $S$ is a simple hereditary right and left noetherian domain, and $A = xS$ is a maximal right ideal. Let $R$ be the subring $\mathfrak{J}(A) = F + xS$. It follows from § 4 of [**4**] that $R$ is also right and left hereditary and noetherian. Furthermore, $A$ is the only nonzero two-sided ideal of $R$, for if $C$ is any nonzero ideal of $R$, $C \supseteq ACA = x(SCxS) = xS = A$, and $A$ is clearly maximal. Also, the $R$-module $S$ is essential over $R$, so $R$ and $S$ are orders in the same division ring $D$.

We shall examine the localization at the prime ideal $A$ of the ring $R$. Since $R$ is right hereditary and right noetherian, the torsion class $\tau$ has property $(T)$ (see [**1**]), and $R$ is $\tau$-torsion free since $R$ is a domain. It follows from [**4**] that both $S/R$ and $R/A$ are irreducible $R$-modules, and these are not isomorphic since $x$ annihilates $R/A$, but (since $y^2x = xy^2 - 2y \notin R$) $x$ does not annihilate $S/R$. Therefore $\mathrm{Hom}_R(S/R, E_R(R/A)) = 0$, whence $S/R$ is $\tau$-torsion. Since $R \subseteq S \subseteq D = E_R(R)$, $S \subseteq R_A$. For any two-sided ideal $I$ of $R_A$, either $S \cap I = 0$ (whence $I = 0$) or $S \cap I = S$, in which case $I$ contains 1 so $I = R_A$.

Therefore $R_A$ is a simple ring. If $R_A (= R_A/J(R_A))$ were artinian, then $R_A = D$, so $AR_A = D$, and $A = R \cap AR_A = R$, a contradiction.

For a two-sided prime ideal $P$ in a right noetherian ring $R$, we say that $R$ satisfies the *right Ore condition* with respect to $\mathscr{C}(P)$ if, for any $x$ in $R$ and $c$ in $\mathscr{C}(P)$, there exist $x'$ in $R$ and $c'$ in $\mathscr{C}(P)$ such that $xc' = cx'$. Proposition 5.5 of [**3**] shows that, when $\tau(R) = 0$, the right Ore condition with respect to $\mathscr{C}(P)$ holds if and only if every element in $\mathscr{C}(P)$ is a unit in $R_P$.

Our final result is really an addition to Theorem 5.6 of [**3**].

THEOREM 4.6. *Let $P$ be a two-sided prime ideal in the right noetherian ring $R$. Let $R^* = R/\tau(R)$ and $P^* = P/\tau(R)$. Then the following are equivalent:*

(1) *$R$ has the right Ore condition with respect to $\mathscr{C}(P)$;*

(2) *$R^*$ has the right Ore condition with respect to $\mathscr{C}(P^*)$;*

(3) *$\tau$ has property $(T)$ and $R_P P^* R_P \neq R_P$;*

(4) *$P^* R_P = J(R_P)$ and $R_P/P^* R_P$ is simple artinian;*

(5) *$R_P P^* R_P \neq R_P$, $P^* R_P$ is a semimaximal right ideal of $R_P$, and $R_P$ has only one isomorphism type of irreducible module.*

*Proof.* (1) and (2) are equivalent by Lemma 5.1 of [**3**]. Therefore, without loss of generality, we assume $\tau(R) = 0$ and drop the $*$.

Clearly (4) implies (5). Now assume that (5) holds. Then $R_P/Q(P)$ is a homomorphic image of $R_P/PR_P$, so the $R_P$-socle of $R_P/Q(P)$ is not zero. That (3) is satisfied follows now from Theorem 4.3, so (5) implies (3). Also, Theorem 4.3 shows (3) implies (4), and the equivalence of (1) and (4) is shown in the proof of Theorem 5.6 of [**3**].

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