Note on Numerical Integration.

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(Received 17th July 1927. Read 4th November 1927.)

1. All the commonly used rules for the approximate quadrature of areas, such as those of Cotes, Simpson, Tchebychef and Gauss, are based on the assumption that \( y \) can be expressed as a rational integral function of \( x \) with finite coefficients. A tacit assumption is thus made that \( \frac{dy}{dx} \) is not infinite within the range considered, and it is therefore hardly a matter for surprise that the degree of accuracy obtainable by the use of these rules in the case of a curve which touches the end ordinates is very poor.

In such a case, however, the difficulty can be entirely obviated by the use of a rule based on the assumption that the equation of the curve is

\[
y = \sqrt{1-x^2} \left( a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{2n-1} x^{2n-1} \right) \]

where \( n \) is the number of ordinates and the limits of the base are taken to be \( \pm 1 \) as usual.

It is proved in what follows that an expression can be obtained for the exact area of such a curve if the ordinates are measured at the points where

\[
x = \cos \left( \frac{k\pi}{n+1} \right) \quad (k = 1, 2, 3, \ldots, n) \]

The geometrical interpretation of this is shewn in Fig. 1, which is drawn for \( n = 5 \). Thus if we divide the circumference of a semi-circle into \( (n+1) \) equal parts the ordinates must pass through the \( n \) points thus found. The trouble of finding the position of the ordinates, which is the objection to the rules of Tchebychef and Gauss, does not exist in the case of this rule.

The value of the coefficients by which the ordinates have to be multiplied is also easily remembered. Thus, if we write as the expression for the area

\[
A = \sum_{k=1}^{n} (p_k y_k) \times l \]

where \( l \) is the perpendicular distance between the end ordinates, we have

\[
p_k = \frac{\pi}{2(n + 1)} \sin \left( \frac{k\pi}{n + 1} \right) \quad \ldots \ldots \ldots \ldots \ldots (4)
\]

This is easily remembered since \( \sin \left( \frac{k\pi}{n + 1} \right) \) is the length of the corresponding ordinate of the semicircle in Fig. (1) divided by the radius, so that we have

\[
A = \frac{\pi}{n + 1} \Sigma \text{(Ordinate of Semicircle)} \times \text{(Ordinate of Curve)} \quad \ldots \ldots (5)
\]

2. If now our given curve is of any unsymmetrical form, as in Fig. 3, we can always construct a symmetrical curve as in Fig. 2 having the same length for all the ordinates \( a_1 b_1, a_2 b_2, \ldots \), as those in Fig. 3 and the areas of the curves will clearly be equal to one another. Hence the rule will hold good for any curve such as that in Fig. 3.

The accuracy of the rule may be estimated by noting that we have \( 2n \) values of \( a_r \) at our disposal in (1). The area given by (3) is therefore that of a continuous curve which agrees with the given curve in \( 4n \) points in addition to \( A \) and \( B \). Of these points \( 2n \) are arbitrary, and therefore if these are supposed to move into coincidence with the points \( a_1, a_2, \ldots b_1, b_2, \ldots \), we may say that (3) gives the area of a curve which not only passes through all the \( (2n + 2) \) points \( A, a_1, a_2 \ldots B, b_1, b_2 \ldots \) but has a common tangent with the given curve at every one of these points also.

3. The unsuitable nature of the rules usually used when applied to curves of the nature considered can be best shown by considering the case of a circle of radius \( a \).

(1) *Cotes' Rule with Five Ordinates.*

\[
\text{Area} = \frac{b}{90} (7y_1 + 32y_2 + 12y_3 + 32y_4 + 7y_5)
\]

\[
= \frac{2r}{90} (2 \times 7 \times 0 + 2 \times 32 \times 1 \cdot 4142r + 12 \times 2r)
\]

\[
= 2 \cdot 6112r^2
\]

Error = \(-17\%\)
(2) Simpson’s Rule with Five Ordinates.

\[
\text{Area} = \frac{b}{12} \left( y_1 + 4y_2 + 2y_3 + 4y_4 + y_5 \right)
\]

\[
= \frac{2r}{12} \left( 0 + 2 \times 4 \times 1 \cdot 4142 + 2 \times 2 \right) r
\]

\[
= 2 \cdot 5548r^2 \quad \text{Error} = -18.7\%
\]

(3) Cotes’ Rule with Seven Ordinates.

\[
\text{Area} = \frac{b}{840} \left( 41y_1 + 216y_2 + 27y_3 + 272y_4 + 27y_5 + 216y_6 + 41y_7 \right)
\]

\[
= \frac{2r}{840} \left( 2 \times 216 \times 1 \cdot 49071 + 2 \times 27 \times 1 \cdot 88562 + 272 \times 2 \right) r
\]

\[
= 3 \cdot 7097r^2 \quad \text{Error} = -2.25\%
\]

(4) Weddle’s Rule with Seven Ordinates.

\[
\text{Area} = \frac{b}{20} \left( y_1 + 5y_2 + y_3 + 6y_4 + y_5 + 5y_6 + y_7 \right)
\]

\[
= \frac{2r}{20} \left( 2 \times 5 \times 1 \cdot 49071 + 2 \times 1 \cdot 88562 + 6 \times 2 \right) r
\]

\[
= 3 \cdot 06784r^2 \quad \text{Error} = -2.34\%
\]

(5) Simpson’s Rule with Seven Ordinates.

\[
\text{Area} = \frac{b}{18} \left( y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + 4y_6 + y_7 \right)
\]

\[
= \frac{2r}{18} \left( 2 \times 4 \times 1 \cdot 49071 + 2 \times 2 \times 1 \cdot 88562 + 4 \times 2 \right) r
\]

\[
= 3 \cdot 05202r^2 \quad \text{Error} = -2.85\%
\]

(6) Tchebychef’s Rule with Five ordinates.

Here the ordinates are measured along the straight lines

\[x = 0, \pm 0.3740r \text{ and } \pm 0.83250r\]

and the area is given by the base multiplied by the arithmetical mean of the ordinates.

In the case of the circle the lengths of the ordinates are

\[1.10804r, 1.85442r, 2r, 1.85442r, 1.10804r\]

and the area is found to be

\[\frac{1}{3} \times 7.92492r \times 2r = 3.16997r^2 \quad \text{Error} = +0.90\%
\]

(7) Gauss’s Rule with Five ordinates.

Here the ordinates are measured along the straight lines

\[x = 0, \pm 0.53847r, \pm 0.90618r\]
and the corresponding coefficients or weights are
\[0.2844, 0.2393, 0.1184.\]

The lengths of the measured ordinates in the case of a circle are therefore
\[2r, 1.6876r, 0.8458r.\]

The area is thus found to be
\[2r (0.2844 \times 2 + 2 \times 0.2393 \times 1.6876 + 2 \times 0.1184 \times 0.8458)r^2 = 3.15399r^2\]
Error = +0.39%.

The last two rules clearly give much better results than the first five, but owing to the trouble of setting off the ordinates and also to the fact that the values of the coefficients necessitate reference to books, they are hardly ever used in practice. The rule given in this paper gives exact results for a circle, even with only one ordinate, and no difficulty is met with however many ordinates are used, and there is therefore no difficulty in obtaining any desired degree of accuracy with any curve between parallel tangents.

4. As an example of a case in which the rule does not give mathematically exact results we may take the area of a cycloid between successive cusps. Here the equations are
\[x = a (\theta + \sin \theta),\]
\[y = a (1 + \cos \theta),\]
where \(a\) is the radius of the generating circle. The length of the base is \(2\pi a\) and the area is \(3\pi a^2\).

If we take five ordinates we shall have
\[x_1 = \frac{1}{4} \pi a\] and therefore \((\theta_1 + \sin \theta_1) = \frac{\pi}{2} = 1.570796.\]

By trial we find
\[\theta_1 = 0.83171\] and \(\sin \theta_1 = 0.73909\)
\[\therefore y_1 = a (1 + \cos \theta_1) = 1.67360a\]
similarly \(x_2 = \frac{\sqrt{3}}{2} \pi a\) and therefore \((\theta_2 + \sin \theta_2) = \frac{\sqrt{3}}{2} \pi = 2.72070\)
\[\therefore \theta_2 = 1.733985\] and \(\sin \theta_2 = 0.986715\)
\[\therefore y_2 = a (1 + \cos \theta_2) = 0.837535a.\]
Hence the area given by the rule

\[
\frac{\pi}{12} \left( \left( \frac{\sqrt{3}}{2} \times 1.67360 + \frac{1}{2} \times 837535 \right)^2 + 2 \right) \times 2\pi a^2
\]

\[= 3.0035 \pi a^2\]

Error = 0.12%.

The area given by Cotes' Rule with five ordinates

\[
\frac{2\pi a^2}{90} \left( 2 \times 32 \times 1.6736 + 12 \times 2 \right)
\]

\[= 2.9136 \pi a^2\]

Error = -2.88%.

and by Simpson's Rule with five ordinates

\[
\frac{2\pi a^2}{12} \left( 2 \times 4 \times 1.6736 + 2 \times 2 \right)
\]

\[= 2.8981 \pi a^2\]

Error = -3.40%.

**Proof of Formula.**

(1) From the Exponential Expression for \(\cos \theta\) we have

\[
2^{2r} \cos^{2r} \theta = (e^{i\theta} + e^{-i\theta})^{2r}
\]

\[= e^{2ir\theta} + 2re^{2i(r-1)\theta} + \ldots + \binom{2r}{r} \frac{(2r)!}{r! \cdot r!} + \ldots + 2re^{-2i(r-1)\theta} + e^{-2ir\theta}.
\]

Hence if \(\theta = \frac{k\pi}{n+1}\), then \(2^{2r} \cos^{2r} \left( \frac{k\pi}{n+1} \right) = \)

\[
e^{\frac{2irk\pi}{n+1}} + 2r \sum_{k=0}^{n} e^{\frac{2i(r-1)k\pi}{n+1}} + \ldots + \binom{2r}{r} \frac{(2r)!}{r! \cdot r!} + \ldots + 2r e^{-\frac{2i(r-1)k\pi}{n+1}} + e^{-\frac{2irk\pi}{n+1}}
\]

Now if we form a series of equations by putting \(k = 0, 1, 2 \ldots n\) in turn, we get on adding

\[
2^{2r} \sum_{k=0}^{n} \cos^{2r} \left( \frac{k\pi}{n+1} \right) = \sum_{k=0}^{n} e^{\frac{2irk\pi}{n+1}} + 2r \sum_{k=0}^{n} e^{\frac{2i(r-1)k\pi}{n+1}} + \ldots + (n+1) \frac{(2r)!}{r! \cdot r!} + \ldots + 2r \sum_{k=0}^{n} e^{-\frac{2i(r-1)k\pi}{n+1}} + \sum_{k=0}^{n} e^{-\frac{2irk\pi}{n+1}}
\]

Now each of the terms on the right, with the exception of the middle term, represents a Geometrical Progression with \((n+1)\) terms. Thus

\[
\sum_{k=0}^{n} e^{\frac{2irk\pi}{n+1}} = \frac{e^{\frac{2ir\pi}{n+1}} - 1}{e^{\frac{2ir\pi}{n+1}} - 1} = 0 \text{ if } r < (n+1)
\]

since the numerator vanishes for all integral values of \(r\), and the denominator does not vanish unless \(r\) is a multiple of \((n+1)\). Thus
all the terms except the middle term vanish, and we therefore get on dividing out by $2^{2r}(n + 1)$

$$\frac{1}{n + 1} \sum_{k=0}^{n} \cos^{2r} \left( \frac{k\pi}{n + 1} \right) = \frac{1}{2^{2r} r! r!} \frac{(2r)!}{(r + 1)! (r + 1)!} \quad \text{if } r < (n + 1)$$

and similarly

$$\frac{1}{n + 1} \sum_{k=0}^{n} \cos^{2r+2} \left( \frac{k\pi}{n + 1} \right) = \frac{1}{2^{2r+2} (r + 1)! (r + 1)!} \frac{(2r + 2)!}{(r + 1)! (r + 1)!} \quad \text{if } r < n$$

hence we find on subtraction

$$\frac{1}{n + 1} \sum_{k=0}^{n} \sin^{2} \left( \frac{k\pi}{n + 1} \right) \cos^{2r} \left( \frac{k\pi}{n + 1} \right) = \frac{1}{2^{2r+1} r! (r + 1)!} \frac{(2r)!}{(r + 1)! (r + 1)!} \quad \text{if } r < (n + 1)$$

Now when $k = 0$, the corresponding term on the left hand side of this equation vanishes (since $\sin^{2} \left( \frac{k\pi}{n + 1} \right) = 0$ when $k = 0$). Hence we must have

$$\frac{1}{n + 1} \sum_{k=1}^{n} \sin^{2} \left( \frac{k\pi}{n + 1} \right) \cos^{2r} \left( \frac{k\pi}{n + 1} \right) = \frac{1}{2^{2r+1} r! (r + 1)!} \frac{(2r)!}{(r + 1)! (r + 1)!} \quad \text{if } r < (n + 1) \quad \text{(2)}$$

(2) Again, we get on putting $x = \cos \theta$

$$\frac{1}{2} \int_{-1}^{1} x^{2r} \sqrt{1 - x^2} \, dx = \frac{1}{2} \int_{0}^{\pi} \sin^{2} \theta \cos^{2r} \theta \, d\theta = \frac{1}{2} \int_{0}^{\pi} \cos^{2r+2} \theta \, d\theta - \frac{1}{2} \int_{0}^{\pi} \cos^{2r} \theta \, d\theta$$

$$= \frac{(2r - 1)(2r - 3) \ldots 1}{2r(2r - 2) \ldots 2} \frac{\pi}{2} - \frac{(2r + 1)(2r - 1) \ldots 1}{(2r + 2)(2r) \ldots 4} \frac{\pi}{2}$$

$$= \frac{(2r - 1)(2r - 3) \ldots 1}{(2r + 2)(2r) \ldots 4} \frac{\pi}{4}$$

Now $(2r - 1)(2r - 3) \ldots 1 = \frac{(2r)!}{2r}$ and $(2r + 2)2r \ldots 4 = 2r(2r + 1)$!

and we therefore find on substituting

$$\frac{1}{2} \int_{-1}^{1} x^{2r} \sqrt{1 - x^2} \, dx = \frac{\pi}{2r + 2} \frac{(2r)!}{r! (r + 1)!} \quad \text{if } r < (n + 1) \quad \text{(3)}$$

Combining equations (1) and (3) we get finally

$$\sum_{k=1}^{n} \frac{\pi}{2(n + 1)} \sin^{2} \left( \frac{k\pi}{n + 1} \right) \cos^{2r} \left( \frac{k\pi}{n + 1} \right) = \frac{1}{2} \int_{-1}^{1} x^{2r} \sqrt{1 - x^2} \, dx \quad \text{(4)}$$

where $r$ and $n$ are any integers $(r < n)$. 
If now we consider the expression
\[ \sum_{k=1}^{n} \frac{\pi}{2(n+1)} \sin^2 \left( \frac{k\pi}{n+1} \right) \cos^{2r+1} \left( \frac{k\pi}{n+1} \right) \]
we see that
\[ \cos^{2r+1} \left( \frac{\pi}{n+1} \right) = - \cos^{2r+1} \left( \frac{n\pi}{n+1} \right) \]
\[ \cos^{2r+1} \left( \frac{2\pi}{n+1} \right) = - \cos^{2r+1} \left( \frac{(n-1)\pi}{n+1} \right) \]

etc.,
so that the terms cancel out in pairs. Also, if \( n \) is an odd number the middle term will be \( \cos^{2r+1} \frac{\pi}{2} = 0 \). Hence we find
\[ \sum_{k=1}^{n} \frac{\pi}{2(n+1)} \sin^2 \left( \frac{k\pi}{n+1} \right) \cos^{2r+1} \left( \frac{k\pi}{n+1} \right) = 0 \ldots \ldots \ldots \ldots (5) \]

Further, putting \( x = \cos \theta \) we find
\[ \frac{1}{2} \int_{-1}^{1} x^{2r+1} \sqrt{1-x^2} \, dx = \frac{1}{2} \int_{0}^{\pi} \cos^{2r+1} \theta \, d\theta - \frac{1}{2} \int_{0}^{\pi} \cos^{2r+3} \theta \, d\theta = 0 \ldots \ldots (6) \]

Thus it follows that if \( m < 2n \)
\[ \sum_{k=1}^{n} \frac{\pi}{2(n+1)} \sin^2 \left( \frac{k\pi}{n+1} \right) \cos^m \left( \frac{k\pi}{n+1} \right) = \frac{1}{2} \int_{-1}^{1} x^m \sqrt{1-x^2} \, dx \ldots \ldots (7) \]
for all values of \( m \) and \( n \), and therefore if
\[ x_k = \cos \left( \frac{k\pi}{n+1} \right) \quad \text{and} \quad p_k = \frac{\pi}{2(n+1)} \sin \left( \frac{k\pi}{n+1} \right) \ldots \ldots (8) \]
we have
\[ \sum_{k=1}^{n} p_k \sqrt{1-x_k^2} \cdot x_k^m = \frac{1}{2} \int_{-1}^{1} x^m \sqrt{1-x^2} \, dx \quad (m < 2n) \ldots \ldots (9) \]

From this it follows that if
\[ y = \sqrt{1-x^2} \left( a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n-1} x^{2n-1} \right) \]
and if we write \( y_k \) for the value of \( y \) corresponding to \( x = x_k \)
\[ \sum_{k=1}^{n} p_k y_k = \frac{1}{2} \int_{-1}^{1} y \, dx \ldots \ldots \ldots \ldots \ldots (10) \]
provided that \( p_k \) and \( x_k \) have the values given in (8). This follows since equation (7) is satisfied for each term in the expression for \( y \) separately.
Note.—The vital part of this proof consists in shewing that
\[ \frac{1}{n+1} \sum_{k=0}^{n} \sin^2 \left( \frac{k\pi}{n+1} \right) \cos^n \left( \frac{k\pi}{n+1} \right) \ldots \ldots \ldots \ldots (1) \]
is independent of \( n \) if \( m < 2n \).

For it clearly follows from this that
\[ \sum_{k=0}^{n+1} \frac{\pi}{n+1} \sin^2 \left( \frac{k\pi}{n+1} \right) \cos^n \left( \frac{k\pi}{n+1} \right) \ldots \ldots \ldots \ldots (2) \]
is also independent of \( n \) if \( n > \frac{m}{2} \). If now \( n \) is made to increase indefinitely, and we write \( \theta \) for \( \frac{k\pi}{n+1} \), \( \frac{\pi}{n+1} \) becomes \( d\theta \) and (2) becomes
\[ \int_{0}^{\pi} \sin^2 \theta \cos^n \theta \, d\theta = \int_{-1}^{1} \sqrt{1-x^2} x^m \, dx \ldots \ldots \ldots \ldots (3) \]
if \( \cos \theta \) is put equal to \( x \).

**Extension for Moments and Moments of Inertia of Section.**

(1) The Moment of the Area bounded by the curve in Fig. 3 about the central ordinate \( a_3 b_3 \) is given by
\[ M = \frac{l^2}{4} \int_{-1}^{1} y \, dx \]
and if we wish to find this in the form
\[ M = \sum_{k=1}^{n} (p'k' yk) \times l^2 \]
we must clearly make
\[ p'k = \frac{\pi}{2(n+1)} \sin \left( \frac{k\pi}{n+1} \right) \times \frac{1}{2} \cos \left( \frac{k\pi}{n+1} \right) = \frac{\pi}{8(n+1)} \sin \left( \frac{2k\pi}{n+1} \right) \ldots (1) \]

(2) In a similar manner we see that the Moment of Inertia about the same axis is given by
\[ I = \frac{l^3}{8} \int_{-1}^{1} yx^2 \, dx \]
and this will be given by
\[ I = \sum_{k=1}^{n} (p''k' yk) \times l^4 \]
if
\[ p''k = \frac{\pi}{2(n+1)} \sin \left( \frac{k\pi}{n+1} \right) \times \frac{1}{4} \cos^2 \left( \frac{k\pi}{n+1} \right) \]
\[ = \frac{\pi}{32(n+1)} \left( \sin \left( \frac{k\pi}{n+1} \right) + \sin \left( \frac{3k\pi}{n+1} \right) \right) \ldots \ldots (2) \]
(3) From these formulae the Moment of Inertia about an axis through the Centre of Gravity parallel to the ordinates is found by use of the formula

\[ I_{c.g.} = I - \frac{M^2}{A} \]  \hspace{1cm} (3)

where \( I, M \) and \( A \) are found by means of the formulae already given.

Note.—In the case of the formula for the Moments half the coefficients will have a negative sign.

Fig. 1

Fig. 2
Note on Discussion of the Paper.

In the course of the discussion it was suggested (I think by Prof. Whittaker) that a simpler method of dealing with curves of the form considered in the paper would be to apply Simpson’s Rule to the area enclosed between the given curve and a semicircle. This method is still open to the original objection unless the radius of curvature at the two extremities of the base is equal to half the length of the base.

Thus, taking the equation as being

\[ y = \sqrt{1-x^2}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots) \]

the subtraction of the ordinates of the semicircle will leave

\[ y' = \sqrt{1-x^2}((a_0 - 1) + a_1 x + a_2 x^2 + \ldots) \]

which is of the same form as before, though the error will be reduced in the ratio \((A_2 - A_1)/A_2\), where \(A_2\) is the area of the given curve and \(A_2\) that of the semicircle. Further, I am doubtful if any saving in time would be effected.