INFINITE PACKINGS OF DISKS

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1. Let U be the closed disk in the plane, centred at the origin, and of unit radius. By a solid packing, or briefly a packing, $\mathbf{C}$ of U we shall understand a sequence $\{D_n\}$, $n = 1, 2, \ldots$, of open proper disjoint subdisks of U, such that the plane Lebesgue measures of U and of $\bigcup_{n=1}^{\infty} D_n$ are the same. If r_n is the radius of D_n and the complex number c_n represents its centre, then the conditions for $\mathbf{C}$ to be a packing are

$$|c_i| < 1, \quad |c_i - c_j| \geqslant r_i + r_j, \quad \sum_{n=1}^{\infty} r_n^{\,2} = 1 \qquad (i, j = 1, 2, \ldots ; i \neq j).$$

It was proved by Mergelyan **(3)** that for any packing the sum of the radii diverges:

$$\sum_{n=1}^{\infty} r_n = \infty. \tag{1}$$

Mergelyan's demonstration of (1) is somewhat involved and leans heavily on the machinery of functions of a complex variable. An elegant direct proof of (1) is given by Wesler **(5)**, who uses the technique of projecting the boundaries of the disks of the packing on a diameter I of U. It is then shown that almost all points of I are projected upon infinitely often, which leads directly to (1). Wesler's proof lends itself to extensive generalizations: packings of balls in E^n by open disjoint sub-balls, packings of convex bodies by sequences of other convex bodies, and so on. The Mergelyan–Wesler theorem (1) suggests the introduction of the sum

$$M_a(\mathbf{C}) = \sum_{n=1}^{\infty} r_n^{\,a}$$

for any packing $\mathbf{C}$, and the investigation of its convergence or divergence for various values of a. Since $r_n < 1$ for all n, we have $M_a(\mathbf{C}) > M_b(\mathbf{C})$ if $a < b$ and $M_a(\mathbf{C})$ is finite; also, if $M_b(\mathbf{C}) = \infty$ and $a < b$, then $M_a(\mathbf{C}) = \infty$. Hence for each packing $\mathbf{C}$, there exists a uniquely defined number $e(\mathbf{C})$, such that $1 \leqslant e(\mathbf{C}) \leqslant 2$, and $M_a(\mathbf{C})$ converges for a $a > e(\mathbf{C})$ and diverges for $a < e(\mathbf{C})$. The number $e(\mathbf{C})$ will be called the exponent of the packing $\mathbf{C}$.

It is our object to prove two theorems, together with some generalizations, relative to the exponents $e(\mathbf{C})$. The first implies that the case $e(\mathbf{C}) = 2$ may occur and it proves a little more. The second is concerned with a special class of packings in which every disk from the second one onward is the largest possible, and it proves that for every such packing $1.035 < e(\mathbf{C}) < 1.999971$.

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2. We begin by extending the concept of an exponent $e(\mathbf{C})$ of a packing $\mathbf{C}$ to that of a local exponent $e(\mathbf{C}, x)$. Let $\mathbf{C} = \{D_n\}$ be a packing and let

$$x \in U - \bigcup_{n=1}^{\infty} D_n.$$

Let $D(x, r)$ denote the open disk of radius r about x as centre and put

$$M_a(\mathbf{C}, x, r) = \sum_{D_n \subset D(x,r)} r_n^{\,a},$$

r_n being the radius of D_n. It is then easily shown that there exists a number $e = e(\mathbf{C}, x, r)$ such that $M_a(\mathbf{C}, x, r)$ converges for $a > e$ and diverges for $a < e$. Since for every $r > 0$ there are infinitely many values of n for which $D_n \subset D(x, r)$, it follows that $e(\mathbf{C}, x, r) \geqslant 0$. As a matter of fact, the Mergelyan–Wesler theorem (1) could be used to show that $e(\mathbf{C}, x, r) \geqslant 1$, but we shall not need this. If $r_1 \leqslant r_2$, then

$$M_a(\mathbf{C}, x, r_1) \leqslant M_a(\mathbf{C}, x, r_2)$$

so that $e(\mathbf{C}, x, r)$ is non-decreasing in r and bounded from below. Hence there exists the limit

$$e(\mathbf{C}, x) = \lim_{r \to +0} e(\mathbf{C}, x, r),$$

which we call the local exponent of $\mathbf{C}$ at x. Our first result is

THEOREM 1. *There exists a packing* $\mathbf{C} = \{D_n\}$ *such that* $e(\mathbf{C}, x) = 2$ *for every*

$$x \in U - \bigcup_{n=1}^{\infty} D_n.$$

Let

$$\mathbf{C}_{ni} = \{D_{ni}\} \qquad (i = 1, 2; n = 1, 2, \ldots)$$

be two packings of U, not necessarily distinct. Let r_{ni} be the radius of the disk D_{ni} and let the complex number c_{ni} represent its centre. By iterating $\mathbf{C}_2$ over D_{n1} in $\mathbf{C}_1$ we mean the following operation: from the packing $\mathbf{C}_1$ we remove the disk D_{n1} and we replace it by a suitably scaled down replica of $\mathbf{C}_2$. The result is a new set of disks

$$\mathbf{C} = \{D_{11}, D_{21}, \ldots, D_{n-1,1}, D_{n+1,1}, \ldots\} \cup \{D'_1, D'_2, \ldots\},$$

where the disk D'_m has the radius $r_{m2} \cdot r_{n1}$ and is centred at $c_{n1} + r_{n1} c_{m2}$. $\mathbf{C}$ is again a packing of U and

$$M_b(\mathbf{C}) = M_b(\mathbf{C}_1) + r_{n1}^{\,b}[M_b(\mathbf{C}_2) - 1]. \tag{2}$$

Similarly, let $\mathbf{C} = \{D_m\}$ be a packing, and let $\{\mathbf{C}_i\}$ $(i = 1, 2, \ldots)$ be a sequence of packings with $\mathbf{C}_i = \{D_{ni}\}$. Then by iterating $\{\mathbf{C}_i\}$ over $\mathbf{C}$ we mean iterating $\mathbf{C}_i$ over D_i simultaneously for all i; this leads to a new packing $\mathbf{C}'$.

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In the special case when $\mathbf{C}_1 = \mathbf{C}_2 = \ldots = \mathbf{C}$ we call the above process the squaring of $\mathbf{C}$ and we denote the result by $\mathbf{C}^2$. By (2)

$$M_b(\mathbf{C}^2) = M_b(\mathbf{C}) + [M_b(\mathbf{C}) - 1]\sum_{n=1}^{\infty} r_n^{\,b} = [M_b(\mathbf{C})]^2. \tag{3}$$

It may be mentioned here that on account of the absolute convergence of a convergent series of positive terms the questions of enumerating and re-enumerating the disks of packings and of iterated packings are unimportant.

From (3), we obtain

$$\sup_{\mathbf{C}} M_b(\mathbf{C}) = \infty, \qquad b < 2, \tag{4}$$

for we can always start with any packing $\mathbf{C} = \mathbf{C}_0$ and define recursively $\mathbf{C}_{n+1} = \mathbf{C}_n^{\,2}$; since $b < 2$, we have then $M_b(\mathbf{C}) = d > 1$ and therefore $M(\mathbf{C}_n) = d^{2^n}$, which proves (4).

Next we prove

LEMMA 1. *There exists a packing* $\mathbf{C}'$ *with* $e(\mathbf{C}') = 2$.

Let $b_n = 2 - n^{-1} (n = 1, 2, \ldots)$ and let $\mathbf{C} = \{D_n\}$ be an arbitrary packing of U, r_n being the radius of D_n. Let $\{\mathbf{C}_m\}$ $(m = 1, 2, \ldots)$ be a sequence of packings such that

$$M_{b_m}(\mathbf{C}_m) > r_m^{\,-b_m}; \tag{5}$$

the existence of such a sequence is guaranteed by (4). Let $\mathbf{C}'$ be the result of iterating $\{\mathbf{C}_m\}$ over $\mathbf{C}$. Choose now any $b < 2$; then

$$M_b(\mathbf{C}') = \sum_{n=1}^{\infty} r_n^{\,b} M_b(\mathbf{C}_n). \tag{6}$$

Let p be the smallest integer $\geqslant (2 - b)^{-1}$ such that $b_n \geqslant b$ for $n \geqslant p$, and therefore

$$M_b(\mathbf{C}_n) \geqslant M_{b_n}(\mathbf{C}_n) \qquad \text{for } n \geqslant p.$$

Now, by (5), (6), and the above,

$$M_b(\mathbf{C}') \geqslant \sum_{n=p}^{\infty} r_n^{\,b} M_b(\mathbf{C}_n) \geqslant \sum_{n=p}^{\infty} r_n^{\,b} M_{b_n}(\mathbf{C}_n) \geqslant \sum_{n=p}^{\infty} r_n^{\,b-b_n} \geqslant \sum_{n=p}^{\infty} 1 = \infty.$$

Hence $M_b(\mathbf{C}') = \infty$ for any $b < 2$, while of course $M_2(\mathbf{C}') = 1$; therefore $e(\mathbf{C}') = 2$, which was to be shown.

We are now in a position to prove Theorem 1. By Lemma 1, there exists a packing $\mathbf{C}_1 = \{D'_n\}$ with $e(\mathbf{C}_1) = 2$. We suppose that the disks D'_n have their radii $r'_1, r'_2, \ldots$ in non-increasing order. Let $\rho_1, \rho_2, \ldots$ be any strictly decreasing sequence of positive numbers such that $\lim_{n\to\infty} \rho_n = 0$. Let n_1 be the greatest integer such that $r'_{n_1} \geqslant \rho_1$; if no such integer exists, take $n_1 = 0$. Iterate $\mathbf{C}_1$ over all the disks

$$D'_{n_1+1}, D'_{n_1+2}, \ldots,$$

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thus obtaining a new packing $\mathbf{C}_2 = \{D''_n\}$. Again, we suppose that the radii $r''_1, r''_2, \ldots$ are arranged in non-increasing order, and we let n_2 be the greatest integer such that $r''_{n_2} \geqslant \rho_2$, or 0 if no such integer exists. Next, we iterate $\mathbf{C}_1$ over the disks

$$D''_{n_2+1}, D''_{n_2+2}, \ldots .$$

This yields a new packing $\mathbf{C}_3$. Repeating the operation indefinitely, we obtain a sequence of packings $\mathbf{C}_1, \mathbf{C}_2, \ldots$, which converge to a limiting packing $\mathbf{C}$ since after N stages of the process all the disks of radius $\geqslant \rho_N$ are permanently fixed. Let $\mathbf{C} = \{D_n\}$, let

$$x \in U - \bigcup_{n=1}^{\infty} D_n,$$

let $r > 0$, and consider the disk $D(x, r)$. At some stage of the iteration process the initial packing $\mathbf{C}_1$ must have been iterated over some small disk lying entirely in $D(x, r)$ so that $e(\mathbf{C}, x, r) = 2$. Since r and x are arbitrary, it follows that $e(\mathbf{C}, x) = 2$.

3. Theorem 1 can be generalized in various ways, and in this section we mention briefly a few such generalizations. In the first place, U can be replaced by a more general set V of sufficient regularity, for instance, by another plane convex body or by a plane closed piecewise smooth Jordan curve together with its interior. A packing of V is then defined, as before, to be a sequence $\{D_n\}$ of open disjoint disks lying in V, such that the sets V and $\bigcup_{n=1}^{\infty} D_n$ have the same plane Lebesgue measure. The definition of exponents and of local exponents goes over without change, and there is no difficulty in obtaining the obvious generalization of Theorem 1. To prove it, we can take any packing of V, replacing each disk by a suitable replica of the packing $\mathbf{C}$ of Theorem 1.

Further, we have an m-dimensional version of Theorem 1. Here U is replaced by the closed ball B in E^m, centred at the origin and of unit radius. Let $\{D_n\}$ be a sequence of open disjoint sub-balls of B; if c_n is the vector representing the centre of D_n and r_n is its radius, then the necessary and sufficient conditions for $\{D_n\}$ to be a packing are

$$|c_i| < 1, \quad |c_i - c_j| \geqslant r_i + r_j, \quad \sum_{n=1}^{\infty} r_n^{\,m} = 1 \qquad (i, j = 1, 2, \ldots; i \neq j).$$

Wesler's generalization of (1) is now

$$\sum_{n=1}^{\infty} r_n^{\,m-1} = \infty .$$

Exponents and local exponents are as before, and the proof of Theorem 1 can be adapted at once to show that there exist packings $\mathbf{C} = \{D_n\}$ such that $e(\mathbf{C}, x) = m$ for every

$$x \in B - \bigcup_{n=1}^{\infty} D_n.$$

We can also consider packings of unbounded regions. For instance, let P be the whole plane; a sequence $\{D_n\}$ of open disjoint disks in P is a packing of P if the plane measure of $P - \bigcup_{n=1}^{\infty} D_n$ is 0. Here a particularly simple proof of the analogue of Theorem 1 is the following. Let $\mathbf{C} = \{D_n\}$ be the packing of U, of Theorem 1. Let x be a point on the circumference of U, which is not in the boundary of any disk D_n. Let A be a circle about x. Under the inversion with respect to A, U goes over into a half-plane H and the disks $\{D_n\}$ go over into the disks $\{F_n\}$; the latter provide a packing of H. Now the family $\{F_n\}$, together with its reflexion in the line bounding H, provides a packing of P for which Theorem 1 holds.

Finally, it is possible to pack a set with sets other than disks. For instance, let U be any plane convex body and let $\mathbf{C} = \{D_n\}$ be a sequence of interiors of convex bodies, which are disjoint, lie in the interior of U, and provide a packing in the sense that U and $\bigcup_{n=1}^{\infty} D_n$ have the same plane measure. It is necessary to place some limitation on the sets D_n, to prevent the possibility of finite packings. We may require, for instance, that the closure of each D_n is a strictly convex set. The sums

$$M_a(\mathbf{C}) = \sum_{n=1}^{\infty} r_n{}^a$$

are now replaced by the sums

$$M_a(\mathbf{C}) = \sum_{n=1}^{\infty} [\text{diam } (D_n)]^a,$$

and again we have both the Mergelyan–Wesler theorem (1) and an equivalent of our Theorem 1.

4. In this section we consider a special class of packings in which every disk, beginning with the second, is largest possible. More specifically, a packing $\mathbf{C} = \{D_n\}$ of U is called osculatory if its disks are determined in the following manner: D_1 is a disk of radius r_1, $0 < r_1 < 1$, internally tangent to U; D_2 is the largest disk fitting into $U - D_1$; D_3 and D_4 are respectively the two largest disks fitting into the two curvilinear triangles making up the boundary of $U - \bigcup_{n=1}^{2} D_n$; $D_5, \ldots, D_{10}$ are the six largest disks fitting into the six curvilinear triangles forming the boundary of $U - \bigcup_{n=1}^{4} D_n$; and so on. In the following it will often be convenient to deal with the curvature of the various disks, rather than with their radii. We remark that at this point it is not quite obvious that an osculatory packing is solid, i.e., that the sum of the areas of its disks equals the area of U. As a matter of fact, there appears to be no very easy direct proof of this. However, we shall prove later on a much stronger proposition; for the time being we consider the osculatory packings by themselves, without claiming as yet that they are indeed packings. It may be added that all the results obtained for osculatory packings hold for a somewhat more general class of packings of U, in which we start with any finite number of disjoint subdisks of U, and from then on continue adding each time the largest disk that fits into the area left uncovered. We need first an elementary result due to Soddy **(4)**, proved also in **(1)**; it is

LEMMA 2. *Let A, B, and C be three pairwise tangent circles in the plane, with curvatures a, b, and c. Let D be either of the two circles tangent to A, B, and C, and let d be its curvature. Then*

$$d = a + b + c + 2(ab + ac + bc)^{\frac{1}{2}}. \tag{7}$$

The following conventions are to be observed. If A, B, and C are pairwise externally tangent, then a, b, and c are taken as positive; otherwise, with B and C tangent internally to A, the sign of a is taken as negative and those of b and c as positive. In the latter case the two values of the square root in (7) lead to two values of d, say d_1 and d_2, both necessarily positive, and corresponding to the two circles tangent internally to A and externally to B and C. In the case of external tangency of A, B, and C, max (d_1, d_2) is positive and corresponds to the circle inscribed into the concave curvilinear triangle contained between A, B, and C; min (d_1, d_2) may be positive (another circle tangent externally to A, B, and C), or 0 (a common tangent line to A, B, and C), or negative (a circle tangent to A, B, and C, and enclosing them). Using this lemma, we have

LEMMA 3. *If $\mathbf{C}$ is an osculatory packing, then*

$$e(\mathbf{C}) \geqslant \log 3/\{\log [(1 + 5^{\frac{1}{2}} + (2 + 20^{\frac{1}{2}})^{\frac{1}{2}})/2]\} = 1.035\ldots. \tag{8}$$

In the osculatory packing $\mathbf{C}$, single out any three pairwise externally tangent disks, call them D_1, D_2, and D_3, and assign to each the order -1. In filling out the area between them we obtain successively one disk of order 0, say D_4; then three disks of order 1, namely D_5, D_6, and D_7; and, generally, 3^n disks of order n, for any $n \geqslant 0$. Let the radii of the latter be r_{jn} $(j = 1, 2, \ldots, 3^n)$ and put $M_n = (\min_j r_{jn})^{-1}$. Each disk of positive order is tangent to exactly three disks of lower order, say p, q, and r, with $p \leqslant q \leqslant r$. Further, either $p = -1$ or $0 \leqslant p < q < r$; this follows easily from the fact that no two disks of the same positive order can touch. Since M_n is increasing with n, we have, by applying (7),

$$M_{n+3} \leqslant M_{n+2} + M_{n+1} + M_n + 2(M_{n+2}M_{n+1} + M_{n+2}M_n + M_{n+1}M_n)^{\frac{1}{2}}, \qquad n = 0, 1, \ldots. \tag{9a}$$

Here and in what follows we take the positive root, by the conventions following (7).

Let $p_i = M_i$ $(i = 0, 1, 2)$ and define recursively

$$p_{n+3} = p_{n+2} + p_{n+1} + p_n + 2(p_{n+2}p_{n+1} + p_{n+2}p_n + p_{n+1}p_n)^{\frac{1}{2}}, \qquad n = 0, 1, \ldots. \tag{9b}$$

Then by (9a), $p_n \geqslant M_n$ for all $n \geqslant 0$. Our immediate goal is to prove an estimate of the form

$$p_n \leqslant Kt^n, \qquad K > 0;\, t \geqslant 1;\, n = 0, 1, \ldots . \tag{10}$$

This will be done by induction on n. Put $K = \max(p_0, p_1, p_2)$ and assume that there is a fixed $t \geqslant 1$ such that $p_m \leqslant Kt^m$ for $m = 0, 1, \ldots, n+2$. This is certainly true for $n = 0$. By the induction hypothesis and by (9b), we have

$$p_{n+3} \leqslant Kt^n[t^2 + t + 1 + 2(t^3 + t^2 + t)^{\frac{1}{2}}]. \tag{11}$$

Therefore we shall have the desired induction step, namely $p_{n+3} \leqslant Kt^{n+3}$, provided that t is chosen so that

$$t^2 + t + 1 + 2(t^3 + t^2 + t)^{\frac{1}{2}} \leqslant t^3.$$

We therefore examine the real roots of the corresponding equation

$$x^2 + x + 1 + 2(x^3 + x^2 + x)^{\frac{1}{2}} = x^3,$$

or after rearranging and squaring

$$x^6 - 2x^5 - x^4 - 4x^3 - x^2 - 2x + 1 = 0. \tag{12}$$

Because of its symmetry, the sextic (12) reduces to quadratics. We divide by x^3 and put $x + x^{-1} = u$; then (12) becomes

$$u(u^2 - 2u - 4) = 0.$$

Thus (12) can be solved and it turns out that there is exactly one real root greater than 1:

$$[1 + 5^{\frac{1}{2}} + (2 + 20^{\frac{1}{2}})^{\frac{1}{2}}]/2 = 2.890 \ldots .$$

Therefore, with t equal to the above number, the induction course is completed and (10) is proved. Hence, by the definition of M_n and p_n,

$$\min r_{jn} \geqslant ct^{-n}, \qquad c > 0,\, n = 0, 1, \ldots .$$

Let the radii of the disks of the osculatory packing **C** be $r_1, r_2, \ldots$; then by the above

$$M_a(\mathbf{C}) = \sum_{k=1}^{\infty} r_k^{\,a} \geqslant \sum_{n=1}^{\infty} \sum_{j=1}^{3^n} r_{jn}^{\,a} \geqslant c^a \sum_{n=1}^{\infty} 3^n\, t^{-na};$$

but the sum on the right clearly diverges for any $a < \log 3/\log t$, which proves (8).

Lemma 3 extends to some of the generalizations of our packings, considered in §3. As an example, we have

THEOREM 2. *Let J be any plane closed piecewise smooth Jordan curve, other than the circle, and let the interior I of J be packed with an infinite sequence*

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$S_1, S_2, \ldots,$ of open circular disks, selected so that S_n is the largest disk fitting into $I - \bigcup_{i=1}^{n-1} S_i$, $n = 1, 2, \ldots$. If r_n is the radius of S_n, then the sum

$$\sum_{n=1}^{\infty} r_n{}^a$$

diverges for any a such that

$$a < \log 3/\{\log[(1 + 5^{\frac{1}{2}} + (2 + 20^{\frac{1}{2}})^{\frac{1}{2}})/2]\} = 1.035\ldots.$$

For it is clear that the sequence $S_1, S_2, \ldots$ must contain a subsequence of the type used in the proof of Lemma 3.

5. The upper bound on the exponent of an osculatory packing, to be derived in the next section, is harder to obtain than the lower bound (8) of Lemma 3, and we need some preliminary results. The first one, which may be of some independent interest, is an extension of Soddy's formula (7).

LEMMA 4. *Let A, B, and $C = C_0$ be three pairwise tangent circles, with curvatures a, b, and $c = c_0$, satisfying $a \leqslant b \leqslant c$. Let $C_1, C_2, \ldots$ be a sequence of circles such that C_n is the smaller one of the two circles tangent to A, B, and C_{n-1}, and let c_n be its curvature. Then*

$$c_n = (a + b)n^2 + 2n[(a + b)c_0 + ab]^{\frac{1}{2}} + c_0, \qquad n = 0, 1, \ldots, \tag{13}$$

where the sign of the square root is positive, as are those of b and c_0, and the sign of a is taken as negative if B and C are internally tangent to A, and positive otherwise.

By (7) we have

$$c_{n+1} = a + b + c_n + 2(ab + ac_n + bc_n)^{\frac{1}{2}}, \qquad n = 0, 1, \ldots.$$

Let

$$f(x) = a + b + x + 2(ax + bx + ab)^{\frac{1}{2}}; \tag{14}$$

we then obtain

$$c_{n+1} = f(c_n), \qquad c_n = f_n(c_0),$$

where $f_n(x)$ is the nth iterate of $f(x)$. To find an explicit expression for $f_n(x)$, we use the method of conjugate functions, searching for an easily iterable function conjugate to $f(x)$. Put

$$(15) \quad \phi(x) = \alpha x + \beta, \quad \alpha = a + b, \beta = -ab/(a + b),$$
$$g(x) = x^2, \quad \psi_t(x) = x + t, \quad \omega(x) = \phi(g(x)).$$

Then

$$\phi^{-1}(f(\phi(x))) = x + 2x^{\frac{1}{2}} + 1 = g(\psi_1(g^{-1}(x)))$$

and therefore

$$f(x) = \omega(\psi_1(\omega^{-1}(x))).$$

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Hence

$$f_2(x) = \omega(\psi_1(\omega^{-1}(\omega(\psi_1(\omega^{-1}(x)))))) = \omega(\psi_2(\omega^{-1}(x))),$$

and generally

$$f_n(x) = \omega(\psi_n(\omega^{-1}(x))) = \omega(n + \omega^{-1}(x)).$$

Substituting into the above from (15), we obtain

$$f_n(x) = (a + b)n^2 + 2n(ax + bx + ab)^{\frac{1}{2}} + x,$$

which yields (13) at once.

In our applications of this lemma, A, B, and C will be the boundaries of three pairwise tangent disks of an osculatory packing. Therefore, in the case of external tangency of A, B, and C, we make no special assumptions on a, b, and c; but in the case of internal tangency, the containing circle A is the boundary of the unit disk U and so $a = -1$.

LEMMA 5. *With the terminology of Lemma* 4

$$c_{n+1}/c_n \leqslant k = 3 + 2\cdot 3^{\frac{1}{2}}, \qquad n = 0, 1, \ldots . \tag{16}$$

In the case of external tangency we have by (7)

$$c_1/c_0 = a/c_0 + b/c_0 + 1 + 2(ab/c_0^2 + a/c_0 + b/c_0)^{\frac{1}{2}}$$

with $a \leqslant b \leqslant c_0$. Therefore the maximum of the right-hand side occurs for $a = b = c_0$, which gives us (16) with $n = 0$. To prove this for general n, we merely replace c_0 and c_1 by c_n and c_{n+1}. In the case of internal tangency, we put $a = -1$ and so by (7)

$$c_1 = b + c_0 - 1 + 2(bc_0 - b - c_0)^{\frac{1}{2}},$$

which is increasing in b. Since $b \leqslant c_0$, the maximum of the right-hand side occurs for $b = c_0$ and so

$$c_1 \leqslant 2c_0 - 1 + 2(c_0^2 - 2c_0)^{\frac{1}{2}};$$

and similarly for any $n \geqslant 0$

$$c_{n+1} \leqslant 2c_n - 1 + 2(c_n^2 - 2c_n)^{\frac{1}{2}}$$

or

$$c_{n+1}/c_n \leqslant 2 - 1/c_n + 2(1 - 2/c_n)^{\frac{1}{2}}.$$

Since c_n is positive, it follows that $c_{n+1}/c_n \leqslant 4$, which is less than k, and so the proof is complete.

LEMMA 6. *Let Δ be the area of the curvilinear triangle T contained between the circles A, B, and C of Lemma* 4 (*in the case of internal tangency we take T to be the smaller of two such triangles*). *Then*

$$\Delta \leqslant \pi(2k + 1)^2 \sum_{n=1}^{\infty} c_n^{-2}. \tag{17}$$

Let C' be the circle concentric with C_n and having the radius $(2k+1)$ times that of C_n. Then it follows from (16) that C'_n contains both C_{n-1} and C_{n+1} for $n \geqslant 1$. As a consequence, no point of T can lie outside all the circles $C'_1, C'_2, \ldots$ and hence

$$\Delta \leqslant \sum_{n=1}^{\infty} \text{Area } (C'_n),$$

which is (17).

Let the situation be the same as in Lemmas 4, 5, and 6. By Lemma 5, it is possible to inscribe in T at least one circle, C_1, of curvature $\leqslant kc_0$. The complement of the interior of C_1 with respect to T consists of three smaller curvilinear triangles, and we inscribe into these further circles, and continue the process, as long as the curvatures of all such inscribed circles are $\leqslant kc_0$. The process comes to a stop after a finite number of inscriptions, and our central lemma (10) will assert that there exists an absolute constant $\delta > 0$ such that the area Γ of all the circles thus inscribed exceeds $\delta\Delta$. There are two cases to distinguish, depending on whether b is not too much smaller than c_0, so that the number of inscribed circles is small, or $b \ll c_0$, in which case there will be many such circles. These two cases necessitate two separate estimates of Γ/Δ from below. Both estimates will be in terms of the ratio b/c_0, one will be effective for b/c_0 small and the other one for b/c_0 large, and δ will be obtained by combining the two estimates.

LEMMA 7.

$$\Gamma/\Delta \geqslant \pi b/8k^2 c_0. \tag{18}$$

Since at least one circle C_1 of curvature $\leqslant kc_0$ can be inscribed in T, we have

$$\Gamma \geqslant \pi c_1^{-2}. \tag{19}$$

To obtain a bound on Δ, consider first the case of external tangency, and let p, q, and r be the points of tangency of A with B, A with C, and B with C, respectively. The area of the triangle pqr is at most $\frac{1}{2}|pr|\,|qr|$, and

$$|pr| \leqslant 2/b, \; |qr| \leqslant 2/c_0.$$

Since T is contained in pqr, we have $\Delta \leqslant 2/bc_0$. In the case of internal tangency, Δ is less than $(|pr| + |qr|)$ times the distance from r to A, and the latter is less than $2/b$; therefore here

$$\Delta \leqslant (2/b + 2/c_0)2/b \leqslant 8/bc_0.$$

Hence by (19)

$$\Gamma/\Delta \geqslant (\pi/8)(b/c_0)(c_0/c_1)^2,$$

and by (16), $c_0/c_1 \geqslant k^{-1}$, which gives us (18).

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Suppose now that the ratio b/c_0 is small so that the number of circles inscribed in T and having curvature $\leqslant kc_0$ is large. In particular, the number of those circles of the sequence $C_1, C_2, \ldots$ of Lemma 4 of curvature $\leqslant kc_0$ will be large. More specifically, let

$$c_m/c_0 \leqslant k < c_{m+1}/c_0 \tag{20}$$

so that all the circles $C_1, C_2, \ldots, C_m$ contribute to Γ, and

$$\Gamma \geqslant \pi \sum_{n=1}^{m} c_n^{-2}.$$

By Lemma 6, we have therefore

$$\Gamma/\Delta \geqslant (2k+1)^{-2} \sum_{n=1}^{m} c_n^{-2} \Big/ \sum_{n=1}^{\infty} c_n^{-2}. \tag{21}$$

Put for brevity

$$S = \sum_{n=1}^{m} c_n^{-2} \Big/ \sum_{n=1}^{\infty} c_n^{-2}. \tag{22}$$

LEMMA 8.

$$S \geqslant k^{-2}(1 - c_1/c_m). \tag{23}$$

By Lemma 5, we have $c_{n+1} \leqslant kc_n$ for $n \geqslant 0$ and therefore

$$\sum_{n=1}^{\infty} c_n^{-2} \leqslant k^2 \sum_{n=2}^{\infty} c_n^{-2}$$

so that

$$S \geqslant k^{-2} \sum_{n=1}^{m} c_n^{-2} \Big/ \sum_{n=2}^{\infty} c_n^{-2}. \tag{24}$$

By Lemma 4, $c_n = F(n)$, where

$$F(x) = (a+b)x^2 + 2x(ac_0 + bc_0 + ab)^{\frac{1}{2}} + c_0.$$

Since $F(x)$ is steadily increasing for $x \geqslant 1$, we can approximate the sums in (24) by integrals:

$$\sum_{n=1}^{m} c_n^{-2} \geqslant \int_1^m F^{-2}(x)\,dx, \qquad \sum_{n=2}^{\infty} c_n^{-2} \leqslant \int_1^{\infty} F^{-2}(x)\,dx;$$

hence by (24)

$$S \geqslant k^{-2} \int_1^m F^{-2}(x)\,dx \Big/ \int_1^{\infty} F^{-2}(x)\,dx.$$

Making the substitution $u = c_1/F(x)$, we obtain after some calculations

$$S \geqslant k^{-2} \int_{c_1/c_m}^{1} G(u)\,du \Big/ \int_0^1 G(u)\,du,$$

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where

$$G(u) = u^{\frac{1}{2}}[abu + (a + b)c_1]^{-\frac{1}{2}}.$$

Since $G(u)$ is increasing and positive for $0 < u \leqslant 1$, we have (23).

LEMMA 9.

(25) $$\Gamma/\Delta \geqslant (2k^2 + k)^{-2}[1 - h(b/c_0)/k]$$

where

(26) $$h(x) = [1 + 3(x^2 + 2kx)^{\frac{1}{2}} + 2(x^2 + 2x)^{\frac{1}{2}}][1 + 2x + 2(x^2 + 2x)^{\frac{1}{2}}].$$

By (20),

(27) $$c_1/c_m = (c_1/c_0)(c_{m+1}/c_m)(c_0/c_{m+1}) < (c_1/c_0)(c_{m+1}/c_m)/k.$$

Since, by (7),

$$c_1 = a + b + c_0 + 2(ab + ac_0 + bc_0)^{\frac{1}{2}}$$

and, by hypothesis,

(28) $$a \leqslant b \leqslant c_0,$$

we have the following upper bound on c_1/c_0 in terms of b/c_0:

(29) $$c_1/c_0 \leqslant 1 + 2b/c_0 + 2[(b/c\ )^2 + 2b/c_0]^{\frac{1}{2}}.$$

To obtain a similar upper bound on c_{m+1}/c_m, we have first, by (13) and (20),

$$(a + b)m^2 + 2m(ac_0 + bc_0 + ab)^{\frac{1}{2}} + c_0 \leqslant kc_0;$$

therefore m does not exceed the larger root of the quadratic equation in m, obtained by equating the two sides in the above; hence

$$(a + b)m \leqslant (kac_0 + kbc_0 + ab)^{\frac{1}{2}} - (ab + ac_0 + bc_0)^{\frac{1}{2}}.$$

Therefore by (28),

(30) $$(a + b)m/c_0 \leqslant [(b/c_0)^2 + 2kb/c_0]^{\frac{1}{2}}.$$

Next, by (13),

$$c_{m+1}/c_m = 1 + (c_{m+1} - c_m)/c_m < 1 + [(a + b)(2m + 1) + 2(ab + ac_0 + bc_0)^{\frac{1}{2}}]/c_0.$$

Since $m \geqslant 1$, we have $2m + 1 \leqslant 3m$ so that by (28) and (30) the above yields

(31) $$c_{m+1}/c_m \leqslant 1 + 3[(b/c\ )^2 + 2kb/c_0]^{\frac{1}{2}} + 2[(b/c_0)^2 + 2b/c_0]^{\frac{1}{2}}.$$

Now (25) follows from (21), (22), (23), (27), (29), and (31).

LEMMA 10.

(32) $$\Gamma/\Delta > 62 \times 10^{-6}.$$

We first simplify the bound (25), making it somewhat worse but better adapted for our computation. Since the parameter $b/c_0 = x$ satisfies $0 < x \leqslant 1$ by (28), we have $x^2 \leqslant x \leqslant x^{\frac{1}{2}}$, and therefore from (26)

(33) $h(x) \leqslant g(x) = \{1 + [3(2k+1)^{\frac{1}{2}} + 2.3^{\frac{1}{2}}]x^{\frac{1}{2}}\}[1 + 2(1 + 3^{\frac{1}{2}})x^{\frac{1}{2}}].$

Hence by (25), (26), and (33),

$$\Gamma/\Delta \geqslant (2k^2 + k)^{-2}[1 - g(x)/k].$$

Since (18) may be written as

$$\Gamma/\Delta \geqslant (\pi/8k^2)x,$$

we have

$$\Gamma/\Delta \geqslant \max\{(2k^2 + k)^{-2}[1 - g(x)/k], (\pi/8k^2)x\},$$

where k is given by (16). We now equate the two functions appearing in the above estimate and solve the resulting equation

$$8(2k+1)^{-2}[1 - g(x)/k] = \pi x$$

which is quadratic in $x^{\frac{1}{2}}$. The only positive root is $x = 0.00666\ldots$, and it follows that $\Gamma/\Delta \geqslant (\pi/8k^2)(0.00666) > 62 \times 10^{-6}$, proving (32).

6. LEMMA 11. *An osculatory packing* **C** *is solid and its exponent satisfies*

$$e(\mathbf{C}) < 1.999971.$$

Put $\delta = 62 \times 10^{-6}$, $\alpha = k^{-2}$, and let r_n and A_n denote respectively the radius and the area of the disk D_n. Let D_0 and D_1 satisfy $0 < r_0 < 1$, $r_1 = 1 - r_0$; by symmetry we may assume that $1/2 \leqslant r_0 < 1$. **C** is obtained by starting with the disks D_0 and D_1, and making use of the already employed inscription condition: whenever disks are to be inscribed into a curvilinear triangle T' bounded by three pairwise tangent disks, we inscribe successively all those disks whose area is $\geqslant \alpha$ times the area of the smallest of the three disks bounding T'. Lemma 5 guarantees that one such disk at least will always exist.

Let T be one of the two congruent curvilinear triangles making up $U - D_0 - D_1$. Let $j(1) = 1$, and inscribe in T disks $D_2, D_3, \ldots, D_{j(2)}$, subject to the above inscription condition. Then continue the process on each of the $2j(2) - 1$ curvilinear triangles composing $T - \cup_{i=2}^{j(2)} D_i$, obtaining new disks $D_{j(2)+1}, \ldots, D_{j(3)}$, and so on. We have therefore

(34) $$A_k \geqslant \alpha^{n-1}A_1 \qquad \text{for } k \leqslant j(n).$$

By Lemma 10, at each new stage of the process, i.e. adding the disks $D_{j(n)+1}, \ldots, D_{j(n+1)}$, we have ensured that the area left uncovered is $< (1 - \delta)$ times the area left uncovered at the end of the previous stage:

$$\left(\pi - A_0 - A_1 - 2\sum_{i=2}^{j(n+1)} A_i\right) < (1-\delta)\left(\pi - A_0 - A_1 - 2\sum_{i=2}^{j(n)} A_i\right);$$

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therefore by induction

$$(35)\qquad \left(\pi - A_0 - A_1 - 2\sum_{i=2}^{j(n+1)} A_i\right) < (\pi - A_0 - A_1)(1-\delta)^{n-1}.$$

Hence **C** is indeed a packing because the residual area of U, left uncovered after n stages of the inscription process, tends, by (35), to 0 as n tends to infinity.

Choose now $\tau < 1$, such that

$$(36)\qquad \alpha^{\tau-1}(1-\delta) < 1.$$

Then

$$(37)\qquad M_{2\tau}(\mathbf{C}) = \pi^{-\tau}\left(A_0^{\tau} + A_1^{\tau} + 2\sum_{n=2}^{\infty} A_n^{\tau}\right)$$

and

$$\sum_{n=2}^{\infty} A_n^{\tau} = \sum_{i=1}^{\infty}\sum_{k=j(i)+1}^{j(i+1)} A_k A_k^{\tau-1}.$$

Since the areas $A_1, A_2, \ldots$ are non-increasing and $\tau < 1$, the powers $A_1^{\tau-1}$, $A_2^{\tau-1}, \ldots$ are non-decreasing and therefore

$$\sum_{n=2}^{\infty} A_n^{\tau} \leqslant \sum_{i=1}^{\infty} A_{j(i+1)}^{\tau-1} \sum_{k=j(i)+1}^{j(i+1)} A_k \leqslant \sum_{i=1}^{\infty} A_{j(i+1)}^{\tau-1} \sum_{k=j(i)+1}^{\infty} A_k.$$

Hence by (34) and (35)

$$\sum_{n=2}^{\infty} A_n^{\tau} \leqslant c\sum_{i=1}^{\infty} (A_1\alpha^i)^{\tau-1}(1-\delta)^i \qquad \text{with } 0 < c < \infty.$$

Therefore all the sums converge by (36). But then (37) implies that $e(\mathbf{C}) \leqslant 2\tau$. Solving from (36) for τ, we obtain

$$e\mathbf{C} \leqslant 2\{1 + [\log(1 - 62\times 10^{-6})]/(2\log k)\} < 1.999971,$$

which completes the proof of Lemma 11.

Lemmas 3 and 11 can be combined into

THEOREM 3. *An osculatory packing* **C** *is a packing and its exponent satisfies*

$$1.035 < e(\mathbf{C}) < 1.999971.$$

Theorems 1 and 3, together with some accessory evidence, suggest strongly that the exponent $e(\mathbf{C})$ attains its minimum for an osculatory packing; further, that there exists a universal constant S (=the exponent of any osculatory packing), satisfying $1.035 < S < 1.999971$, and such that the set

$$\{e(\mathbf{C}) : \mathbf{C} \text{ is a packing}\}$$

is the closed interval $[S, 2]$. According to some numerical evidence from **(2)**, it is likely that the true value of S is somewhere near 1.3. While our bounds on S could be somewhat improved, it appears certain that some quite different method must be employed to find S.

7. Besides the hypotheses at the end of §6, our work raises several other questions relative to exponents. Is it always true that $M_{e(\mathbf{C})}(\mathbf{C})$ is finite? What is the exponent of "most" packings? Is there an m-dimensional generalization of our conjecture concerning the universal constant S? Is there any connection between the exponent of a packing $C = \{D_n\}$ and the Hausdorff dimension of the residual set $U - \bigcup_{n=1}^{\infty} D_n$?

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