APPROXIMATIONS TO THE NORM OF THE SINGULAR PART OF A MEASURE

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(Received 6th August 1974)

Let G be a non-discrete LCA group with dual group Γ. Denote by $M(G)$ the usual convolution algebra of bounded Borel measures on G and $M_a(G)$ those $\mu \in M(G)$ which are absolutely continuous with respect to m_G—the Haar measure on G.

The Fourier-Stieltjes transform $\hat{\mu}$ of a measure $\mu \in M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \quad (\gamma \in \Gamma).$$

A closed subset $\mathscr{R}$ of Γ is called a Riesz (or small) set if whenever supp $\hat{\mu} \subset \mathscr{R}$ then $\mu \in M_a(G)$. In this paper we pursue a line of research originated by R. Doss in (3) and subsequently explicated by him in (5).

Suppose $p(x) = \sum_n a_n(-x, \gamma_n)$ is a trigonometric polynomial on G with $\| p \|_\infty \leqq 1$. We shall concern ourselves with calculating the "limit" of $\sum_n a_n\hat{\mu}(\gamma_n)$ off the Riesz set $\mathscr{R}$. We need to make our idea for taking such a limit precise:

For any subset $\mathscr{S} \subset \Gamma$ denote by $P_{\mathscr{S}}$ the set of all trigonometric polynomials on G of the form $p(x) = \sum_n a_n(-x, \gamma_n)$ with $\| p \|_\infty \leqq 1$ and $\gamma_n \notin \mathscr{S}$. Define

$$\| \mu \|_{\mathscr{S}} = \sup_{p \in P_{\mathscr{S}}} |\sum_n a_n \hat{\mu}(\gamma_n)|.$$

Then by the limit of $\sum_n a_n\hat{\mu}(\gamma_n)$ off $\mathscr{S}$ we mean

$$\lim_{K \to \infty} \| \mu \|_{\mathscr{S} \cup K} = \inf_K \| \mu \|_{\mathscr{S} \cup K} \quad (K \text{ compact in } \Gamma).$$

For $\mathscr{R}$, a Riesz set, put

$$\lim_{\mathscr{R} \to \infty} \| \mu \|_{\mathscr{R}} = \inf_{\mathscr{R}} \| \mu \|_{\mathscr{R}}.$$

Note that in this general setting the methods of (5) are not available. Our main result now follows; the proof is pleasantly simple.

Theorem 1. *Let $\mu \in M(G)$ and let $\mu = \mu_a + \mu_s$ be the canonical Lebesgue decomposition of μ with $\mu_a \in M_a(G)$ and μ_s singular with respect to m_G. Then*

$$\lim_{\mathscr{R} \to \infty} \| \mu \|_{\mathscr{R}} = \lim_{K \to \infty} \| \mu \|_{\mathscr{R} \cup K} = \| \mu_s \|.$$

https://doi.org/10.1017/S0013091500015789 Published online by Cambridge University Press

Proof. Let $\mathscr{R}$ be any Riesz set. Then

$$\| \mu \|_{\mathscr{R}} \leqq \| \mu_s \| + \| \mu_a \|_{\mathscr{R}}$$

and since $\lim_{\mathscr{R} \to \infty} \| \mu_a \|_{\mathscr{R}} = 0$ (this is easy to check), we gather

$$\lim_{\mathscr{R} \to \infty} \| \mu \|_{\mathscr{R}} \leqq \| \mu_s \|. \tag{1}$$

To prove the opposite inequality we obtain via Proposition 2 and the Main Theorem of (4) a measure $\lambda_{\mathscr{R}} \in M(G)$ such that

$$\hat{\lambda}_{\mathscr{R}} = \hat{\mu} \quad \text{off } \mathscr{R} \tag{2}$$

and

$$\| \lambda_{\mathscr{R}} \| \leqq \| \mu \|_{\mathscr{R}}. \tag{3}$$

Observe that the identity

$$\mu_s + \mu_a - (\mu - \lambda_{\mathscr{R}}) = \lambda_{\mathscr{R}}$$

when combined with (2) yields the inequality

$$\| \mu_s \| \leqq \| \lambda_{\mathscr{R}} \|.$$

By (3) we must have $\| \mu_s \| \leqq \| \mu \|_{\mathscr{R}}$. Thus

$$\| \mu_s \| \leqq \| \mu \|_{\mathscr{R}} \quad \text{for all Riesz sets } \mathscr{R}. \tag{4}$$

Hence (1) and (4) give

$$\lim_{\mathscr{R} \to \infty} \| \mu \|_{\mathscr{R}} = \| \mu_s \|.$$

Finally, let K be any compact subset of Γ. We claim that $\mathscr{R} \cup K$ is a Riesz set. For suppose supp $\hat{v} \subset \mathscr{R} \cup K$, $v \in M(G)$. Then there is an $f \in L^1(G)$ such that $\hat{f}(K) = 1$. Since $v - f * v \in M_a(G)$, this implies $v \in M_a(G)$. We conclude therefore, by (4), that

$$\| \mu_s \| \leqq \| \mu \|_{\mathscr{R} \cup K}.$$

Since $\lim_{K \to \infty} \| \mu_a \|_{\mathscr{R} \cup K} = 0$, it follows that $\lim_{K \to \infty} \| \mu \|_{\mathscr{R} \cup K} = \| \mu_s \|$ and the theorem is proved.

Corollary 1. *A measure $\mu \in M(G)$ is absolutely continuous if and only if*

$$\lim_{K \to \infty} \| \mu \|_{\mathscr{R} \cup K} = 0.$$

Corollary 2. *A measure $\mu \in M(G)$ is singular if and only if*

$$\lim_{K \to \infty} \| \mu \|_{\mathscr{R} \cup K} = \| \mu \|.$$

Next we give an extension of a Theorem of I. Glicksberg (6, p. 419):

Theorem 2. *Let $\mathscr{S}$ be any closed subset of Γ such that $\mathscr{S} \cap (\gamma - \mathscr{S})$ has finite Haar measure for a dense subset of $\gamma \in \Gamma$ and let $\mu_i \in M(G)$ $i \in \{1, 2\}$. Suppose*

$$\lim_{K \to \infty} \| \mu_i \|_{\mathscr{S} \cup K} = 0.$$

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Then $\mu_1 * \mu_2$ is absolutely continuous with respect to m_G.

Proof. Given $\varepsilon > 0$, there correspond compact $K_{i\varepsilon}$ such that for all trigonometric polynomials of the form $p(x) = \sum_n c_n(-x, \gamma_n)$ with $\| p \|_\infty \leqq 1$ and $\gamma_n \notin \mathscr{S} \cup K_{i\varepsilon}$ the inequality

$$| \Sigma c_n \hat{\mu}_i(\gamma_n)| < \varepsilon \tag{5}$$

obtains.

Now (5), Proposition 2 and the Main Theorem of (**4**) give a measure $\mu_{i\varepsilon} \in M(G)$ such that

$$\hat{\mu}_i = \hat{\mu}_{i\varepsilon} \quad \text{off } \mathscr{S} \cup K_{i\varepsilon} \tag{6}$$

and

$$\| \mu_{i\varepsilon} \| \leqq \varepsilon. \tag{7}$$

Recall that there exist $f_i \in L^1(G)$ such that $\hat{f}_i(K_{i\varepsilon}) = 1$. Observe that

$$\operatorname{supp} \{(\mu_i - \mu_{i\varepsilon}) - f_i * (\mu_i - \mu_{i\varepsilon})\}^\wedge \subset \mathscr{S}$$

which implies by Glicksberg's Theorem that

$$\{(\mu_1 - \mu_{1\varepsilon}) - f_1 * (\mu_1 - \mu_{1\varepsilon})\} * \{(\mu_2 - \mu_{2\varepsilon}) - f_2 * (\mu_2 - \mu_{2\varepsilon})\} \in M_a(G).$$

Thus

$$(\mu_1 - \mu_{1\varepsilon}) * (\mu_2 - \mu_{2\varepsilon}) \in M_a(G). \tag{8}$$

Let $\varepsilon \to 0$ in (8). From (7) we infer that $\mu_1 * \mu_2 \in M_a(G)$. This completes the proof.

Remarks. (1) Theorem 1 and proof (with obvious modification) holds true for compact groups (i.e. not necessarily abelian).

(2) To obtain Doss' results in (**3**) and (**5**), it suffices to take $\mathscr{R}$ compact in Theorem 1.

(3) For some interesting examples of Riesz sets when $G = \boldsymbol{T}$ the reader is referred to (**1**) and (**7**).

(4) Using the same technique as in Theorem 2, one can give the following extension of a Theorem of Doss (**2**, p. 81):

Suppose G is a locally compact abelian group whose dual Γ is ordered, $\mu \in M(G)$ and $\lim_{K \to \infty} \| \mu \|_{\mathscr{P} \cup K} = 0$ where $\mathscr{P}$ is the positive cone of Γ. Then $\hat{\mu}_s$ is of analytic type and $\hat{\mu}_s(0) = 0$.

(5) Let $-N$ be the negative integers and S any set of integers $\{n_k\}$ where $\lim_{k \to \infty} (n_{k+p} - n_k) = 0$. Suppose

$$\lim_{K \to \infty} \| \mu \|_{-N \cup S \cup K} = 0.$$

Then it can be shown that the $p+1$ convolution power $\mu^{p+1} \in M_a(T)$ (see (**8**)).

The author takes pleasure in thanking Sadahiro Saeki for helpful conversations.

https://doi.org/10.1017/S0013091500015789 Published online by Cambridge University Press

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