ON NORMALLY EMBEDDED SUBGROUPS OF LOCALLY SOLUBLE FC-GROUPS

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1. Introduction. In his Habilitationsschrift [**3**] B. Fischer introduced the concept of a normally embedded subgroup of a finite group. A subgroup of a finite group G is said to be normally embedded in G if each of its Sylow subgroups is a Sylow subgroup of a normal subgroup of G. Meanwhile this concept has become of considerable importance in the theory of finite soluble groups and has been studied by various authors. However, in infinite group theory, normally embedded subgroups seem to have received little attention. The object of this note is to study normally embedded subgroups of locally soluble FC-groups.

In [**1**], W. Anderson defined a *Fitting set* of a finite soluble group G to be a non-empty collection $\mathcal{F}$ of subgroups of G such that

(i) $N \trianglelefteq F \in \mathcal{F}$ implies $N \in \mathcal{F}$;

(ii) $N, H \in \mathcal{F}$ and $N, H \trianglelefteq NH$ implies $NH \in \mathcal{F}$;

(iii) $F \in \mathcal{F}$ implies $F^x \in \mathcal{F}$ for each $x \in G$.

A combination of Theorems 3.1 and 3.3 of [**1**] leads to the following characterization of normally embedded subgroups of a finite soluble group G.

A subgroup V of G is normally embedded in G if and only if $\mathcal{F} = \{F \mid F \leqslant V^g$ for some $g \in G\}$ is a Fitting set of G.

We shall extend this result to the class of locally soluble FC-groups.

Let G be an arbitrary group. An automorphism α of G is said to be *locally inner* if, for each finite set of elements $g_1, \ldots, g_n$ of G, there is an element $x \in G$ such that $g_i\alpha = g_i^x$ for $i = 1, 2, \ldots, n$. Following Tomkinson [**4**] we denote by Linn(G) the group of locally inner automorphisms of G. Let $\mathbb{P}$ be the set of all primes, and let $p \in \mathbb{P}$. Then a torsion subgroup V of G is called *p-normally embedded* in G if each Sylow p-subgroup of V is also a Sylow p-subgroup of some normal subgroup of G. This is clearly equivalent to saying that if P is a Sylow p-subgroup of V, then P is a Sylow p-subgroup of P^G. If V is p-normally embedded in G for each $p \in \mathbb{P}$, then V is said to be *normally embedded* in G.

A *Fitting set* $\mathcal{F}$ of a group G is a non-empty collection of subgroups of G satisfying the following conditions.

(FS1) If A is an ascendant subgroup of $F \in \mathcal{F}$, then $A \in \mathcal{F}$;

(FS2) $\mathcal{F}$ is N-closed, i.e. the product of any collection of normal $\mathcal{F}$-subgroups is an $\mathcal{F}$-subgroup;

(FS3) If $F \in \mathcal{F}$ and $\alpha \in \mathrm{Linn}(G)$, then $F\alpha \in \mathcal{F}$.

In Section 3, we show that for an FC-group G, conditions (FS1) and (FS3) can be replaced by:

(FS1*) If $A \trianglelefteq F \in \mathcal{F}$, then $A \in \mathcal{F}$;

(FS3*) If $F \in \mathcal{F}$ and $g \in G$, then $F^g \in \mathcal{F}$.

The main result of this note is the following.

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THEOREM. *A torsion subgroup V of the locally soluble* FC-*group G is normally embedded in G if and only if $\mathcal{F} = \{F \mid F \leqslant V\alpha$ for some $\alpha \in \mathrm{Linn}(G)\}$ is a Fitting set of G.*

In particular, this result shows that every normally embedded subgroup of a locally soluble FC-group G is an injector of G for some subgroup-closed Fitting set of G.

In the course of the proof of this theorem, the so-called generalized Sylow bases of a group G play an important role. For each prime $p \in \mathbb{P}$, let $N(p) \trianglelefteq G$. Then a set $\{V_p\}_{p\in\mathbb{P}}$ of subgroups of G is said to be a *generalized Sylow basis of G associated with the normal subgroups* $\{N(p)\}_{p\in\mathbb{P}}$ if $V_p \in \mathrm{Syl}_p(N(p))$ and $V_pV_q = V_qV_p$ for all $p, q \in \mathbb{P}$. In the case when G is a finite soluble group, the existence of generalized Sylow bases was proved by B. Fischer; see [**2**, Theorem 2.7].

In Section 2 the existence and local conjugacy of generalized Sylow bases in locally soluble FC-groups is established. Section 3 is devoted to the basic facts about Fitting sets which are needed in Section 4 for the proof of the theorem.

For the relevant facts about FC-groups, we refer the reader to Tomkinson [**4**].

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2. Generalized Sylow bases of locally soluble FC-groups. Let G be an FC-group. By a result of B. H. Neumann (see [**4**, 1.6]) the set T of elements in G of finite order form a subgroup which contains G'. Hence, by [**4**, 5.22], G is locally soluble if and only if T has a Sylow basis. Moreover, if $\{S_p\}_{p\in\mathbb{P}}$ is a Sylow basis of G and $N \trianglelefteq G$, then $\{S_p \cap N\}_{p\in\mathbb{P}}$ is a Sylow basis of N [**4**, 5.16].

In the case of finite soluble groups it was shown by Fischer (see [**2**, 2.7]) that generalized Sylow bases exist, and Chambers [**2**, 2.8] proved that any two such systems are conjugate. In the following these results are extended to the class of locally soluble FC-groups.

PROPOSITION 2.1. *Let G be a locally soluble* FC-*group, and let $N(p) \trianglelefteq G$ for each $p \in \mathbb{P}$. Then there exist generalized Sylow bases of G associated with $\{N(p)\}_{p\in\mathbb{P}}$ and any two of them are locally conjugate. Furthermore, if $\{V_p\}_{p\in\mathbb{P}}$ is a generalized Sylow basis associated with $\{N(p)\}_{p\in\mathbb{P}}$, then $\langle V_p \mid p \in \mathbb{P}\rangle$ is a normally embedded subgroup of G.*

Proof. Let M be the torsion subgroup of G, and let $\mathcal{S} = \{S_p\}_{p\in\mathbb{P}}$ be a Sylow basis of M. For each $p \in \mathbb{P}$, let $L(p) = N(p) \cap M$ be the torsion subgroup of $N(p)$. By [**4**, 5.16], $\mathcal{S} \cap L(p)$ is a Sylow basis of $L(p)$. Let $V_p = S_p \cap L(p)$. Then $V_p \in \mathrm{Syl}_p(L(p))$. To see that V_pV_q is a subgroup of G for all $p, q \in \mathbb{P}$, observe that

$$S_pS_q \cap V_pL(q) = V_p(S_pS_q \cap L(q)) = V_p(S_p \cap L(q))(S_q \cap L(q)) = V_p(S_p \cap L(q))V_q,$$

and, similarly, $S_pS_q \cap V_qL(p) = V_q(S_q \cap L(p))V_p$. Therefore,

$$V_pV_q = V_q(S_q \cap L(p))V_p \cap V_p(S_p \cap L(q))V_q,$$

and so $\{V_p\}_{p\in\mathbb{P}}$ is a generalized Sylow basis of G associated with $\{N(p)\}_{p\in\mathbb{P}}$.

Suppose that $\{V_p\}_{p\in\mathbb{P}}$ and $\{W_p\}_{p\in\mathbb{P}}$ are generalized Sylow bases of G associated with $\{N(p)\}_{p\in\mathbb{P}}$. Let Σ be a local system of finitely generated normal subgroups of G and, for each $S \in \Sigma$, let $M(S) = M \cap S$ be the torsion subgroup of S. We note that $M(S)$ is a finite normal subgroup of G. Also $V_p \cap S$ and $W_p \cap S$ are finite groups, since they are contained in $M(S)$. Furthermore, $V_p \cap S = V_p \cap M(S)$ and $W_p \cap S = W_p \cap M(S)$ are Sylow p-subgroups of the normal subgroup $N(p) \cap M(S)$ of G. Hence $\{V_p \cap M(S)\}_{p \in \mathbb{P}}$ and $\{W_p \cap M(S)\}_{p \in \mathbb{P}}$ are generalized Sylow bases of $M(S)$ associated with $\{N(p) \cap M(S)\}_{p \in \mathbb{P}}$. By a result of Chambers [**2**, 2.8] there is an element $t \in M(S)$ such that $(W_p \cap M(S))^t = V_p \cap M(S)$ for all $p \in \mathbb{P}$. Now let $\mathscr{A}_S$ be the set of automorphisms of S which are induced by inner automorphisms of G and which map $W_p \cap S = W_p \cap M(S)$ onto $V_p \cap S = V_p \cap M(S)$. Then the sets $\mathscr{A}_S$ satisfy the conditions of Theorem 4.16 of [**4**] and so there exists $\alpha \in \mathrm{Linn}(G)$ such that $(W_p \cap S)\alpha = V_p \cap S$ for each $S \in \Sigma$ and each $p \in \mathbb{P}$. Hence $W_p\alpha = \bigcup_{S \in \Sigma} (S \cap W_p)\alpha = \bigcup_{S \in \Sigma} (S \cap V_p) = V_p$ for each $p \in \mathbb{P}$.

The last statement of the proposition follows immediately.

Proposition 2.2. *Let G be an* FC-*group, and let Σ be a local system of finitely generated normal subgroups of G. Let p be a prime and let V be a torsion subgroup of G. Then V is p-normally embedded in G if and only if $V \cap S$ is p-normally embedded in S for each $S \in \Sigma$.*

Proof. The necessity is obvious. Conversely, assume that $V \cap S$ is normally embedded in S for each $S \in \Sigma$. Let $V_p \in \mathrm{Syl}_p(V)$, and, for each $S \in \Sigma$, let $H(S)$ be a normal subgroup of S such that $V_p \cap S \in \mathrm{Syl}_p(H(S))$. Put

$$H^*(S) = \bigcap_{\substack{T \in \Sigma \\ S \subseteq T}} H(T).$$

Then $H^*(S) \leq H^*(T)$ for each pair $S, T \in \Sigma$ with $S \leq T$. Now define $H = \bigcup_{S \in \Sigma} H^*(S)$. Then $H \trianglelefteq G$. Note that $V_p \cap H \in \mathrm{Syl}_p(H)$; for $V_p \cap H = \bigcup_{S \in \Sigma} (V_p \cap H^*(S))$ and $V_p \cap H^*(S) \in \mathrm{Syl}_p(H^*(S))$. Since

$$V_p = \bigcup_{S \in \Sigma} (V_p \cap S) = \bigcup_{S \in \Sigma} (V_p \cap H(S)) = \bigcup_{S \in \Sigma} (V_p \cap H^*(S)) = V_p \cap H,$$

one obtains $V_p \in \mathrm{Syl}_p(H)$, as required.

In the next lemma it is shown that normally embedded subgroups of locally soluble FC-groups are pronormal. Here a subgroup H of a group G is said to be *pronormal in G* if H and H^g are conjugate in their join $\langle H, H^g \rangle$ for all $g \in G$.

Lemma 2.3. *Let N be a normal subgroup of the locally soluble* FC-*group G. If V is a normally embedded subgroup of G, then $V \cap N$ is pronormal in G.*

Proof. If V is normally embedded in G, so is $V \cap N$; thus we may assume $N = G$. Further, it is enough to show that V is locally pronormal; that is, V and $V\alpha$ are locally conjugate in their join $\langle V, V\alpha \rangle$ for each $\alpha \in \mathrm{Linn}(G)$. The assertion is then a consequence of Lemma 4.24 of [**4**]. Now let $\{V_p\}_{p \in \mathbb{P}}$ be a Sylow basis for V, and let $\{M(p)\}_{p \in \mathbb{P}}$ be a family of normal subgroups of G such that $V_p \in \mathrm{Syl}_p(M(p))$. Let $\alpha \in \mathrm{Linn}(G)$ and put $L = \langle V, V\alpha \rangle$. Then we must show that there is an element $\theta \in \mathrm{Linn}(L)$ with $V\theta = V\alpha$. Now V_p and $V_p\alpha$ belong to $\mathrm{Syl}_p(L \cap M(p))$, and $M(p) \cap L \trianglelefteq L$. Therefore, $\{V_p\}_{p \in \mathbb{P}}$ and $\{V_p\alpha\}_{p \in \mathbb{P}}$ are generalized Sylow bases of L associated with the normal subgroups $\{M(p) \cap L\}_{p \in \mathbb{P}}$ of L. By Proposition 2.1 there exists $\theta \in \mathrm{Linn}(L)$ such that $V_p\theta = V_p\alpha$ for each $p \in \mathbb{P}$. Hence $V\theta = V\alpha$, and V is locally pronormal in G.

Since pronormality is one of the properties which makes the Frattini agrument work, we obtain from Lemma 2.3 the following.

COROLLARY 2.4 (Frattini argument). *If V is a normally embedded subgroup of the locally soluble* FC*-group G, then $G = N_G(V \cap N)N$ for any $N \trianglelefteq G$.*

3. Fitting sets of locally soluble FC-groups. Let G be an FC-group with a local system $\{S \mid S \in \Sigma\}$ of finitely generated normal subgroups of G, and let $\mathcal{F}$ be a set of subgroups of G. Of course, conditions (FS1*) and (FS3*) are consequences of (FS1) and (FS3), respectively. Assume now that $\mathcal{F}$ satisfies (FS2), (FS1*) and (FS3*). Let A be an ascendant subgroup of $F \in \mathcal{F}$. Then $S \cap A$ is a subnormal subgroup of F for each $S \in \Sigma$. By (FS1*), $S \cap A \in \mathcal{F}$. But then A is the product of the normal $\mathcal{F}$-subgroups $S \cap A$ and so $A \in \mathcal{F}$ by (FS2). This shows that $\mathcal{F}$ satisfies (FS1). Now let $F \in \mathcal{F}$, $\alpha \in \mathrm{Linn}(G)$ and $S \in \Sigma$. It follows from Corollary 1.5 of [**4**] that $F \cap S$ is finitely generated and so $(F \cap S)\alpha = (F \cap S)^g$ for some $g \in G$. Then (FS3*) implies that $(F \cap S)\alpha \in \mathcal{F}$, since $F \cap S \in \mathcal{F}$ by (FS1*). Hence $F\alpha$ is the product of normal $\mathcal{F}$-subgroups $F\alpha \cap S$ and so $F\alpha \in \mathcal{F}$ by (FS2). Thus $\mathcal{F}$ satisfies also (FS3). In the remainder of this section, we shall make use of the equivalence established here.

Let $\mathcal{F}$ be a Fitting set of an arbitrary FC-group G, and let $H \leqslant G$. Then $\mathcal{F}(H) = \{F \mid F \leqslant H \text{ and } F \in \mathcal{F}\}$ is a Fitting set of H. Furthermore, if H is an ascendant subgroup of G, then $\mathcal{F}(H) = \mathcal{F} \cap H = \{F \cap H \mid F \in \mathcal{F}\}$. The join $G_{\mathcal{F}}$ of normal $\mathcal{F}$-subgroups of G is called the *$\mathcal{F}$-radical* of G. By (FS2), $G_{\mathcal{F}}$ is the unique maximal normal $\mathcal{F}$-subgroup of G. An *$\mathcal{F}$-injector* of G is an $\mathcal{F}$-subgroup X of G such that $X \cap A$ is $\mathcal{F}$-maximal in A for each ascendant subgroup A of G. If X is an $\mathcal{F}$-injector of G and A is an ascendant subgroup of G, then $X \cap A$ is an $\mathcal{F}(A)$-injector of A. $\mathcal{F}$ is said to be a *torsion Fitting set* of G provided that each $F \in \mathcal{F}$ is a torsion subgroup of G. Throughout this paper we consider for the most part only torsion Fitting sets of G. $\mathrm{Inj}_{\mathcal{F}}(G)$ will denote the set of all $\mathcal{F}$-injectors of G.

Using arguments similar to those given in Theorems 6.37 and 6.38 of [**4**] along with Lemmas 4.18 and 4.24 of [**4**], one obtains the following result.

LEMMA 3.1. *Let G be a locally soluble* FC*-group with torsion subgroup T, and let $\mathcal{F}$ be a torsion Fitting set of G. Then*

(i) $\mathrm{Inj}_{\mathcal{F}}(G)$ *is non-empty and any two $\mathcal{F}$-injectors of G are locally conjugate*;
(ii) $\mathrm{Inj}_{\mathcal{F}(T)}(T) = \mathrm{Inj}_{\mathcal{F}}(G)$;
(iii) *if $X \in \mathrm{Inj}_{\mathcal{F}}(G)$ and $\alpha \in \mathrm{Linn}(G)$, then $X\alpha \in \mathrm{Inj}_{\mathcal{F}}(G)$*;
(iv) *if $X \in \mathrm{Inj}_{\mathcal{F}}(G)$ and $N \trianglelefteq G$, then $X \cap N$ is a pronormal subgroup of G*;
(v) *if $X \in \mathrm{Inj}_{\mathcal{F}}(G)$ and $X \leqslant H \leqslant G$, then $X \in \mathrm{Inj}_{\mathcal{F}(H)}(H)$*;
(vi) *if $X \in \mathrm{Inj}_{\mathcal{F}}(G)$ and A is an ascendant subgroup of G, then $X \cap A \in \mathrm{Inj}_{\mathcal{F}(A)}(G)$.*

LEMMA 3.2. *Let G be a locally soluble* FC*-group, and let $\mathcal{F}$ be a torsion Fitting set of*

G. For each $p \in \mathbb{P}$, *let* $\mathcal{F}_p = \{H \leq G \mid$ *if* $X \in \mathrm{Inj}_{\mathcal{F}(H)}(H)$ *and* $X_p \in \mathrm{Syl}_p(X)$ *then* $X_p \in \mathrm{Syl}_p(H)\}$. *Then*

(i) $\mathcal{F}_p$ *is a Fitting set of* G;

(ii) *if* $X \in \mathrm{Inj}_{\mathcal{F}}(G)$ *and* $X_p \in \mathrm{Syl}_p(X)$, *then X is p-normally embedded in G if and only if* $X_p \in \mathrm{Syl}_p(G_{\mathcal{F}_p})$.

Proof. (i) Suppose that A is a normal subgroup of $H \in \mathcal{F}_p$. By part (i) of Lemma 3.1, $\mathrm{Inj}_{\mathcal{F}(H)}(H)$ is a non-empty set of subgroups of H. Let $X \in \mathrm{Inj}_{\mathcal{F}(H)}(H)$ and $X_p \in \mathrm{Syl}_p(X)$. Then $X_p \cap A$ is a Sylow p-subgroup of both $X \cap A$ and A, and $X \cap A \in \mathrm{Inj}_{\mathcal{F}(A)}(A)$ by part (vi) of Lemma 3.1. Thus $A \in \mathcal{F}_p$ and so (FS1*) is fulfilled.

Let N be the product of normal $\mathcal{F}_p$-subgroups N_λ, $\lambda \in \Lambda$. Let $Y \in \mathrm{Inj}_{\mathcal{F}(N)}(N)$ and let $Y_p \in \mathrm{Syl}_p(Y)$. Then, by part (vi) of Lemma 3.1, $Y \cap N_\lambda \in \mathrm{Inj}_{\mathcal{F}(N_\lambda)}(N_\lambda)$, and $Y_p \cap N_\lambda$ is a Sylow p-subgroup of $Y \cap N_\lambda$, hence also of N_λ. Therefore, $Y_p = \prod_{\lambda \in \Lambda} (Y_p \cap N_\lambda) \in \mathrm{Syl}_p(N)$ and so $N \in \mathcal{F}_p$. This means that (FS2) is satisfied.

Finally, let $H \in \mathcal{F}_p$ and $g \in G$. Let $X \in \mathrm{Inj}_{\mathcal{F}(H)}(H)$ and $X_p \in \mathrm{Syl}_p(X)$. Then $X_p \in \mathrm{Syl}_p(H)$ and so X_p^g is a Sylow p-subgroup of both H^g and X^g. Since $X^g \in \mathrm{Inj}_{\mathcal{F}(H^g)}(H^g)$, one has $H^g \in \mathcal{F}_p$, which shows that (FS3*) is fulfilled.

(ii) To prove the necessity, let $X \in \mathrm{Inj}_{\mathcal{F}}(G)$, let $X_p \in \mathrm{Syl}_p(X)$, and let $N(p) \trianglelefteq G$ such that $X_p \in \mathrm{Syl}_p(N(p))$. By part (vi) of Lemma 3.1, $X \cap N(p) \in \mathrm{Inj}_{\mathcal{F}(N(p))}(N(p))$ and clearly $X_p \in \mathrm{Syl}_p(X \cap N(p))$. Therefore $N(p) \in \mathcal{F}_p$, and so $N(p) \leq G_{\mathcal{F}_p}$. This implies that $X_p \in \mathrm{Syl}_p(G_{\mathcal{F}_p})$, since $X \cap G_{\mathcal{F}_p} \in \mathrm{Inj}_{\mathcal{F}_p(G_{\mathcal{F}_p})}(G_{\mathcal{F}_p})$ and $G_{\mathcal{F}_p} \in \mathcal{F}_p$.

The converse is obvious.

The following results are technical lemmas which are needed to prove the theorem.

LEMMA 3.3. *Let G be an* FC-*group, and let Σ be a local system of finitely generated normal subgroups of G. For each $S \in \Sigma$, let $\mathcal{F}_S$ be a Fitting set of S such that* $\mathcal{F}_T \cap S = \mathcal{F}_S$, *whenever* $S \leq T \in \Sigma$. *Then*

$$\mathcal{F} = \{F \leq G \mid F \cap S \in \mathcal{F}_S \text{ for each } S \in \Sigma\}$$

is a Fitting set of G and $\mathcal{F} \cap S = \mathcal{F}_S$ *for each* $S \in \Sigma$.

Proof. Condition (FS1*) is obviously fulfilled. To see that condition (FS2) holds, let N be the product of normal $\mathcal{F}$-subgroups N_λ, $\lambda \in \Lambda$. We must show that $N \in \mathcal{F}$. By a well-known result of B. H. Neumann [**4**, Corollary 1.5], each $S \in \Sigma$ satisfies the maximum condition for subgroups, which implies that $N \cap S$ is finitely generated. Let $S \in \Sigma$. Then there exist $T \in \Sigma$ and $\lambda_1, \ldots, \lambda_r \in \Lambda$ such that $N \cap S \leq (N_{\lambda_1} \cap T)(N_{\lambda_2} \cap T) \ldots (N_{\lambda_r} \cap T)$. Let $L = (N_{\lambda_1} \cap T)(N_{\lambda_2} \cap T) \ldots (N_{\lambda_r} \cap T)$. By assumption, each $N_{\lambda_i} \cap T \in \mathcal{F}_T$ and so $L \in \mathcal{F}_T$; for $\mathcal{F}_T$ is a Fitting set of T. Since $N \cap S \trianglelefteq L$, it follows that $N \cap S \in \mathcal{F}_T$. Hence $N \cap S \in \mathcal{F}_T \cap S = \mathcal{F}_S$ for each $S \in \Sigma$, so that $N \in \mathcal{F}$.

Let $H \in \mathcal{F}$, $S \in \Sigma$ and $g \in G$. Then there exists $T \in \Sigma$ such that T contains both S and g. Since $H \cap T \in \mathcal{F}_T$, it follows that $H^g \cap T \in \mathcal{F}_T$, and so $H^g \cap S \in \mathcal{F}_T \cap S = \mathcal{F}_S$. This implies that $H^g \in \mathcal{F}$, and so condition (FS3*) is fulfilled. Therefore, $\mathcal{F}$ is a Fitting set of G.

It is clear that $\mathcal{F} \cap S = \mathcal{F}_S$ for each $S \in \Sigma$; thus the lemma is proved.

LEMMA 3.4. *Let V be a subgroup of the* FC-*group G, and let Σ be a local system of finitely generated normal subgroups of G. Then*

$$\mathcal{F} = \{F \leqslant G \mid F \leqslant V\alpha \text{ for some } \alpha \in \mathrm{Linn}(G)\}$$

is a Fitting set of G if and only if $\mathcal{F}_S = \mathcal{F} \cap S$ is a Fitting set of S for each $S \in \Sigma$.

Proof. Assume that $\mathcal{F}$ is a Fitting set of G, and let $S \in \Sigma$. Since $S \trianglelefteq G$, $\mathcal{F} \cap S = \mathcal{F}(S)$ is a Fitting set.

Conversely, assume that $\mathcal{F}_S = \mathcal{F} \cap S$ is a Fitting set of S for each $S \in \Sigma$. By Lemma 3.3, $\mathcal{X} = \{H \leqslant G \mid H \cap S \in \mathcal{F}_S \text{ for each } S \in \Sigma\}$ is a Fitting set of G. We show that $\mathcal{F} = \mathcal{X}$. The inclusion $\mathcal{F} \subseteq \mathcal{X}$ follows easily and hence it suffices to verify that $\mathcal{X} \subseteq \mathcal{F}$. Suppose that $H \in \mathcal{X}$; then $H \cap S \in \mathcal{F}_S$ for each $S \in \Sigma$. Let $S \in \Sigma$. Then $H \cap S \leqslant (V \cap S)^g$ for some $g \in G$, since $V \cap S$ is finitely generated. Denote by $\mathcal{A}_S$ the set of all automorphisms θ of S induced by inner automorphisms of G such that $H \cap S \leqslant (V \cap S)\theta$. Then the sets $\mathcal{A}_S$ satisfy the conditions of Theorem 4.16 of [**4**] and so there is an element $\alpha \in \mathrm{Linn}(G)$ such that $H \cap S \leqslant (V \cap S)\alpha$ for each $S \in \Sigma$. Thus one has

$$H = \bigcup_{S \in \Sigma} (H \cap S) \leqslant \bigcup_{S \in \Sigma} (V \cap S)\alpha = V\alpha,$$

and hence $H \in \mathcal{F}$. This completes the proof of the lemma.

4. Proof of the theorem. We begin with some preliminary remarks. Let G be a locally soluble FC-group, and let Σ be a local system of finitely generated normal subgroups of G. Let V be a torsion subgroup of G. For each $S \in \Sigma$, put $\mathcal{F}_S = \mathcal{F} \cap S$ where

$$\mathcal{F} = \{F \leqslant G \mid F \leqslant V\alpha \text{ for some } \alpha \in \mathrm{Linn}(G)\}.$$

Let M be the torsion subgroup of G and, for each $S \in \Sigma$, let $M(S) = M \cap S$ be the torsion subgroup of S. Recall that $M(S)$ is a finite soluble normal subgroup of G. Then

$$\begin{aligned}\mathcal{F}_S &= \{F \mid F \leqslant S, F \leqslant V\alpha \text{ for some } \alpha \in \mathrm{Linn}(G)\}\\ &= \{F \mid F \leqslant (V \cap S)\alpha \text{ for some } \alpha \in \mathrm{Linn}(G)\}\\ &= \{F \mid F \leqslant (V \cap M(S))^g \text{ for some } g \in G\}.\end{aligned}$$

Assume first that V is a normally embedded subgroup of G. We have to show that $\mathcal{F}$ is a Fitting set of G. Since V is normally embedded in G, it follows from Proposition 2.2 that $V \cap M(S)$ is normally embedded in $M(S)$ for each $S \in \Sigma$. By Corollary 2.4, $G = N_G(V \cap M(S))M(S)$ for each $S \in \Sigma$. Therefore,

$$\mathcal{F}_S = \{F \mid F \leqslant (V \cap M(S))^m \text{ for some } m \in M(S)\}.$$

By Theorem 3.1 of [**1**], $\mathcal{F}_S$ is a Fitting set of $M(S)$ for each $S \in \Sigma$. Since $\mathcal{F}_S$ is also a Fitting set of S for each $S \in \Sigma$, it follows from Lemma 3.4 that $\mathcal{F}$ is a Fitting set of G.

To prove the converse, assume that $\mathcal{F}$ is a Fitting set of G. Then $\mathcal{F}_S = \mathcal{F}(S) = \{F \mid F \leqslant (V \cap M(S))^g \text{ for some } g \in G\}$ is a torsion Fitting set of G for each $S \in \Sigma$. By part (i) of Lemma 3.1, G has an $\mathcal{F}(S)$-injector X, say. There exists $g \in G$ such that

$X=(V\cap M(S))^g$. Thus, $V\cap M(S)$ is an $\mathcal{F}(S)$-injector of G. By part (iv) of Lemma 3.1, $V\cap M(S)$ is pronormal in G and hence $G=N_G(V\cap M(S))M(S)$ by Corollary 2.4 This implies that

$$\mathcal{F}_S=\mathcal{F}(S)=\{F \mid F\leq (V\cap M(S))^m \text{ for some } m\in M(S)\}.$$

Again, by Theorem 3.1 of [**1**], $V\cap M(S)$ is normally embedded in $M(S)$ for each $S\in\Sigma$. Let $S\in\Sigma$ and $p\in\mathbb{P}$, and let P be a Sylow p-subgroup of $V\cap M(S)$. Further, let

$$\mathcal{F}_p(S)=\{H\leq S \mid \textit{if } X\in \mathrm{Inj}_{\mathcal{F}(H)}(H) \text{ and } X_p\in \mathrm{Syl}_p(X) \text{ then } X_p\in\mathrm{Syl}_p(H)\}.$$

Then, by part (i) of Lemma 3.2, $\mathcal{F}_p(S)$ is a Fitting set of S. Since $V\cap M(S)\in \mathrm{Inj}_{\mathcal{F}(M(S))}(M(S))$ and $\mathcal{F}(S)=\mathcal{F}(M(S))$, it follows from part (ii) of Lemma 3.2 that P is a Sylow p-subgroup of $M(S)_{\mathcal{F}_p(S)}$. But $M(S)_{\mathcal{F}_p(S)}\trianglelefteq S$, since $M(S)\trianglelefteq S$. This shows that $V\cap S=V\cap M(S)$ is p-normally embedded in S. Thus for each $S\in\Sigma$ and each $p\in\mathbb{P}$, $V\cap S$ is p-normally embedded in S. By Proposition 2.2, V is then normally embedded in G. This completes the proof of the theorem.

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